Multipliers of locally compact quantum groups and Hilbert C*-modules
3. Quantum groups

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Locally compact quantum groups

Such a thing is a von Neumann algebra $L^\infty(\mathbb{G})$ together with a unital, injective, normal $*$-homomorphism $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$ such that $(\Delta\otimes\iota)\Delta = (\iota\otimes\Delta)\Delta$.

The pre-adjoint Δ_* induces an associative product on $L^1(\mathbb{G}) = L^\infty(\mathbb{G})_*$. Then $L^1(\mathbb{G})$ becomes a (completely contractive) Banach algebra.

We assume the existence of left and right normal, faithful, semifinite weights. Using the left weight, we construct a Hilbert space $L^2(\mathbb{G})$ upon which $L^\infty(\mathbb{G})$ acts in standard position. There is a unitary map W (whose existence proof needs the right weight!) such that

$$\Delta(x) = W^*(1\otimes x)W \qquad (x\in L^\infty(\mathbb{G})).$$

If we define $C_0(\mathbb{G})$ to be the norm closure of

$$\{(\iota\otimes\omega)W : \omega\in B(L^2(\mathbb{G}))_*\}$$

then $C_0(\mathbb{G})$ is a C*-algebra with $C_0(\mathbb{G})'' = L^\infty(\mathbb{G})$, and such that Δ restricts to a map $C_0(\mathbb{G}) \to M\big(C_0(\mathbb{G})\otimes C_0(\mathbb{G})\big)$. Furthermore, the weights restrict to "nice" weights on $C_0(\mathbb{G})$. We think of $(C_0(\mathbb{G}),\Delta)$ as being the C*-algebraic counterpart to $L^\infty(\mathbb{G})$.

Duality

If we let $C_0(\hat{\mathbb{G}})$ be the norm closure of

$$\{(\omega\otimes\iota)W : \omega\in B(L^2(\mathbb{G}))_*\}$$

then this is a C*-algebra; let $L^\infty(\hat{\mathbb{G}}) = C_0(\hat{\mathbb{G}})''$. There exists a coassociative $\hat\Delta : L^\infty(\hat{\mathbb{G}}) \to L^\infty(\hat{\mathbb{G}})\overline{\otimes}L^\infty(\hat{\mathbb{G}})$. Also $L^\infty(\hat{\mathbb{G}})$ admits left and right invariant weights, and so becomes a locally compact quantum group in its own right.

We have that $\hat{\hat{\mathbb{G}}} = \mathbb{G}$ canonically.

In fact, the map

$$\lambda : L^1(\mathbb{G}) \to C_0(\hat{\mathbb{G}}); \quad \omega\mapsto(\omega\otimes\iota)W,$$

makes sense, and is a (completely) contractive homomorphism.

We have that:

$$W \in L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\hat{\mathbb{G}})$$
$$W \in M\big(C_0(\mathbb{G})\otimes B_0(L^2(\mathbb{G}))\big)$$
$$W \in M\big(B_0(L^2(\mathbb{G}))\otimes C_0(\hat{\mathbb{G}})\big)$$
$$W \in M\big(C_0(\mathbb{G})\otimes C_0(\hat{\mathbb{G}})\big).$$

Multipliers: from abstract to concrete

Theorem

Let $(L,R)\in M_{cb}(L^1(\hat{\mathbb{G}}))$. There exists $a\in M(C_0(\mathbb{G}))$ such that

$$\hat\lambda(L(\hat\omega)) = a\hat\lambda(\hat\omega), \quad \hat\lambda(R(\hat\omega)) = \hat\lambda(\hat\omega)a \qquad (\hat\omega\in L^1(\hat{\mathbb{G}})).$$

Proof.

Kraus and Ruan showed this for Kac algebras; not so hard to adapt the ideas to locally compact quantum groups.
Idea is to firstly define a as a (possibly unbounded) densely defined operator:

$$a\hat\lambda(\hat\omega)\xi = \hat\lambda(L(\hat\omega))\xi \qquad (\xi\in L^2(\mathbb{G}), \hat\omega\in L^1(\hat{\mathbb{G}})).$$

That this is well-defined needs the existence of R (and doesn't use any complete boundedness).
You then show that $(R^*\otimes\iota)(\hat W) = \hat W(1\otimes a)$ on some dense subspace of $L^2(\mathbb{G})\otimes L^2(\mathbb{G})$. As $\hat W$ is unitary, it follows that a must be bounded. □

One-sided Multipliers: from abstract to concrete

A corollary of the Junge, Neufang, Ruan representation result is:

Theorem (JNR)

Let $R \in M^r_{cb}(L^1(\hat{\mathbb{G}}))$. Then there exists $a \in L^\infty(\mathbb{G})$ such that

$$\hat\lambda(R(\hat\omega)) = \hat\lambda(\hat\omega)a \qquad (\hat\omega \in L^1(\hat{\mathbb{G}})).$$

Here $M^r_{cb}(L^1(\hat{\mathbb{G}}))$ is the space of completely bounded right multipliers. That is, bounded maps $R : L^1(\hat{\mathbb{G}}) \to L^1(\hat{\mathbb{G}})$ such that $R(ab) = aR(b)$ and with $R^* : L^\infty(\hat{\mathbb{G}}) \to L^\infty(\hat{\mathbb{G}})$ being completely bounded.

Towards a quantum version of Gilbert's Theorem

Theorem

Let $f \in C^b(G)$. The following are equivalent:

1. *$f \in M_{cb}A(G)$;*
2. *there exists a Hilbert space K, and bounded continuous maps $\alpha, \beta : G \to K$, such that $f(st^{-1}) = (\alpha(s)|\beta(t))$ for $s, t \in G$.*

Remember that when $A = C_0(G)$,

$$C_0(G, K) = A \otimes K, \quad C^b(G, K) = \mathcal{L}(A, A \otimes K).$$

In the previous talk, given $\alpha \in \mathcal{L}(A, A \otimes K)$, we constructed $\Delta * \alpha$ (somehow analogous to "applying Δ pointwise") in $\mathcal{L}(A \otimes A, A \otimes A \otimes K)$. Then we get:

Theorem

Let $f \in M(A)$. The following are equivalent:

1. *$f \in M_{cb}A(G)$;*
2. *there exists a Hilbert space K and $\alpha, \beta \in \mathcal{L}(A, A \otimes K)$ with $(1 \otimes \beta)^*(\Delta * \alpha) = f \otimes 1$ in $\mathcal{L}(A \otimes A) = M(A \otimes A)$.*

On the left

Theorem

Let $a \in M(C_0(\mathbb{G}))$, and let $L \in \mathcal{CB}(L^1(\hat{\mathbb{G}}))$ be defined by $L^*(\cdot) = \tilde\beta^*(\cdot \otimes 1)\tilde\alpha$, for some $\alpha, \beta \in \mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$. Then following are equivalent:

1. *$(1 \otimes \beta)^*(\Delta * \alpha) = a \otimes 1$;*
2. *L is a left multiplier, represented by a in the sense that $\hat\lambda(L(\hat\omega)) = a\hat\lambda(\hat\omega)$.*

Proof.

As $\Delta(\cdot) = W^*(1 \otimes \cdot)W$, we can recast the first condition as

$$(1 \otimes \tilde\beta^*)W^*_{12}(1 \otimes \tilde\alpha)W = a \otimes 1.$$

Then use that $\hat W = \sigma W^* \sigma$ to show that

$$(L \otimes \iota)(\hat W) = (1 \otimes a)\hat W,$$

which is equivalent to the second condition. (And then reverse the argument). □

On the left, harder

Theorem

Let $L \in M^l_{cb}(L^1(\hat{\mathbb{G}}))$ be represented by $a \in L^\infty(\mathbb{G})$. (Always true by [JNR].) Then there exist $\alpha, \beta \in \mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ with $(1 \otimes \beta)^*(\Delta * \alpha) = a \otimes 1$. So $a \in M(C_0(\mathbb{G}))$, and $L^*(x) = \tilde\beta^*(x \otimes 1)\tilde\alpha$.

Proof.

Basic idea is as in Jolissaint's proof of Gilbert's Theorem. We replace a unitary representation of G by a unitary corepresentation of $C_0(\mathbb{G})$. The complication comes that we need to use "universal quantum groups", see the paper of Kustermans. (For example, for non-amenable G, we really need to work with $C^*(G)$ and not $C^*_r(G)$). □

On the right

There exists an antilinear isometry $J : L^2(\mathbb{G}) \to L^2(\mathbb{G})$ connected to the Tomita-Takesaki theory of $L^\infty(\mathbb{G})$. Remarkably, this gives a map

$$\hat{\kappa} : L^\infty(\hat{\mathbb{G}}) \to L^\infty(\hat{\mathbb{G}}); \quad x \mapsto Jx^*J$$

which restricts to $C_0(\hat{\mathbb{G}})$, and satisfies $\hat{\Delta}\hat{\kappa} = \sigma(\hat{\kappa} \otimes \hat{\kappa})\hat{\Delta}$. So the pre-adjoint $\hat{\kappa}_* : L^1(\hat{\mathbb{G}}) \to L^1(\hat{\mathbb{G}})$ is an anti-homomorphism. Furthermore,

$$\hat{\lambda} \circ \hat{\kappa}_* = \kappa \circ \hat{\lambda}.$$

Theorem (Really, a lemma!)

Let L be a map on $L^1(\hat{\mathbb{G}})$, and let $R = \hat{\kappa}_* L \hat{\kappa}_*$. Then the following are equivalent:

1. *L is a completely bounded left multiplier represented by $a \in L^\infty(\mathbb{G})$ (or in $M(C_0(\mathbb{G}))$);*
2. *R is a completely bounded right multiplier represented by $\kappa(a) \in L^\infty(\mathbb{G})$ (or in $M(C_0(\mathbb{G}))$);*

So, end of story for right multipliers?

Using the oppposite algebra

Let $\hat{\mathbb{G}}^{\mathrm{op}}$ be the locally compact quantum group which has $L^\infty(\hat{\mathbb{G}}^{\mathrm{op}}) = L^\infty(\hat{\mathbb{G}})$ but with $\hat{\Delta}^{\mathrm{op}} = \sigma\hat{\Delta}$. We swap the left and right invariant weights.

We have that $L^1(\hat{\mathbb{G}}^{\mathrm{op}})$ is just the *opposite* Banach algebra to $L^1(\hat{\mathbb{G}})$. So left and right multipliers get swapped.

But what is the dual of $\hat{\mathbb{G}}^{\mathrm{op}}$? It is $\mathbb{G}'$, where $L^\infty(\mathbb{G}') = L^\infty(\mathbb{G})'$, the commutant. Tomita theory tells us that $L^\infty(\mathbb{G}') = JL^\infty(\mathbb{G})J$. We can similarly build a coproduct and weights. The multiplicative unitary is

$$W' = (J \otimes J)W(J \otimes J),$$

from which it follows that also $C_0(\mathbb{G}') = JC_0(\mathbb{G})J$.

There is a bijection between $\alpha' \in \mathcal{L}(C_0(\mathbb{G}'), C_0(\mathbb{G}') \otimes K)$ and $\alpha \in \mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ (we need to pick an "involution" J_K on K, so this isn't totally canonical).

Then $(1 \otimes \beta')^*(\Delta' * \alpha') = JaJ \otimes 1$ if and only if $(1 \otimes \beta)^*(\Delta * \alpha) = a \otimes 1$.

Swapping things about

Pick $R \in M^r_{cb}(L^1(\hat{\mathbb{G}}))$, and consider $R^{\mathrm{op}} \in M^l_{cb}(L^1(\hat{\mathbb{G}}^{\mathrm{op}}))$. Let a and JbJ "represent" R and R^{op}:

$$\hat{\lambda}(R(\hat{\omega})) = \hat{\lambda}(\hat{\omega})a, \quad \hat{\lambda}^{\mathrm{op}}(R^{\mathrm{op}}(\hat{\omega})) = JbJ\hat{\lambda}^{\mathrm{op}}(\hat{\omega}).$$

We can find α', β' associated to R^{op}; this leads to α, β with

$$b \otimes 1 = (1 \otimes \beta)^*(\Delta * \alpha).$$

It turns out that

$$\kappa(a) \otimes 1 = (1 \otimes \alpha)^*(\Delta * \beta).$$

We call a pair (α, β) "invariant" if $(1 \otimes \beta)^*(\Delta * \alpha) \in M(C_0(\mathbb{G})) \otimes 1$.

So we've naturally found that (α, β) is invariant (say for b) if and only if (β, α) is invariant (say for $\kappa(a)$). What's the relationship between a and b?

The (unbounded) antipode

The *antipode* S on $\mathbb{G}$ is an unbounded operator: it plays the role of the inverse in quantum group theory. For Kac algebras, we have that $S = \kappa$, but even for compact quantum groups, S may be unbounded and only densely defined. For example, we have

$$S\big((\iota \otimes \omega)(W)\big) = (\iota \otimes \omega)(W^*) \qquad (\omega \in B(H)_*).$$

We can "split off" the unbounded part of S. There is a strong*-continuous automorphism group $(\sigma_t)_{t\in\mathbb{R}}$ of $L^\infty(\mathbb{G})$ (which restricts to a norm continuous automorphism group of $C_0(\mathbb{G})$). You can analytically extend this to unbounded maps; then

$$S = \kappa\tau_{-i/2}.$$

Then, if $(\alpha, \beta) \to b$ and $(\beta, \alpha) \to \kappa(a)$, then $b = \tau_{-i/2}(a^*)$.

Taking a coordinate approach

Let (e_i) be an orthonormal basis for K. Recall that we can view $C_0(\mathbb{G}) \otimes K$ as being those families (a_i) in $C_0(\mathbb{G})$ with $\sum_i a_i^* a_i$ converging in norm.

Similarly, we can view $\mathcal{L}(C_0(\mathbb{G}), C_0(\mathbb{G}) \otimes K)$ as being those families (a_i) in $\mathcal{L}(C_0(\mathbb{G})) = M(C_0(\mathbb{G}))$ with $\sum_i a_i^* a_i$ converging in the *strict* topology.

So let α be associated to (α_i), and similarly for β. Then (α, β) is invariant for b if and only if

$$\sum_i (1 \otimes \beta_i^*)\Delta(\alpha_i) = b \otimes 1.$$

Theorem

Let $\overline{S}$ be the strict extension of S to $M(C_0(\mathbb{G}))$. If $a \in M(C_0(\mathbb{G}))$ and we have families (α_i), (β_i) with

$$a \otimes 1 = \sum_i \Delta(\alpha_i)(1 \otimes \beta_i),$$

then $a \in D(\overline{S})$ and

$$\overline{S}(a) \otimes 1 = \sum_i (1 \otimes \alpha_i)\Delta(\beta_i).$$

Universal quantum groups

For a group G, we let $C^*(G)$ be the universal (or full) group C*-algebra: this is the completion of $L^1(G)$ under the largest C*-norm.

We can do a similar thing for quantum groups: but firstly we need to turn $L^1(\mathbb{G})$ into a $*$-algebra. The standard way to do this is to use the involution, so we need to restrict to a subalgebra $L^1_\sharp(\mathbb{G})$ of $L^1(\mathbb{G})$ where this is bounded. This then leads to $C_u(\hat{\mathbb{G}})$. Denote the $*$-representation by

$$\lambda_u : L^1_\sharp(\mathbb{G}) \to C_u(\hat{\mathbb{G}}).$$

The dual of $C_u(\hat{\mathbb{G}})$ is a dual Banach algebra (follow an argument of [Runde]) which contains $L^1(\hat{\mathbb{G}})$ as an ideal. So we get an inclusion $C_u(\hat{\mathbb{G}})^* \to M_{cb}(L^1(\hat{\mathbb{G}}))$.

Aside: If $\hat{\mathbb{G}}$ is *co-amenable* ($L^1(\hat{\mathbb{G}})$ has a bai) then $C_u(\hat{\mathbb{G}}) = C_0(\hat{\mathbb{G}})$ and $M_{cb}(L^1(\hat{\mathbb{G}})) = C_0(\hat{\mathbb{G}})^*$. Is the converse true? (This is Losert's Theorem for $A(G)$.)

Universal quantum groups: in our framework

Pick $\mu \in C_u(\hat{\mathbb{G}})^*$ and choose a representation $\theta : C_u(\hat{\mathbb{G}}) \to B(K)$ and $\xi, \eta \in K$ with

$$\mu(a) = (\xi | \theta(a)\eta) \qquad (a \in C_u(\hat{\mathbb{G}})).$$

[Kustermans]$\Rightarrow$ there is a unitary $U \in M(C_0(\mathbb{G}) \otimes B_0(K))$ with

$$\theta(\lambda_u(\omega)) = (\omega \otimes \iota)(U), \quad (\Delta \otimes \iota)(U) = U_{13}U_{23}.$$

Associate U with $\mathcal{U} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K)$, and let

$$\alpha = \mathcal{U}^*(\iota \otimes \xi)^*, \quad \beta = \mathcal{U}^*(\iota \otimes \eta)^*.$$

Then (α, β) induces the left multiplier induced by μ. (Indeed, if μ were actually in $L^1(\hat{\mathbb{G}})$ already, we could take $U = W$.)

Double multipliers

Theorem

Let $(L, R) \in M_{cb}(L^1(\hat{\mathbb{G}}))$. There exists a Hilbert space K with an involution J_K, $\mathcal{T} \in \mathcal{L}(C_0(\mathbb{G}) \otimes K)$ and $\xi, \eta \in K$ such that:

- *with $\alpha = \mathcal{T}(\iota \otimes \xi)^*$ and $\beta = \mathcal{T}(\iota \otimes \eta)^*$, we have that (α, β) induces L;*
- *with $\alpha = \mathcal{T}(\iota \otimes J_K\eta)^*$ and $\beta = \mathcal{T}(\iota \otimes J_K\xi)^*$, we have that (α, β) induces $\hat{\kappa}_* R \hat{\kappa}_*$ (and thus R).*

Proof.

The proof "glues" two Hilbert spaces together, but this isn't entirely trivial: you definitely need that (L, R) is a double multiplier (and not just unconnected left and right multipliers).. □

If (L, R) is induced by $C_u(\hat{\mathbb{G}})^*$, then we can take $\mathcal{T}$ to be unitary. Is the converse true? (Probably equivalent to my earlier question!)