Banach Algebras with an algebraic structure of Kakutani-Kodaira flavour

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Recall: Multipliers on Banach algebras

Let A be a Banach algebra with a faithful multiplication.

- $\mu : A \longrightarrow A$ is a left multiplier if $\mu(ab) = \mu(a)b$,
  a right multiplier if $\mu(ab) = a\,\mu(b)$.
- For $a \in A$, $\ell_a : x \longmapsto ax$ is a left multiplier,
  $r_a : x \longmapsto xa$ is a right multiplier.
- $LM(A) :=$ the left multiplier algebra of A $(\subseteq B(A))$,
  $RM(A) :=$ the right multiplier algebra of A $(\subseteq B(A)^{op})$.
  Then $LM(A)$ and $RM(A)$ are Banach algebras.
  The multiplier algebra $M(A)$ of A is also defined.
- $a \longmapsto \ell_a$ and $a \longmapsto r_a$ are injective and contractive.

Recall: Multipliers on Banach algebras

- If A has a bounded approximate identity (BAI), then
$$\|\cdot\|_{LM(A)} \sim \|\cdot\|_A \sim \|\cdot\|_{RM(A)} \text{ on } A.$$
In this case, A is identified with a left closed ideal in $LM(A)$, and a right closed ideal in $RM(A)$.

- For $\mu \in LM(A)$, we write $\mu \in A$ if $\mu = \ell_a$.
For $\mu \in RM(A)$, we write $\mu \in A$ if $\mu = r_a$.

Question: How can A be characterized inside $LM(A)$, $RM(A)$?

Motivation – a range space problem

For $\mathbb{G} = L_\infty(G)$, $VN(G)$, in the representation

$$\Theta^r : M^r_{cb}(L_1(\mathbb{G})) \cong CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

by Neufang-Ruan-Spronk (08),

$$\Theta^r(L_1(\mathbb{G})) = CB^{\sigma, (L_\infty(\mathbb{G}), ?)}_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G}))) \ ?$$

One may ask the same question for the representation of general LCQGs by Junge-Neufang-Ruan (09).

- Using a measure theoretic proof, N-R-S showed that

$$\Theta^r(L_1(G)) = CB^{\sigma, (L_\infty(G), C_b(G))}_{VN(G)}(B(L_2(G))) .$$

- The question was open for $\mathbb{G} = VN(G)$ (N.-R.-S.).
- We will consider a **Banach algebraic** approach to this range space problem.

Banach algebraic approach

- Any Banach algebra A is a *right* $RM(A)$-module via

$$A \times RM(A) \longrightarrow A, \ (a, \mu) \longmapsto \mu(a) \, .$$

  Then A^*, A^{**} are naturally left, right $RM(A)$-modules, resp.

- $\langle AA^* \rangle = \overline{\mathrm{span}\{a \cdot f : \ a \in A, \ f \in A^*\}}^{\|\cdot\|} \subseteq \langle RM(A) \cdot A^* \rangle \subseteq A^*$.

- Recall: a Banach space X is weakly sequentially complete (**WSC**) if every w-Cauchy sequence in X is w-convergent.

- The predual of a von Neumann algebra is WSC.

Proposition. (H.-N.-R.) Let A be a WSC Banach algebra with a sequential BAI. Then for $\mu \in RM(A)$, T.F.A.E.

(i) $\mu \in A$.

(ii) $\mu \cdot A^* \subseteq \langle AA^* \rangle$.

(iii) $\exists\, m \in A^{**}$ such that $n \cdot \mu = n \Diamond m \ \ (n \in A^{**})$.

The left version also holds.

- The following inequality is crucial in the proof:

  card(BAI) $\leq$ a cardinal level of weak completeness of A.

- However, A can have a BAI but without any sequential BAI.

Banach algebraic approach: More general situation

- $L_1(G)$ has a sequential BAI $\Longleftrightarrow$ G is metrizable.
- $A(G)$ has a sequential BAI $\Longleftrightarrow$ G is amenable σ-compact.

More general, it can be shown that

- $\min\{\text{card}(J) : (e_j)_{j\in J} \text{ is a BAI of } L_1(G)\}$
  $=$ the local weight $\chi(G)$ of G.
- $\min\{\text{card}(J) : (e_j)_{j\in J} \text{ is a BAI of } A(G)\}$
  $=$ the compact covering number $\kappa(G)$ of G.

Our approach: Consider Banach algebras A with a "Large" family of "Small" subalgebras. More precisely, A has a family $\{A_i\}$ of subalgebras such that each A_i is WSC with a sequential BAI, and $\{A_i\}$ is large so that each $\mu \in RM(A)$ is determined by its behavior on these subalgebras.

Banach algebras of type (M) – definition

Definition. (H.-N.-R.) Let A be a Banach algebra with a BAI. Suppose that for every $\mu \in RM(A)$, there is a closed subalgebra B of A with a BAI satisfying the following conditions.

(1) $\mu|_B \in RM(B)$.

(2) $f|_B \in BB^*$ for all $f \in AA^*$.

(3) There is a family $\{B_j\}$ of closed right ideals in B such that

- (i) each B_j is WSC with a sequential BAI;
- (ii) for all j, there is a left B_j-module projection from B onto B_j;
- (iii) $\mu \in A$ if $\mu|_{B_j} \in B_j$ for all j.

Then A is said to be of type (RM).

Similarly, Banach algebras of type (LM) are defined.

- A is of type (M) if A is both of type (LM) and of type (RM).

The classical Kakutani-Kodaira theorem.

Let G be a σ-compact locally compact group. Then
$\forall$ sequence (U_n) of neighborhoods of e,
$\exists$ a compact normal subgroup N of G such that

$$N \subseteq \bigcap U_n \quad \text{and} \quad G/N \text{ is metrizable.}$$

A generalized Kakutani-Kodaira theorem

Theorem. (H. 05) Let G be a locally compact group. Then
$\forall$ family $(U_j)_{j \in J}$ of neighborhoods of e with card$(J) \leq \kappa(G)\aleph_0$,
$\exists$ a compact normal subgroup N of G such that

$$N \subseteq \bigcap U_j \quad \text{and} \quad \chi(G/N) \leq \kappa(G)\aleph_0 .$$

- In fact, this generalized K-K theorem was motivated by its dual version (H. 02), which was used to study the **ENAR** of $A(G)$ in the sense of Granirer (96). We give below a unified K-K theorem in the setting of Kac algebras.
- Recently, this generalized K-K theorem is used by Filali-Neufang-SanganiMonfared (09) and Losert-Neufang-Pachl-Steprāns in their study of topological centres of $A(G)$ and $M(G)$, resp.

Recall: Kac algebras and reduced Kac algebras

Let $\mathbb{K} = (\mathcal{M}, \Gamma, \kappa, \varphi)$ be a **Kac algebra**, where $\kappa : \mathcal{M} \longrightarrow \mathcal{M}$ is an involutive anti-automorphism satisfying

$$(\kappa \otimes \kappa) \circ \Gamma \; = \; \Sigma \circ \Gamma \circ \kappa \, .$$

Let $p \in \mathcal{M}$ be a central projection such that

$$\Gamma(p) \; \geq \; p \otimes p \quad \text{and} \quad \kappa(p) \; = \; p \, .$$

Let $r : \mathcal{M} \longrightarrow \mathcal{M}_p$ be the canonical surjection $x \longmapsto xp$. Then $\mathbb{K}_p = (\mathcal{M}_p, \Gamma_p, \kappa_p, \varphi_p)$ is a **reduced Kac algebra** of $\mathbb{K}$, where

$$\Gamma_p(r(x)) \; = \; (r \otimes r)\Gamma(x) \quad \text{and} \quad \kappa_p(r(x)) \; = \; r(\kappa(x)) \, ,$$

and φ_p is obtained by reduction from φ.

Recall: Kac algebras and reduced Kac algebras

- $L_\infty(G)$ and $VN(G)$ are Kac algebras, and $\widehat{L_\infty(G)} = VN(G)$.

- It is known (Takesaki-Tatsuuma 71) that

  - For $\mathbb{K} = L_\infty(G)$,

    $\mathbb{K}_P$ is a reduced Kac algebra of $L_\infty(G)$ iff

    $\mathbb{K}_P = L_\infty(H)$ for some open subgroup H of G.

  - For $\mathbb{K} = VN(G)$,

    $\mathbb{K}_P$ is a reduced Kac algebra of $VN(G)$ iff

    $\mathbb{K}_P = VN(G/N)$ for some comp. normal subgroup N of G.

Recall: Decomposability number

- For a von Neumann algebra $\mathcal{M}$, the **decomposability number** $\operatorname{dec}(\mathcal{M})$ of $\mathcal{M}$ is the greatest carnality of a family of pairwise orthogonal non-zero projections in $\mathcal{M}$.

  E.g., $\operatorname{dec}(B(H)) = \dim(H)$ and $\operatorname{dec}(B(H)^{**}) = 2^{2^{\dim(H)}}$.

Theorem. (H.-N. 06) Let G be an infinite LCG. Then

(i) $\operatorname{dec}(L_\infty(G)) = \kappa(G)\aleph_0$.

(ii) $\operatorname{dec}(VN(G)) = \chi(G)\aleph_0$.

A Kac algebraic Kakutani-Kodaira theorem

Definition (H. 05) For a Kac algebra $\mathbb{K}$ and a cardinal α, the α^{th} **Kakutani-Kodaira number** $\delta_\alpha(\mathbb{K})$ of $\mathbb{K}$ is the least cardinal κ such that

$\forall$ family $(\mathcal{U}_i)_{j\in J}$ of w^*-nbhds of $id_{\mathbb{K}}$ with card$(J) \leq \alpha$,
$\exists$ a reduced Kac algebra $\mathbb{K}_p$ of $\mathbb{K}$ such that

$$p \in \bigcap \mathcal{U}_j \quad \text{and} \quad \mathrm{dec}\,(\mathbb{K}_p) \leq \kappa\,.$$

Then $\delta_\alpha(\mathbb{K}) \leq \delta_\beta(\mathbb{K})$ if $\alpha \leq \beta$. We denote $\delta_1(\mathbb{K})$ by $\delta(\mathbb{K})$.

Theorem. (H. 05) If $\mathbb{K} = L_\infty(\mathbb{G})$ or $VN(G)$, then

$$\delta(\mathbb{K}) \leq \mathrm{dec}(\widehat{\mathbb{K}}),$$

and the equality holds for many $\mathbb{K}$ with uncountable $\mathrm{dec}(\widehat{\mathbb{K}})$.

Equivalently, we have $\delta_{\mathrm{dec}(\widehat{\mathbb{K}})}(\mathbb{K}) \leq \mathrm{dec}(\widehat{\mathbb{K}})$.

More on quantitative description of duality

More on dual relation between $\mathbb{K}$ and $\widehat{\mathbb{K}}$ can be described quantitatively in terms of these Kac algebraic invariants. For example, we have the following.

Theorem. (H. 05) For $\mathbb{K} = L_\infty(\mathbb{G})$ or $VN(G)$, there exists a one-to-one correspondence between the families

$$\{\text{maximally decomposable sub Kac algebras of } \mathbb{K}\}$$

and

$$\{\text{norm closed } \widehat{\mathbb{K}}\text{-invariant } *\text{-subalgebras } \mathcal{A} \text{ of } L_1(\widehat{\mathbb{K}}) \text{ with } \mathrm{dense}(\mathcal{A}) = \mathrm{dec}(\mathbb{K})\}.$$

Banach algebras of type (M)

Using the Kac algebraic Kakutani-Kodaira theorem, we showed that the class of Banach algebras of type (M) includes:

- group algebras $L_1(G)$;
- weighted convolution (Beurling) algebras $L_1(G, \omega)$;
- Fourier algebras $A(G)$ of amenable G.

This class also includes:

- WSC Banach algebras A with a central BAI and A being an ideal in A^{**} ;
- WSC Banach algebras with a sequential BAI, in particular, quantum group algebras $L_1(\mathbb{G})$ of co-amenable $\mathbb{G}$ with $L_1(\mathbb{G})$ separable.

- It turns out that Banach algebras A of type (M) behave well regarding multipliers and structures on A^{**}.

Theorem. (H.-N.-R.) Let A be a Banach algebra of type (RM). Then for $\mu \in RM(A)$, T.F.A.E.

(i) $\mu \in A$.

(ii) $\mu \cdot A^* \subseteq \langle AA^* \rangle$.

(iii) $\exists\, m \in A^{**}$ such that $n \cdot \mu = n \Diamond m \ (n \in A^{**})$.

The left version holds for A of type (LM) and $\mu \in LM(A)$.

Theorem. (H.-N.-R.) Let $\mathbb{G}$ be a LCQG and let

$$\Theta^r : M^r_{cb}(L_1(\mathbb{G})) \cong CB^{\sigma, L_\infty(\mathbb{G})}_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G})))$$

be the completely isometric representation by J.-N.-R.

If $L_1(\mathbb{G})$ is of type (M) (e.g., $\mathbb{G}$ is co-amenable and $L_1(\mathbb{G})$ is separable), then

$$\Theta^r(L_1(\mathbb{G})) = CB^{\sigma, (L_\infty(\mathbb{G}), RUC(\mathbb{G}))}_{L_\infty(\widehat{\mathbb{G}})}(B(L_2(\mathbb{G}))),$$

where $RUC(\mathbb{G}) := \langle L_1(\mathbb{G}) \star L_\infty(\mathbb{G}) \rangle$.

- In particular, for every amenable LCG G, we have

$$\Theta(A(G)) \;=\; CB^{\sigma,\,(VN(G),\,UC(\widehat{G}))}_{L_\infty(G)}(B(L_2(G)))\,.$$

where $UC(\widehat{G}) = \langle VN(G) \cdot A(G) \rangle = \langle A(G) \cdot VN(G) \rangle$.

This answers the open question by N.-R.-S. (08).

- The converse of the above is also true. That is, if we let

$$A_\Theta(G) \;:=\; \{\mu \in M_{cb}A(G) :\; \mu \cdot VN(G) \subseteq UC(\widehat{G})\},$$

then we have

Corollary. G is amenable $\iff$ $A_\Theta(G) \;=\; A(G)$.

Recall: Arens products

- The **left Arens product** $\Box$ on A^{**} is naturally defined when A is considered as a *left* A-module:

  for $a, b \in A$, $f \in A^*$, and $m, n \in A^{**}$, we have

  $$\langle f \cdot a, b\rangle = \langle f, ab\rangle,\ \langle n \Box f, a\rangle = \langle n, f \cdot a\rangle,\ \langle m \Box n, f\rangle = \langle m, n \Box f\rangle.$$

- The **right Arens product** $\Diamond$ on A^{**} is defined similarly.

- Equivalently,

  $$m \Box n = w^*\text{-}\lim_{\alpha} \lim_{\beta} a_\alpha b_\beta \quad \text{and} \quad m \Diamond n = w^*\text{-}\lim_{\beta} \lim_{\alpha} a_\alpha b_\beta$$

  whenever (a_α), (b_β) are nets in A w^*-convergent to m, n.

- Both $\Box$ and $\Diamond$ extend the multiplication on A.

Recall: Arens regularity

- A is said to be **Arens regular** if $\Box$ and $\Diamond$ coincide.
- Every operator algebra (in particular, every C^*-algebra) and every quotient algebra thereof are Arens regular.

It is known that

(i) $L_1(G)$ is Arens regular $\Longleftrightarrow$ G is finite (Young 73).

(ii) for amenable G, $A(G)$ is Arens regular $\Longleftrightarrow$ G is finite (Lau 81).

- (i) and (ii) can be seen dual to each in the setting of LCQG, noticing that $L_\infty(G)$ is always co-amenable, and $VN(G)$ is co-amenable iff G is amenable.
- It is still open whether (ii) holds for all LCGs G.

Recall: Topological centres

- $(A^{**}, \Box)$ is a **right topological semigroup** under w^*-top:

  for any fixed $m \in A^{**}$, $n \longmapsto n\Box m$ is w^*-w^* cont.

- Similarly, $(A^{**}, \Diamond)$ is a **left topological semigroup.**

- The topological centres of $(A^{**}, \Box)$ and $(A^{**}, \Diamond)$ are

$$\mathfrak{Z}_t(A^{**}, \Box) = \{m \in A^{**} : n \longmapsto m\Box n \text{ is } w^*\text{-}w^* \text{ cont.}\},$$

$$\mathfrak{Z}_t(A^{**}, \Diamond) = \{m \in A^{**} : n \longmapsto n\Diamond m \text{ is } w^*\text{-}w^* \text{ cont.}\},$$

simply called the **left** and **right** topological centres of A^{**}.

Recall: Topological centres

- $A \subseteq \mathfrak{Z}_t(A^{**}, \Box) \subseteq A^{**}$; $\quad A \subseteq \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A^{**}$.

- $\mathfrak{Z}_t(A^{**}, \Box) = A^{**} \iff A$ is AR $\iff \mathfrak{Z}_t(A^{**}, \Diamond) = A^{**}$.

- A is said to be **left strongly Arens irregular** (LSAI) if $\mathfrak{Z}_t(A^{**}, \Box) = A$ (Dales-Lau 05).

  Similarly, RSAI and SAI are defined.

- Every group algebra $L_1(G)$ is SAI (Lau-Losert 88).

Theorem. (H.-N.-R.) Let A be a Banach algebra of type (M). Then for $m \in A^{**}$, T.F.A.E.

(i) $m \in A$.

(ii) $m \in \mathfrak{Z}_t(A^{**}, \Box)$ and $m \cdot A \subseteq A$.

(iii) $m \in \mathfrak{Z}_t(A^{**}, \Diamond)$ and $A \cdot m \subseteq A$.

Corollary. Let A be a Banach algebra of type (M). Then

(1) A is LSAI $\iff$ $\mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A$;

(2) A is RSAI $\iff$ $A \cdot \mathfrak{Z}_t(A^{**}, \Diamond) \subseteq A$.

Recall: The left quotient algebra $\langle A^*A\rangle^*$ of A^{**}

- $\langle A^*A\rangle$ is an A-submodule of A^* and is **left introverted** in A^* (i.e., a *left* $(A^{**},\Box)$-submodule of A^*).

- $\Box$ on A^{**} induces a product on $\langle A^*A\rangle^*$ such that the canonical quotient map $A^{**} \longrightarrow \langle A^*A\rangle^*$ yields

$$(\langle A^*A\rangle^*,\Box) \;\cong\; (A^{**},\Box)/\langle A^*A\rangle^{\perp}\,.$$

- $\langle A^*A\rangle^*$ is a also right topological semigroup under the w^*-topology. Its topological centre is defined by

$$\mathfrak{Z}_t(\langle A^*A\rangle^*) \;=\; \{m \in \langle A^*A\rangle^* : n \longmapsto m\Box n \text{ is } w^*\text{-}w^* \text{ cont.}\}.$$

- For every LCG G, $\mathfrak{Z}_t(LUC(G)^*) \;=\; M(G)$ (Lau 86).

Some asymmetry phenomena

- Let $q : A^{**} \longrightarrow \langle A^*A \rangle^*$ be the canonical quotient. Then

$$q(\mathfrak{Z}_t(A^{**}, \Box)) \subseteq \mathfrak{Z}_t(\langle A^*A \rangle^*) .$$

- If A has a BRAI, then $RM(A) \hookrightarrow \mathfrak{Z}_t(\langle A^*A \rangle^*) \subseteq \langle A^*A \rangle^*$.

Proposition. (H.-N.-R.) If A has a BRAI, then

$$\mathfrak{Z}_t(\langle A^*A \rangle^*) = RM(A) \iff A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A .$$

- Recall from the above: If A is of type (M), then

$$\mathfrak{Z}_t(A^{**}, \Box) = A \iff \mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A .$$

In next lecture, we shall explain this asymmetry and show what is missing here.

The answer to an open question by Lau-Ülger

For $m \in A^{**}$, let $m_R : A^* \longrightarrow A^*$ be the map $f \longmapsto f \diamondsuit m$.

- **Question** (Lau-Ülger 96):

  For a WSC Banach algebra A with a BAI, if $m \in \mathfrak{Z}_t(A^{**}, \Box)$,

  are $\ker(m_R)$ and $m_R(\mathrm{ball}(A^*))$ w^*-closed in A^*?

- **Answer**: It can be **negative** for A of type (M) with **Property (X)** (Godefroy-Talagrand 81).

A special case for the answer is as follows.

Proposition. (H.-N.-R.) Let $\mathcal{M}$ be a von Neumann algebra with $A = \mathcal{M}_*$ separable with a BAI. Then, for any $m \in \mathfrak{Z}_t(A^{**}, \Box) \setminus A$,

either $\ker(m_R)$ or $m_R(\mathrm{ball}(A^*))$ is not w^*-closed in A^*.

An outline of the proof:

$A = \mathcal{M}_*$ is separable $\implies$ A has the **Mazur property**

(i.e., for $m \in A^{**}$, we have $m \in A$ if m is sequentially w^*-cont).

In this case,

$$\ker(m_R) \text{ and } m_R(\mathrm{ball}(A^*)) \text{ are both } w^*\text{-closed in } A^*$$

$\iff$ $m_R : A^* \longrightarrow A^*$ is w^*-w^* cont (Godefroy 89).

Let $m \in \mathfrak{Z}_t(A^{**}, \Box) \setminus A$. By our characterization of A inside $\mathfrak{Z}_t(A^{**}, \Box)$ given above, we have

$$m \cdot A \not\subseteq A; \text{ i.e., } m_R^*(A) \not\subseteq A.$$

Therefore, $m_R : A^* \longrightarrow A^*$ is not w^*-w^* cont.

A question by the referee

Question: In the above Proposition, which of the sets $\ker(m_R)$ and $m_R(\mathrm{ball}(A^*))$ is not w^*-closed in A^*?

Answer: Both are possible.

Example. Let $\lhd$ be the multiplication on $T(\ell_2(\mathbb{Z}))$ induced by the left fundamental unitary W of $\ell_\infty(\mathbb{Z})$. Then $(T(\ell_2(\mathbb{Z})), \lhd)^{op}$ is just the convolution algebra $(T(\ell_2(\mathbb{Z})), *)$ introduced by Neufang (00). It is known from Auger-Neufang (07) the right topological centre of $(T(\ell_2(\mathbb{Z})), *)$ is $\ell_1(\mathbb{Z}) \oplus \ell_\infty(\mathbb{Z})^\perp$. Then

$$\mathfrak{Z}_t(T(\ell_2(\mathbb{Z}))^{**}, \Box_\lhd) = \ell_1(\mathbb{Z}) \oplus \ell_\infty(\mathbb{Z})^\perp \supsetneqq T(\ell_2(\mathbb{Z})).$$

Let A be the unitization of $(T(\ell_2(\mathbb{Z})), \lhd)$. Then

$$\mathfrak{Z}_t(A^{**}, \Box) = \ell_1(\mathbb{Z}) \oplus \ell_\infty(\mathbb{Z})^\perp \oplus \mathbb{C} \supsetneqq A.$$

The answer to the question by the referee

Proposition. (H.-N.-R.) Let A be the same as above. Then A is a unital Banach algebra with A^* a von Neumann algebra.

Let $s \in \ell_\infty(\mathbb{Z})^\perp \setminus \ell_\infty(\mathbb{Z})_\perp$ and $m = (s, \alpha) \in \mathfrak{Z}_t(A^{**}, \Box) \setminus A$.

(i) If $\alpha \neq 0$, then $\ker(m_R) = \{0\}$ is obviously w^*-closed in A^*, but $m_R(\mathrm{ball}(A^*))$ is not w^*-closed in A^*.

(ii) If $\alpha = 0$, then $\ker(m_R)$ is not w^*-closed in A^*. In this case, $m_R(\mathrm{ball}(A^*))$ is w^*-closed in A^* iff $\|m\|$ is attainable.