ON MULTIPLIERS AND COMPLETELY BOUNDED MULTIPLIERS – THE CASE $SL(2,\mathbb{R})$

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$A(G)$ Fourier algebra of a locally compact group G. $B(G)$ Fourier-Stieltjes algebra.

$A(G)''$ bidual of $A(G)$ with (first) Arens product $\odot$.

$M(A(G))$ multipliers of $A(G)$ with norm $\| \ \|_M$. Every $f \in M(A(G))$ is given by (and identified with) a bounded continuous function on G. It extends to $A(G)''$ and this is again denoted by $f \odot \xi$ for $\xi \in A(G)''$ (bidual mapping).

$M_0(A(G))$ completely bounded multipliers of $A(G)$ with norm $\| \ \|_{M_0}$ (see [CH] for basic properties).

$VN(G)$ group von Neumann algebra (generated by the left regular representation on $L^2(G)$), we use the standard identification with the dual space $A(G)'$.

$C_0(G)$ continuous functions on G vanishing at infinity.

$\mathcal{B}(\mathcal{H})$ bounded linear operators on a Hilbert space $\mathcal{H}$.

For $G = SL(2,\mathbb{R})$ (real 2x2-matrices of determinant one), let K be the subgroup of rotations $k_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$ and H the subgroup of matrices $\begin{pmatrix} a & 0 \\ b & \frac{1}{a} \end{pmatrix}$ with $a > 0,\ b \in \mathbb{R}$. Recall (part of the Iwasawa decomposition) that $G = KH$, the decomposition of the elements $x = kh$ being unique. We parametrize the dual group $\widehat{K}$ of the compact abelian group K by $\chi_j(k_\varphi) = e^{ij\varphi}$ $(j \in \mathbb{Z},\ \varphi \in \mathbb{R})$.

Theorem. *For $G = SL(2,\mathbb{R})$ we have $M(A(G)) = M_0(A(G))$. There exists $\zeta \in A(G)''$ with $\|\zeta\| = 1$ such that*

$$\|f \odot \zeta\| = \|f\|_M = \|f\|_{M_0} \qquad \text{holds for all} \ \ f \in M(A(G)).$$

$A(G)$ is dense in $M(A(G)) \cap C_0(G)$ with respect to $\| \ \|_M$. Put $f_{mn} = \chi_m * f * \chi_n$. For $f \in M(A(G)) \cap C_0(G)$, we have that $(f_{mn} \mid H)_{m,n\in\mathbb{Z}}$ defines an element of the predual of $VN(H)\bar{\otimes}\mathcal{B}(l^2(\mathbb{Z}))$ whose norm equals $\|f\|_M$.

For general $f \in M(A(G))$, we have that $\lambda = \lim_{x\to\infty} f(x)$ exists. Then $f - \lambda \in M(A(G)) \cap C_0(G)$ and $\|f\|_M = \|f - \lambda\|_M + |\lambda|$.

The Theorem holds similarly for all connected groups G that are locally isomorphic to $SL(2,\mathbb{R})$ and have finite centre. With some modifications, one can find

Summary of talks given at the School of Mathematics, University of Leeds, 28 May– 2 June 2010.

presumably also a version for the universal covering group of $SL(2,\mathbb{R})$. The state ζ will arise from a representation of the C*-algebra $VN(G)$ on some ultraproduct of Hilbert spaces.

For general G, we have $A(G) \subseteq B(G) \subseteq M_0(A(G)) \subseteq M(A(G))$. When G is amenable (e.g. abelian or compact), $M(A(G)) = B(G)$ holds. When G is non-amenable (e.g., $SL(2,\mathbb{R})$ or the discrete free group F_2), it is known that $B(G)$ is a proper subspace of $M_0(A(G))$. For a general discrete group G, containing F_2 as a subgroup, Bozejko (1981) has shown that $M_0(A(G))$ is a proper subspace of $M(A(G))$.

If K is a compact subgroup of some locally compact group G, a function f on G is called *radial* (with respect to K) or K–bi-invariant, if $f(k_1 x k_2) = f(x)$ holds for all $x \in G,\ k_1, k_2 \in K$. If there exists a closed amenable subgroup H of G such that $G = KH$ holds set-theoretically, then for a radial function f, Cowling and Haagerup [CH] have shown that the following conditions are equivalent:

$$\text{(i)}\ \ f \in M(A(G)) \qquad \text{(ii)}\ \ f \in M_0(A(G)) \qquad \text{(iii)}\ \ f \mid H \in B(H)$$

(with equality of norms). This applies, in particular, for a semisimple Lie group G with finite centre, K a maximal compact subgroup.

For $G = SL(2,\mathbb{R})$ and $m, n \in \mathbb{Z}$, using our notation above, we call f (m,n)-radial, if $f(k_1 x k_2) = \chi_m(k_1) f(x) \chi_n(k_2)$ holds for all $x \in G,\ k_1, k_2 \in K$. Then the same equivalence as above holds for (m,n)-radial functions f and for $(m,n) \neq (0,0)$ one even gets (by our Theorem) $f \mid H \in A(H)$.

On the following pages, we indicate the PROOF of the Theorem:
In one direction, assume that $(f_{mn} \mid H)_{m,n\in\mathbb{Z}}$ defines an element of the predual of $VN(H)\bar{\otimes}\mathcal{B}(l^2(\mathbb{Z}))$ whose norm equals c. Then it is not so hard to show that $f \in M_0(A(G))$ and $\|f\|_{M_0} \leq c$ using that such a functional is represented by a trace class operator on $L^2(G) \otimes l^2(\mathbb{Z})$ and the following Proposition (compare condition (iv) in [CH] p. 508).

Proposition 1. *Let G be a locally compact group, K a compact subgroup, $\mathcal{H}$ a separable Hilbert space, $f : G \to \mathbb{C}$ continuous, $P, Q : G \to \mathcal{H}$ a.e. defined and Borel measurable.*
Assume that $c_P = \operatorname*{ess\,sup}_{x\in G} \int_K \|P(kx)\|^2\, dk < \infty$ *and similarly* $c_Q < \infty$.
If $f(y^{-1}x) = (\, P(x) \mid Q(y)\,)$ *holds a.e. on* $G \times G$, *then* $f \in M_0(A(G))$ *and* $\|f\|_{M_0} \leq \sqrt{c_P c_Q}$.

(a.e. refers to Haar measure on G or $G \times G$, dk refers to normalized Haar measure on K, $(\ \mid\ )$ denotes the inner product of $\mathcal{H}$).

For the other direction, we start by recalling the description of the irreducible unitary **representations** (going back to Bargmann). For simplicity, we confine to representations of $PSL(2,\mathbb{R}) = SL(2,\mathbb{R})/\{\pm I\}$ (projective special linear group; $\{\pm I\}$ being the centre of $SL(2,\mathbb{R})$). We use (essentially) the notations (and parametrization) of Vilenkin [V].

Put $\mathcal{H} = L^2(\mathbb{R})$ (for ordinary Lebesgue measure), $g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$,

$$\big(T_l(g)f\big)\,(x) = f\Big(\frac{\alpha x+\gamma}{\beta x+\delta}\Big)\,|\beta x+\delta|^{2l} \qquad \text{for} \quad f \in \mathcal{H}\,.$$

For $l = -\frac{1}{2} + i\lambda$ with $\lambda \in \mathbb{R}$ this gives unitary (strongly continuous, irreducible) representations of $SL(2,\mathbb{R})$ (*principal series*). $-\frac{1}{2} \pm i\lambda$ gives equivalent representations, hence it will be enough to consider $\lambda \geq 0$.

For $l \in \mathbb{Z}$ one gets the *discrete series* (but here the inner product has to be changed to make T_l unitary, changing $\mathcal{H}$ too; see below).

Further cases for unitary representations are $l \in]-1,0[$, which gives the complementary series (again with a different inner product) and, finally, there is also the trivial (one-dimensional) representation.

T_l arises from the right action of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ (and the corresponding action on the projective line). In the notation of [V] this is T_χ with $\chi = (l,0)$ (the second parameter can be used to describe further representations of $SL(2,\mathbb{R})$ and other covering groups). Integer case: for $l \geq 0$, we take T_l to be only the part T_χ^- (notation of [V]) and for $l < 0$ the part T_χ^+. Thus T_{-l-1} is (equivalent to) the conjugate representation of T_l.

Multiplication in $A(G)$ and $B(G)$ corresponds to **tensor products** of representations. For $SL(2,\mathbb{R})$ the decompositions have been determined by Pukanszky (1961). A completed and better accessible account has been given by Repka [R].

For $l_j = -\frac{1}{2} + i\lambda_j$ $\qquad T_{l_1} \otimes T_{l_2} \sim 2 \int_{\mathbb{R}^+}^{\oplus} T_{-\frac{1}{2}+i\lambda}\,d\lambda \oplus \sum_{l\in\mathbb{Z}} T_l\,.$

For $l_1 = -\frac{1}{2} + i\lambda_1\,,\ l_2 \in \mathbb{N}_0$ $\qquad T_{l_1} \otimes T_{l_2} \sim \int_{\mathbb{R}^+}^{\oplus} T_{-\frac{1}{2}+i\lambda}\,d\lambda \oplus \sum_{l\geq 0} T_l\,.$

For $l_j \in \mathbb{N}_0$ $\qquad T_{l_1} \otimes T_{l_2} \sim \sum_{l > l_1+l_2} T_l\,.$

Similarly in the remaining cases.

To get **coefficients** for the unitary representations, we use (corresponding to [V]) an ortho*normal* basis (e^l_m) of the Hilbert space $\mathcal{H}_l$ of T_l. For $l = -\frac{1}{2} + i\lambda$ (principal series), we have $\mathcal{H}_l = \mathcal{H}$ and the basis is indexed by $m \in \mathbb{Z}$. For $l \in \mathbb{N}_0$, the range is $m > l$ and for integers $l < 0$: $m \leq l$.

The basis vectors satisfy $\quad T_l(k_\varphi)\,e^l_m = e^{2mi\varphi} e^l_m = \chi_{2m}(\varphi)\,e^l_m$ ("elliptic basis").

We put $t^l_{mn}(g) = (T_l(g)e^l_n \mid e^l_m)$. This gives the unitary matrix coefficients of $T_l(g)$. t^l_{mn} is $(2m, 2n)$-radial (we get only even integers, since we restrict to representations of $PSL(2,\mathbb{R})$).
For $l = -\frac{1}{2} + i\lambda$, we have $t^l_{mn} \in B(G)$ for all $m, n \in \mathbb{Z}$ (it even belongs to the reduced Fourier-Stieltjes algebra $B_\rho(G)$, i.e., the w*-closure of $A(G)$ in $B(G)$).
For $l \in \mathbb{Z}$, the representations T_l are *square-integrable*, thus $t^l_{mn} \in A(G) \cap L^2(G)$ for $l \in \mathbb{N}_0,\ m, n > l$ and for $l < 0,\ m, n \le l$.
For $l = -\frac{1}{2} + i\lambda$, the "non-radial component" of t^l_{mn} is described by $\mathfrak{P}^l_{mn}(\operatorname{ch} 2\tau) = t^l_{mn}\begin{pmatrix} e^\tau & 0 \\ 0 & e^{-\tau}\end{pmatrix}$ for $\tau \ge 0$ (ch denoting the hyperbolic cosine). In [V] the functions $\mathfrak{P}^l_{mn}$ are defined (and investigated) for all $l \in \mathbb{C}$, but (apart of the principal series) using a non-normalized orthogonal basis for the matrix representation. For the discrete series, the corresponding functions arising from the *unitary* coefficients are denoted by $\mathcal{P}^l_{mn}$ in [VK] ($l \in \mathbb{Z}$). For $l \in \mathbb{N}_0,\ m, n > l$ they are related by $\mathfrak{P}^l_{mn} = \Big(\dfrac{(m-l-1)!\,(n+l)!}{(m+l)!\,(n-l-1)!}\Big)^{\frac{1}{2}} \mathcal{P}^l_{mn}$.

Technically, the continuous part in the decomposition of tensor products is more difficult to handle (and the appearance of multiplicities causes additional complications). Therefore we restrict to the discrete part.
For $l_1 = -\frac{1}{2} + i\lambda,\ l_2 \in \mathbb{N}_0$, we define the *Clebsch-Gordan coefficients* by

$$e^{l_1}_j \otimes e^{l_2}_m = \sum_{l \ge 0} C(l_1, l_2, l; j, m, j+m)\, e^l_{j+m} + \text{ cont. part}\ .$$

The same for $l_1 \in \mathbb{Z}$ with $l_1 \ge -l_2 - 1$ (for $l_1 < -l_2 - 1$ the discrete part of $T_{l_1} \otimes T_{l_2}$ contains only T_l with $l < 0$). We put $C(l_1, l_2, l; j, m, j+m) = 0$ when $j + m \le l$ (in addition, for $l_1 \in \mathbb{Z}$, the coefficients will be 0 outside the range $l > l_1 + l_2$ for $l_1 \in \mathbb{N}_0$ and outside $0 \le l \le l_1 + l_2$ for $l_1 < 0$). The isomorphism between T_l and a component of $T_{l_1} \otimes T_{l_2}$ is determined only up to a factor of modulus 1. This is fixed by requiring that $C(l_1, l_2, l; l - l_2, l_2 + 1, l + 1) > 0$ (of course, in the integer case this refers only to those l that have not been excluded above).
For l_1, l_2 as above, this gives a decomposition of products in $B(G)$

(1) $$t^{l_1}_{jj'}\, t^{l_2}_{mm'} = \sum_{l \ge 0} \overline{C(l_1, l_2, l; j, m, j+m)}\, C(l_1, l_2, l; j', m', j'+m')\, t^l_{j+m\, j'+m'} + \text{ cont. part}\,.$$

Now, we consider the behaviour for **large** l_2.

Proposition 2 (Asymptotics of CG-coefficients)**.** *For fixed* $l_1 = -\frac{1}{2} + i\lambda\,,\ j,s \in \mathbb{Z}$ *and finite* $\kappa \geq 1$*, we have*

$$\lim_{\substack{l_2 \to \infty \\ \frac{m}{l_2} \to \kappa}} C(l_1, l_2, l_2 + s; j, m, j+m) = \mathfrak{P}^{l_1}_{s\,j}(\kappa)\ .$$

For $\kappa = 1,\ j = s$, one has to add the restriction $m > l_2$. Corresponding results hold for $l_1 \in \mathbb{Z}$ (discrete series), e.g., when $l_1 \in \mathbb{N}_0\,,\ j,s > l_1$, the limit is $\mathcal{P}^{l_1}_{s\,j}(\kappa)$. Similarly for the complementary series and unitary representations of covering groups. This is the counterpart of a classical result of Brussaard, Tolhoek (1957) on the CG-coefficients of $SU(2)$.

Since $(\mathfrak{P}^{l_1}_{s\,j}(\kappa))$ is the matrix of a unitary operator, its column vectors have norm 1 (in $l^2(\mathbb{Z})$). From $\|e^{l_1}_j \otimes e^{l_2}_m\| = 1$, it follows by orthogonality that the norm of the continuous part in the decomposition of $e^{l_1}_j \otimes e^{l_2}_m$ tends to 0 for $l_2 \to \infty$ (with l_1, j fixed, $\frac{m}{l_2} \to \kappa$). The same holds for the decomposition of $t^{l_1}_{jj'}\, t^{l_2}_{mm'}$ in (1).

It was already noted by Pukanszky that the densities arising in the continuous part are given by analytic functions. Thus (with at most contably many exceptions) all $\lambda \geq 0$ will appear in the decomposition of $e^{l_1}_j \otimes e^{l_2}_m$ (for $l_1 = -\frac{1}{2} + i\lambda_1$). But from a more quantitative viewpoint, most of the product will be concentrated on the (positive part of the) discrete series when l_2 is large.

Idea of Proof. Recall the Fourier inversion formula:

$$h(e) = \int_0^\infty \mathrm{tr}(T_{-\frac{1}{2}+i\lambda}(h))\, \lambda\ \mathrm{th}(\pi\lambda)\, d\lambda + \sum_{l \geq 0} (l + \frac{1}{2})\big(\,\mathrm{tr}(T_l(h)) + \mathrm{tr}(T_{-l-1}(h))\,\big)\ .$$

for $h \in A(PSL(2,\mathbb{R})) \cap L^1(PSL(2,\mathbb{R}))$ and the extensions of the representations to $L^1(PSL(2,\mathbb{R}))$ for an appropriate choice of the Haar measure. This describes also the Plancherel measure.

On the level of coefficients, applied to $(2m,2n)$-radial functions with $m,n \geq 0$, this gives a generalization of the Mehler-Fock transformation

$$g(x) = \sum_{l=0}^{\min(m,n)-1} (l + \frac{1}{2})\, b(l)\, \mathcal{P}^l_{mn}(x) + \int_0^\infty a(\lambda)\, \mathfrak{P}^{-\frac{1}{2}+i\lambda}_{mn}(x)\, \lambda\ \mathrm{th}(\pi\lambda)\, d\lambda$$

with $b(l) = \int_1^\infty g(x)\, \mathcal{P}^l_{mn}(x)\, dx$ for $g \in L^2([1,\infty])$ (convergence in L^2). Thus the discrete part is just the expansion with respect to the orthogonal system $(\mathcal{P}^l_{mn}) \subseteq L^2([1,\infty])$ (m,n fixed) and the coefficients are obtained from inner products.

We apply this to $g = \mathfrak{P}^{l_1}_{ss}\,\mathcal{P}^{l_2}_{l_2+1\,l_2+1}$ and get for $l = l_2 + s$ by (1)

$$|\,C(l_1,l_2,l_2+s;s,l_2+1,l_2+s+1)|^2 = \\ (l_2+s+\frac{1}{2})\int\limits_1^\infty \mathfrak{P}^{l_1}_{ss}(x)\,\mathcal{P}^{l_2}_{l_2+1\,l_2+1}(x)\,\mathcal{P}^{l_2+s}_{l_2+s+1\,l_2+s+1}(x)\,dx$$

By [V] we have $\mathcal{P}^l_{l+1\,l+1}(x) = \mathfrak{P}^l_{l+1\,l+1}(x) = \Big(\dfrac{2}{x+1}\Big)^{l+1}$. It follows easily that for $l_2 \to \infty$ and $s \in \mathbb{Z}$ fixed, $(l_2+s+\frac{1}{2})\,\mathcal{P}^{l_2}_{l_2+1\,l_2+1}\,\mathcal{P}^{l_2+s}_{l_2+s+1\,l_2+s+1} \to \delta_1$ (point measure) holds weakly with respect to bounded continuous functions on $[1,\infty[$. Since $\mathfrak{P}^{l_1}_{ss}(1) = 1$, this gives $|\,C(l_1,l_2,l_2{+}s;s,l_2{+}1,l_2{+}s{+}1)| \to 1$ (when $l_1 = -\frac{1}{2}+i\lambda$ is fixed) and by our choice of the phase, we get $C(l_1,l_2,l_2{+}s;s,l_2{+}1,l_2{+}s{+}1) \to 1$.

Next we take $g = \mathfrak{P}^{l_1}_{s\,j}\,\mathcal{P}^{l_2}_{l_2+1\,m}$ and get for $l = l_2 + s$ by (1)

$$\underbrace{\overline{C(l_1,l_2,l_2+s;s,l_2+1,l_2+s+1)}}_{\to 1}\,C(l_1,l_2,l_2+s;j,m,j+m) = \\ (l_2+s+\frac{1}{2})\int\limits_1^\infty \mathfrak{P}^{l_1}_{s\,j}(x)\,\mathcal{P}^{l_2}_{l_2+1\,m}(x)\,\mathcal{P}^{l_2+s}_{l_2+s+1\,j+m}(x)\,dx$$

Let $\mu_{l_2 m}$ be the measure on $[1,\infty]$ with density $(l_2+s+\frac{1}{2})\,\mathcal{P}^{l_2}_{l_2+1\,m}\,\mathcal{P}^{l_2+s}_{l_2+s+1\,j+m}$. Again one can use the formulas of [V] for $\mathfrak{P}^l_{l+1\,m}(x)$. With a slight change of coordinates, one gets that $\dfrac{\mu_{l_2 m}}{\|\mu_{l_2 m}\|}$ has a β'-distribution and from the values of expectation and variance one can conclude that $\|\mu_{l_2 m}\| \to 1$ and $\mu_{l_2 m} \to \delta_\kappa$ for $l_2 \to \infty$, $\dfrac{m}{l_2} \to \kappa$. □

In the next step we use **ultraproducts** to work with these limit relations. Such constructions for group representations have been done by Cowling and Fendler.

We take some element $p \in \beta\mathbb{N}\backslash\mathbb{N}$ (Stone-Čech compactification). The ultraproduct of the Hilbert spaces $(\mathcal{H}_l)_{l>0}$ (with respect to p) is denoted by $\mathcal{H}_p$. It consists of equivalence classes of all sequences $(h_l) \in \prod \mathcal{H}_l$ such that $\lim\limits_{l\to p}\|h_l\| < \infty$, factoring by the subspace of sequences with $\lim\limits_{l\to p}\|h_l\| = 0$. We use the notation $\lim\limits_{l\to p} h_l$ to denote the equivalence class of (h_l). $\mathcal{H}_p$ is again a Hilbert space and we get a representation T_p of the C*-algebra $VN(G)$ on $\mathcal{H}_p$ putting $T_p(S)(\lim\limits_{l\to p} h_l) = \lim\limits_{l\to p} T_l(S)h_l$ (for $S \in VN(G)$).

Each function $f\colon \mathbb{N} \to \mathbb{N}$ satisfying $f(l) > l\;\forall\, l$ (or more generally, $\lim\limits_{l\to p} f(l) - l > 0$) defines a unit vector in $\mathcal{H}_p$ by $e(p,f) = \lim\limits_{l\to p} e^l_{f(l)}$. Of course it is enough to require that f is defined for $l \geq l_0$. For functions f, f' we get a coefficient functional by $(t^p_{ff'}\,,\,S) = \big(T_p(S)\,e(p,f') \mid e(p,f)\big)$ for $S \in VN(G)$. Then $t^p_{ff'} \in VN(G)'$ (dual

space) and $t^p_{ff'} = \lim_{l\to p} t^l_{f(l)\,f'(l)}$ (w*-limit).
Recall that $\beta\mathbb{N}\setminus\mathbb{N}$ is a $\mathbb{Z}$-module under addition. Thus we get in the same way Hilbert spaces $\mathcal{H}_{p+s}$ and representations T_{p+s} for all $s\in\mathbb{Z}$.
For f as above, put $\kappa_p(f) = \lim_{l\to p}\dfrac{f(l)}{l}$ (possibly infinite).
Write $\kappa=\kappa_p(f)$, $\kappa'=\kappa_p(f')$. Assuming, $1<\kappa,\kappa'<\infty$, $l_1=-\frac12+i\lambda$, we get from (1) and Proposition 2

$$t^{l_1}_{jj'}\odot t^p_{ff'} = \lim_{l_2\to p} t^{l_1}_{jj'}\, t^{l_2}_{f(l_2)f'(l_2)} = \sum_{s\in\mathbb{Z}} \overline{\mathfrak{P}^{l_1}_{sj}(\kappa)}\,\mathfrak{P}^{l_1}_{sj'}(\kappa')\lim_{l_2\to p} t^{l_2+s}_{f(l_2)+j\;f'(l_2)+j'}$$

(note that $\big(\overline{\mathfrak{P}^{l_1}_{sj}(\kappa)}\,\mathfrak{P}^{l_1}_{sj'}(\kappa')\big)_{s\in\mathbb{Z}}\in l^1$). Put $u(l)=l-1$ for $l\in\mathbb{Z}$, then $\lim_{l_2\to p} t^{l_2+s}_{f(l_2)+j\;f'(l_2)+j'} = t^{p+s}_{f\circ u^s+j\;f'\circ u^s+j'}$ and we arrive at

$$t^{l_1}_{jj'}\odot t^p_{ff'} = \sum_{s\in\mathbb{Z}} \overline{\mathfrak{P}^{l_1}_{sj}(\kappa)}\,\mathfrak{P}^{l_1}_{sj'}(\kappa')\, t^{p+s}_{f\circ u^s+j\;f'\circ u^s+j'}\,. \tag{2}$$

Next, we consider $\overline{\mathcal{H}}_p = \bigoplus_{s\in\mathbb{Z}}\mathcal{H}_{p+s}$ (l^2-sum) and the corresponding representation $\overline{T}_p = \bigoplus_{s\in\mathbb{Z}} T_{p+s}$ of $VN(G)$.
For $1<\kappa<\infty$, $\mathcal{K}_\kappa$ shall be the closed subspace of $\mathcal{H}_p$ generated by the vectors $e(p,f)$, taking all functions f with $\kappa_p(f)=\kappa$. We put $\mathcal{K}=\bigoplus_{1<\kappa<\infty}\mathcal{K}_\kappa$.
$U(\lim_{l\to p+s} h_l) = \lim_{l\to p+s+1} h_{l-1}$ defines an isometric isomorphism of $\mathcal{H}_{p+s}$ and $\mathcal{H}_{p+s+1}$ and this extends to a unitary operator $U\colon\overline{\mathcal{H}}_p\to\overline{\mathcal{H}}_p$ (in particular $U\big(e(p{+}s,f)\big) = e(p{+}s{+}1,f\circ u)$). Let $\overline{\mathcal{K}}_\kappa$ be the closed U-invariant subspace of $\overline{\mathcal{H}}_p$ generated by $\mathcal{K}_\kappa$ (it is generated by the vectors $e(p{+}s,f)$, taking all functions f with $\kappa_{p+s}(f)=\kappa$ for some $s\in\mathbb{Z}$). Clearly, $\overline{\mathcal{K}}_\kappa\perp\overline{\mathcal{K}}_{\kappa'}$ holds for $\kappa\neq\kappa'$ and we write $\overline{\mathcal{K}}=\bigoplus_{1<\kappa<\infty}\overline{\mathcal{K}}_\kappa$ (the closed U-invariant subspace of $\overline{\mathcal{H}}_p$ generated by $\mathcal{K}$). $V\big(e(p{+}s,f)\big) = e(p{+}s,f{+}1)$ defines a unitary operator on $\overline{\mathcal{K}}_\kappa$ (for $1<\kappa<\infty$) and this extends to a unitary operator $V\colon\overline{\mathcal{K}}\to\overline{\mathcal{K}}$ satisfying $VU=UV$. (For $\kappa=1$, V is no longer surjective).

For a fixed function f with $\kappa=\kappa_p(f)$ satisfying $1<\kappa<\infty$, it follows easily that $\{e(p+s,f\circ u^s+j)\} = \{U^sV^je(p,f) : s,j\in\mathbb{Z}\}$ defines an orthonormal system of vectors in $\overline{\mathcal{K}}_\kappa$.
A special case, used below, will be the functions $f_\kappa(l)=[\kappa\, l]$ (integer part), satisfying $\kappa_p(f_\kappa)=\kappa$ for each p and $1<\kappa<\infty$.

Lemma 1. *For $\lambda\in\mathbb{R}$, $j\in\mathbb{Z}$, $1<\kappa<\infty$*
$A^\lambda_j = V^j\sum_{s\in\mathbb{Z}}\mathfrak{P}^{-\frac12+i\lambda}_{sj}(\kappa)\,\big|2s\big|^{i\lambda}\,U^s$ defines a bounded linear operator $\mathcal{K}_\kappa\to\overline{\mathcal{K}}_\kappa$.
Taking $A^\lambda_j=0$ on $\mathcal{K}^\perp$ gives a bounded linear operator $A^\lambda_j\colon\overline{\mathcal{H}}_p\to\overline{\mathcal{H}}_p$.

(Here we adopt $0^{i\lambda}=1$).

Corollary. *Given $e, e' \in \mathcal{K}$ define $t \in VN(G)'$ by $(t,S) = (T_p(S)\,e' \mid e\,)$. Then for $l = -\frac{1}{2} + i\lambda$ ($\lambda \in \mathbb{R}$) and $j, j' \in \mathbb{Z}$ we have $(t^l_{jj'} \odot t, S) = (\overline{T}^p(S)A^\lambda_{j'}e' \mid A^\lambda_j e)$ ($S \in VN(G)$).*

Lemma 2. *$\overline{T}_p(VN(G))$ is w^*-dense in $\prod\limits_{s\in\mathbb{Z}} \mathcal{B}(\mathcal{H}_{p+s})$.*

In particular, this implies that T_p is irreducible and $(T_p, \mathcal{H}_p)$ is the cyclic representation for the state t^p_{ff} (with cyclic vector $e(p,f)$) for every function f as above. Furthermore (slightly more general as in Lemma 2), one has $T_p \not\sim T_{p'}$ for $p \neq p'$. Considering $L^1(G)$ as a (w*-dense) subalgebra of $VN(G)$, it is not hard to see that $T_p(h) = 0$ for $h \in L^1(G)$, hence these are singular representations of $VN(G)$.

For the final step we need a refinement of Lemma 2. Although $\overline{T}_p(VN(G))$ is not a von Neumann algebra, the fact that $VN(G)$ is a von Neumann algebra allows to get a stronger result on the size of $\overline{T}_p(VN(G))$.
Recall that the representations T_l are *square integrable* for $l \in \mathbb{Z}$. Thus they are equivalent to subrepresentations of the left regular representation on $L^2(G)$ and we can consider $\prod\limits_{l\geq 0} \mathcal{B}(\mathcal{H}_l)$ as a subalgebra of $VN(G)$.
For $1 \leq \alpha < \beta \leq \infty$ let $P_{\alpha\beta} \in VN(G)$ be the orthogonal projection on the closed subspace of $\bigoplus\limits_{l>0} \mathcal{H}_l$ generated by $\{e^l_m : \alpha < \dfrac{m}{l} < \beta,\ l > 0\}$. For $\alpha < \beta \leq \alpha' < \beta'$, it follows that $P_{\alpha\beta}P_{\alpha'\beta'} = P_{\alpha'\beta'}P_{\alpha\beta} = 0$. For $\alpha < \kappa < \beta$ we have $\overline{\mathcal{K}}_\kappa \subseteq \mathrm{im}\big(\overline{T}_p(P_{\alpha\beta})\big)$.

Lemma 3. *Assume that $\alpha_n \nearrow \infty$. For $n \geq 1$,*

E_n ($\subseteq \overline{\mathcal{H}}_p$) shall be a finite dimensional subspace of $\mathrm{im}\big(\overline{T}_p(P_{\alpha_n\alpha_{n+1}})\big)$,
$S_n \in \mathcal{B}(\overline{\mathcal{H}}_p)$ such that $\|S_n\| \leq 1$, $S_n(E_n) \subseteq \mathrm{im}\big(\overline{T}_p(P_{\alpha_n\alpha_{n+1}})\big)$ and $S_n(\mathcal{H}_{p+s}) \subseteq \mathcal{H}_{p+s}$ for all $s \in \mathbb{Z}$.

Then there exists $S \in VN(G)$ such that $\overline{T}_p(S) \mid E_n = S_n$ for all n.

At the Harmonic Analysis Conference in Istanbul 2004, I talked about the case $G = SU(2)$. For that group, one could use a limit of averages of states t^p_{ff} (for $f = f_\kappa$; approaching Lebesgue measure on $[-1,1]$) to get a singular state $\zeta \in VN(G)'$ satisfying $\|f \odot \zeta\| = \|f\|$ for all $f \in A(G)$. This cannot exist for $G = SL(2,\mathbb{R})$, because of non-amenability. Instead of this, we will use another type of asymptotics.
Now, we fix $p \in \beta\mathbb{N} \setminus \mathbb{N}$ and write $\overline{T}$ for $\overline{T}_p$. We choose $p_1 \in \beta\mathbb{N} \setminus \mathbb{N}$ satisfying $(2^n) \in p_1$ (a sufficiently "thin" ultrafilter). $(\overline{\mathcal{H}}_p)_{p_1}$ shall denote the ultrapower of $\overline{\mathcal{H}}_p$ with respect to p_1. If $(h^{(n)})$ is a bounded sequence in $\overline{\mathcal{H}}_p$, we write, as before,

$\lim\limits_{n\to p_1} h^{(n)}$ for the corresponding equivalence class, defining an element of $(\overline{\mathcal{H}}_p)_{p_1}$. The representation $\overline{T}$ of $VN(G)$ on $\overline{\mathcal{H}}_p$ defines a representation $\overline{\overline{T}}$ of $VN(G)$ on $(\overline{\mathcal{H}}_p)_{p_1}$. We define $\bar{\bar{e}} \in (\mathcal{H}_p)_{p_1} \subseteq (\overline{\mathcal{H}}_p)_{p_1}$ and $\zeta \in VN(G)'$ by

$$\bar{\bar{e}} = \lim_{n\to p_1} \frac{1}{n} \sum_{r=1}^{n^2-1} e(p, f_{\mathrm{ch}(n+\frac{r}{n})}) , \qquad (\zeta, S) = (\,\overline{\overline{T}}(S)\,\bar{\bar{e}} \mid \bar{\bar{e}}\,)$$

For $g \in \mathcal{K}(\,\mathbb{R}\setminus\{0\}\times\mathbb{Z}\,)$ ($\mathcal{K}(\Omega)$: continuous functions with compact support), we put

$$\varphi(g) = \lim_{n\to p_1} \frac{1}{n} \sum_{r=1}^{n^2-1} \sum_{j,s\in\mathbb{Z}} g\big(\frac{2s}{e^c}, j\big)\,(-1)^s \frac{\sqrt{2}}{e^{c/2}}\, U^s V^j\, e(p, f_{\mathrm{ch}\,c}) \qquad \text{with} \quad c = n + \frac{r}{n}$$

Note that the support condition makes the sum finite, furthermore, $s \neq 0$ implies $\varphi(g) \perp (\mathcal{H}_p)_{p_1}$.

Lemma 4. $\varphi(g) \in (\overline{\mathcal{H}}_p)_{p_1}$, $\|\varphi(g)\| = \|g\|_2$.
Thus φ extends to an isometry $\varphi\colon L^2(\mathbb{R}\times\mathbb{Z}) \to (\overline{\mathcal{H}}_p)_{p_1}$.
Putting $\varphi_1(g+\lambda) = \varphi(g) + \lambda\,\bar{\bar{e}}$ defines an isometry $\varphi_1\colon L^2(\mathbb{R}\times\mathbb{Z}) \oplus \mathbb{C} \to (\overline{\mathcal{H}}_p)_{p_1}$.

Let $P \in \mathcal{B}\big((\overline{\mathcal{H}}_p)_{p_1}\big)$ be the orthogonal projection to $\varphi(L^2(\mathbb{R}\times\mathbb{Z}))$. For $S \in VN(G)$, $g,h \in L^2(\mathbb{R}\times\mathbb{Z})$ put $(\psi(S)g \mid h) = (\overline{\overline{T}}(S)\varphi(g) \mid \varphi(h))$. This defines a contractive linear mapping $\psi: VN(G) \to \mathcal{B}(L^2(\mathbb{R}\times\mathbb{Z}))$, $\psi(VN(G))$ being isometrically isomorphic to the dilation $P\,\overline{\overline{T}}(VN(G))\,P$.
Similarly, for P_1 the projection to $\varphi_1(L^2(\mathbb{R}\times\mathbb{Z}))$, one gets $\psi_1 : VN(G) \to \mathcal{B}(L^2(\mathbb{R}\times\mathbb{Z})) \oplus \mathbb{C}$ (note that $(\mathcal{H}_p)_{p_1}$ is invariant under $\overline{\overline{T}}(VN(G))$).

For $m = 2^n$, $\alpha_n = \mathrm{ch}\,2^n$, the m-th term in the limits defining $\bar{\bar{e}}$ and $\varphi(g)$ belong to $\mathrm{im}\big(\overline{T}_p(P_{\alpha_n\alpha_{n+1}})\big)$. This makes it possible to apply Lemma 3.

Lemma 5. *$\psi(VN(G))$ is w^*-dense in $\mathcal{B}\big(L^2(]-\infty,0]\times\mathbb{Z})\big) \oplus \mathcal{B}\big(L^2([0,\infty[\times\mathbb{Z})\big)$.*

Similarly, for ψ_1 one has to add a sum with $\mathbb{C}$. As above, the w*-closure of $\psi(VN(G))$ is isometrically isomorphic to $P\,\overline{\overline{T}}(VN(G))^- P$ ($^-$ denoting the w*-closure in $\mathcal{B}\big((\overline{\mathcal{H}}_p)_{p_1}\big)$). Thus by Kaplansky's density theorem, corresponding density results hold for the image of the unit ball of $VN(G)$.

For the final step, we will use the *Whittaker functions.* They are defined by

$$W_{\lambda,\mu}(z) = \frac{z^{\mu+\frac{1}{2}}\, e^{-\frac{z}{2}}}{\Gamma(\mu-\lambda+\frac{1}{2})} \int_0^\infty e^{-zu}\, u^{\mu-\lambda-\frac{1}{2}}\,(1+u)^{\mu+\lambda-\frac{1}{2}}\, du$$

for $\mathrm{Re}\,z > 0$, $\mathrm{Re}(\mu-\lambda+\frac{1}{2}) > 0$ and then for all $\lambda,\mu \in \mathbb{C}$ by analytic continuation.

Proposition 3 (Approximation of coefficients). *For* $n \in \mathbb{Z},\ l = -\frac{1}{2} + i\lambda$ *fixed*

$$\lim_{m\to\infty} \Big(\mathfrak{P}^l_{mn}(\operatorname{ch}\tau) - \frac{m^{-l-1}}{\Gamma(n-l)}\, W_{n,i\lambda}\big(\frac{4m}{e^\tau}\big) \Big)\, e^{\frac{\tau}{2}} = 0$$

holds uniformly for $\tau \geq 0$ *.*

This complements classical results on the asymptotic behaviour of $\mathfrak{P}^l_{mn}$ for fixed l, m, n; e.g., if $m = n,\ \lambda \neq 0$ one has $\mathfrak{P}^l_{mm}(\operatorname{ch}\tau)\, e^{\frac{\tau}{2}} - \dfrac{2}{\sqrt{\pi\lambda\operatorname{th}(\pi\lambda)}} \cos(\lambda\tau + \eta) \to 0$ for $\tau \to \infty$ (where $\eta \in \mathbb{R}$ depends on λ). The approximation implies also that the row vector $\big(\mathfrak{P}^l_{mn}(\operatorname{ch}\tau)\big)_{m>0}$ can be approximated in l^2 by $\Big(\frac{m^{-l-1}}{\Gamma(n-l)}\, W_{n,i\lambda}\big(\frac{4m}{e^\tau}\big)\Big)$ for $\tau \to \infty$. An approximation for the "lower half" $\big(\mathfrak{P}^l_{mn}(\operatorname{ch}\tau)\big)_{m<0}$ is obtained using the identity $\mathfrak{P}^l_{mn} = \mathfrak{P}^l_{-m\,-n}$.

For $j \in \mathbb{Z},\ \lambda \in \mathbb{R},\ l = -\frac{1}{2} + i\lambda$, we put

$$g_{j,\lambda}(x, j') = \begin{cases} 0 & \text{for } j' \neq j \\ \dfrac{(-1)^j\, 2^{i\lambda}}{\Gamma(j-l)\,\sqrt{x}}\, W_{j,i\lambda}(2x) & \text{for } j' = j\,,\ x > 0 \\ \dfrac{(-1)^j\, 2^{i\lambda}}{\Gamma(-j-l)\sqrt{-x}}\, W_{-j,i\lambda}(-2x) & \text{for } j' = j\,,\ x < 0 \end{cases}$$

Then $g_{j,\lambda} \in L^2(\mathbb{R} \times \mathbb{Z})$.
$A^\lambda_j \in \mathcal{B}(\overline{\mathcal{H}}_p)$ defines a bounded operator on $(\overline{\mathcal{H}}_p)_{p_1}$, again denoted by A^λ_j .

Lemma 6. *We have* $A^\lambda_j\, \bar{\bar{e}} = \varphi(g_{j,\lambda})$ *.*

Corollary. $(t^l_{jj'} \odot \zeta\,, S) = (\psi(S)\, g_{j',\lambda} \mid g_{j,\lambda})$ *(*$S \in VN(G)$*).*

The basis of $L^2(\mathbb{R})$ used by [V] to define the coefficients of T_l for $l = -\frac{1}{2} + i\lambda$ is given by $e^l_m(x) = \dfrac{(-1)^m}{\sqrt{\pi}}\, e^{2\pi i \arctan(x)}(1+x^2)^l$.

We consider the real Fourier transform $\hat{f}(y) = \dfrac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} f(x)\, dx$. Then we have

$$\widehat{e^l_m}(y) = (-1)^m \frac{2^{i\lambda}|y|^{-\frac{1}{2}-i\lambda}}{\Gamma(\operatorname{sgn}(y)m - l)}\, W_{\operatorname{sgn}(y)m,i\lambda}(2|y|) = g_{m,\lambda}(y, m)\, |y|^{-i\lambda} .$$

(The functions e^l_m are not integrable, so strictly speaking, this is the Fourier-Plancherel transform).

For $h = \begin{pmatrix} a & 0 \\ b & \frac{1}{a} \end{pmatrix} \in H$, we have $(T_l(h)f)(x) = |a|^{-2l} f(a^2x + ab)$. Composition with Fourier transform defines equivalent representations (Whittaker model) $\pi_\lambda(g)\hat{f} = (T_l(g)f)\widehat{\ }$. For $h \in H$ this gives $(\pi_\lambda(h)\,\eta)(y) = |a|^{-2l-2}\, e^{iy\frac{b}{a}}\, \eta(\frac{y}{a^2})$.

Put $(\rho_\lambda\eta)(y) = |y|^{i\lambda}\, \eta(y)$. Then $\rho_\lambda : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is an isometric isomorphism and $\pi_\lambda(h) = \rho_\lambda^{-1} \circ \pi_0(h) \circ \rho_\lambda$ (in particular, all T_l and π_λ define equivalent

representations of H). π_0 splits into two irreducible representations (the restrictions to $L^2(]-\infty,0]$ and $L^2([0,\infty[)$ and these are the only infinite dimensional irreducible unitary representations of H (up to equivalence). Thus π_0 defines a normal isomorphism of the von Neumann algebras $VN(H)$ and $\mathcal{B}\big(L^2(]-\infty,0])\big)\oplus\mathcal{B}\big(L^2([0,\infty[)\big)$ and this extends to a normal isomorphism $\tilde\pi_0$ of the von Neumann algebras $VN(H)\bar\otimes\mathcal{B}(l^2(2\mathbb{Z}))$ and $\mathcal{B}\big(L^2(]-\infty,0]\times\mathbb{Z})\big)\oplus\mathcal{B}\big(L^2([0,\infty[\times\mathbb{Z})\big)$.
We have $g_{j,\lambda}(\cdot,j)=\rho_\lambda\,\widehat{e^l_j}$, consequently $\pi_0(S)\,g_{j,\lambda}(\cdot,j)=\rho_\lambda\big(\pi_\lambda(S)\,\widehat{e^l_j}\big)=\rho_\lambda\big((T_l(S)\,e^l_j)\widehat{}\,\big)$, resulting in

$$\big(\pi_0(S)\,g_{j',\lambda}(\cdot,j')\mid g_{j,\lambda}(\cdot,j)\big)=(\,S\,,t^l_{jj'})\quad\text{for}\quad S\in VN(H)\;. \tag{3}$$

For $f\in M(A(G))\cap C_0(G)$ put $\Phi(f)=(f_{mn}\mid H)_{m,n\in\mathbb{Z}}$ with $f_{mn}=\chi_m*f*\chi_n$. For general $f\in M(A(G))$, put $\lambda=\lim_{x\to\infty}f(x)$, $f_0=f-\lambda$, $\Phi_1(f)=\Phi_0(f)+\lambda$.

Lemma 7. *For $f\in M(A(G))\cap C_0(G)$, $\Phi(f)$ defines an element of the predual of $VN(H)\bar\otimes\mathcal{B}(l^2(\mathbb{Z}))$ and we have*

$$(f\odot\zeta\,,S)=\big(\,\tilde\pi_0^{-1}\circ\psi(S)\,,\,\Phi(f)\,\big)\quad\textit{for } S\in VN(G)\;.$$

For general $f\in M(A(G))$, $f_0\in C_0(G)$ holds and $\Phi_1(f)$ defines an element of the predual of $\big(VN(H)\bar\otimes\mathcal{B}(l^2(\mathbb{Z}))\big)\oplus\mathbb{C}$. We have

$$(f\odot\zeta\,,S)=\big(\,(\tilde\pi_0\oplus 1)^{-1}\circ\psi_1(S)\,,\,\Phi_1(f)\,\big)\quad\textit{for } S\in VN(G)\;.$$

Corollary. $\|\Phi_1(f)\|=\|\Phi(f_0)\|+|\lambda|=\|f\odot\zeta\|$ *holds for all $f\in M(A(G))$;.*

As indicated earlier this supplies the remaining step for the proof of the Theorem.

Idea of Proof. Recall that the left and right actions of G on $A(G)$ are continuous and isometric. It follows easily that $f\in M(A(G))$ implies $\mu*f,\ f*\mu\in M(A(G))$ for every bounded measure μ on G, in particular, $f_{mn}\in M(A(G))$ for all $m,n\in\mathbb{Z}$. We will start with the K-finite case (i.e. when only finitely many f_{mn} are non-zero).

For general $f\in M(A(G))$, the same argument as in [CH] gives $f\mid H\in B(H)$, in particular, $f\mid H$ is a weakly almost periodic function. In the case of the (m,n)-radial functions f_{mn}, it follows easily (using $G=HK$) that f_{mn} is weakly almost periodic and for f K-finite, this implies that f is weakly almost periodic. By the results of [Ve] it follows that $\lambda=\lim_{x\to\infty}f(x)$ exists and $f_0\in C_0(G)$. As mentioned before, the unitary dual of H ($ax+b$-group) has a very simple structure and this implies $B(H)=A(H)+B(H/[H,H])$. Thus for K-finite $f\in M(A(G))\cap C_0(G)$, we get (since $[H,H]$ is not compact) $f\mid H\in A(H)$. For general $f\in M(A(G))$ this implies that $f_{mn}\mid H\in A(H)$ for $(m,n)\neq(0,0)$ and there exists $\lambda\in\mathbb{C}$ such that $(f-\lambda)_{00}\mid H=(f_{00}-\lambda)\mid H\in A(H)$.

For $f = t^l_{jj'}$, with $l = -\frac{1}{2} + i\lambda$ the evaluation of $(f \odot \zeta\, , S)$ follows from (3) and the Corollary of Lemma 6. This works in a similar way for the coefficients of discrete series representations (as mentioned before we have restricted to representations of $PSL(2,\mathbb{R})$ and this produces only (m,n)-radial functions with m,n even; the other representations of $SL(2,\mathbb{R})$ give odd values for m,n and this amounts to extend the definition of $\overline{\mathcal{H}}_p\, ,\ \varphi, \dots$ to half-integer j, s). Then (for f a linear combination of such coefficients) $\|\Phi(f)\| = \|f \odot \zeta\|$ follows from Lemma 2 and $\|\Phi(f)\| \geq \|f\|_{M_0}$ using Proposition 1. In particular, $\|f\|_M = \|\Phi(f)\|$. Using approximations (similar as below), the formula then follows for K-finite f belonging to $A(G)$ and further on for its norm closure in $M(A(G))$.

If f_n ($\subseteq B(G)$) are spherical functions from the complementary series (i.e. arising from representations T_{l_n} with $l_n \in]-1, 0[$), then $\|f_n\|_M = 1$ and it was shown in [DH] that they belong to the norm closure of $A(G)$ in $M(A(G))$. Hence the same is true for $f f_n$ for any $f \in M(A(G))$. If $l_n \to 0$ (or -1) for $n \to \infty$, we have $f_n \to 1$ uniformly on compact sets in G and (see also [DH]) this implies that $(f_n \mid H)$ is an approximate unit in $A(H)$. Thus for K-finite $f \in M(A(G)) \cap C_0(G)$, we get that $(f f_n)$ is a Cauchy sequence in $M(A(G))$. Since it converges to f in the strong operator topology, we conclude that $\|f - f f_n\| \to 0$.

For general $f \in M(A(G))$ such that $f_{mn} \mid H \in A(H)$ for all $m,n \in \mathbb{Z}$, one can use approximations (e.g. by Fejer sums) to see that $\Phi(f)$ belongs to the predual. Then, as above, it follows that f belongs to the norm closure of $A(G)$ in $M(A(G))$ (which implies $f \in C_0(G)$). ☐

References

[CH] Cowling, Haagerup, Inventiones Math. **96** (1989).

[DH] De Cannière, Haagerup, American J. Math. **107** (1985).

[R] Repka, American J. Math. **100** (1978).

[V] Vilenkin, Special Functions...,AMS Transl. 1968.

[VK] Vilenkin, Klimyk, Representation of Lie groups..., Vol.1, Kluwer 1991.

[Ve] Veech, Monatsh. Math. **88** (1979).

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