G locally compact group

π representation of G by unitary operators on a Hilbert space $\mathcal{H}_\pi$ strongly continuous

$(\mid)$ inner product of $\mathcal{H}_\pi$

For $f, g \in \mathcal{H}_\pi$

$u_{f,g}(x) = (\pi(x) f \mid g)$ <u>coefficient function</u> of π

$B(G) = \{ u_{f,g} : \pi \text{ as above}, f, g \in \mathcal{H}_\pi \}$

<u>Fourier-Stieltjes algebra</u> of G

(Krein ~ 1940)

$\|u\|_B = \inf \{ \|f\| \|g\| : u = u_{f,g} \}$

$B(G) \subseteq C_b(G)$ subalgebra
contains the constant functions
Banach algebra w.r.t. $\| \ \|_B$

pointwise product

$u_{f,g} \cdot u_{f_1, g_1} = u_{f \otimes f_1, g \otimes g_1}$

coeff. of $\pi \otimes \pi_1$ (repr. on $\mathcal{H} \otimes \mathcal{H}_1$)

tensor product of repr.

$\mathcal{H} = L^2(G)$ (w.r.t. left Haar measure)

$(\lambda(x)f)(y) = f(x^{-1}y)$ $(f \in L^2)$

left regular representation

$A(G) = \{ u_{f,g} : f, g \in L^2(G) \}$ (coeff. of λ)

<u>Fourier algebra</u> of G (Eymard 1964)

$\|u\|_A = \inf \{ \|f\| \|g\| : u = u_{f,g},\ f, g \in L^2(G) \}$

$A(G)$ closed subalgebra (ideal) of $B(G)$

$VN(G)$ group von Neumann algebra

= von Neumann algebra on $L^2(G)$ generated by $\{\lambda(x) : x \in G\}$

(= strong closure in $\mathcal{B}(\mathcal{H})$ of linear subspace generated by this set)

$= \{T \in \mathcal{B}(\mathcal{H}) : T \circ \rho(x) = \rho(x) \circ T \quad \forall x \in G\}$

$\rho(x)$ right translation

$\left((\rho(x) f)(y) = f(yx)\, \Delta(x)^{\frac{1}{2}} \right)$

duality :

$(T, u_{f,g}) = (Tf \mid g)$

Precisely for $T \in VN(G)$ this does not depend on the representation of $u = u_{f,g}$.

It makes $VN(G)$ the <u>dual</u> of the Banach space $A(G)$

$(A(G) =$ predual of $VN(G))$

Similarly :

$B(G)$ is the dual of $C^*(G)$ (= enveloping C^*-algebra of $L^1(G)$)

Convolution:

$$(f*g)(y) = \int_G f(x)\, g(x^{-1}y)\, dx$$

$$f, g \in L^1(G) \Rightarrow f*g \in L^1(G)$$
$$\Rightarrow L^1(G) \text{ Banach algebra}$$

extensions:

$\mu * \nu$ $\quad (\mu, \nu \in M(G)) \Rightarrow M(G)$ Banach algebra
(finite Radon measures on G)

$\mu * f \in L^2(G)$ for $\mu \in M(G)$, $f \in L^2(G)$
defines $\lambda(\mu)$ convolution operator
$\lambda(\mu) \in VN(G) \quad \forall \mu \in M(G)$, $\lambda(\mu * \nu) = \lambda(\mu) \circ \lambda(\nu)$
(in particular $\lambda(w) \in VN(G) \quad \forall w \in L^1(G)$)
$M(G)$ is isomorphic to subalgebra of $VN(G)$

$$(\lambda(w), u_{f,g}) = (\lambda(w) f \mid g) = (w * f \mid g)$$
$$= \int_G \left(\int_G w(x) f(x^{-1}y)\, dx \right) \overline{g(y)}\, dy$$
$$= \int_G w(x) \underbrace{\int_G f(x^{-1}y) \overline{g(y)}\, dy}_{(\lambda(x) f \mid g)}\, dx$$
$$= \int_G w(x)\, u_{f,g}(x)\, dx \qquad \text{(same for } \lambda(\mu))$$

$\Rightarrow$ duality between $VN(G)$, $A(G)$ extends the usual duality between measures and functions on G.

δ_x point measure at x
$\delta_x * f = \lambda(x) f \qquad$ i.e. $\lambda(\delta_x) = \lambda(x)$.

generalization: π repr. of G (as before)
$\pi(\mu) f = \int_G \pi(x) f\, d\mu(x)$ (Bochner integral) extends π to a representation of $M(G)$ on $\mathcal{H}_\pi$
$\mu \in M(G)$, $f \in \mathcal{H}_\pi$

Examples:

$G = \mathbb{Z}$: $A(\mathbb{Z})$ = set of Fourier coefficients of all integrable functions on $T = \mathbb{R}/\mathbb{Z}$ (torus group)

$B(\mathbb{Z})$ = Fourier-Stieltjes coeff. of all measures on T

more generally:

G <u>abelian</u> (loc. comp.)

$\hat{G}$ dual group (characters, 1-dim. repr.)

Fourier transform defines isomorphisms

between $A(G)$ and $L^1(\hat{G})$

$B(G)$ $M(\hat{G})$

non-commutative case

$\hat{G}$: equivalence classes of irreducible repres.

for G compact: $\qquad \pi \in \hat{G}$ finite dimens.

$$A(G) = \Big\{ u \in C(G) : \underbrace{\sum_{\pi \in \hat{G}} \dim \pi \cdot \|\pi(f)\|_{N_1}}_{= \|f\|_A} < \infty \Big\}$$

trace-norm

more generally

G type I, unimodular, second countable

μ Plancherel measure on $\hat{G}$

for $u \in L^1 \cap L^2(G)$: $\quad \int_G |u(x)|^2 dx = \int_{\hat{G}} \operatorname{tr}\left(\pi(u)\pi(u)^*\right) d\mu(\pi)$

extends to isomorphisms between

$L^2(G)$ and $\int_{\hat{G}}^{\oplus} \mathcal{H}_\pi \otimes \mathcal{H}_\pi \, d\mu(\pi)$ $\quad$ direct integrals

Hilbert-Schmidt operators

$VN(G) \qquad \int_{\hat{G}}^{\oplus} \mathcal{B}(\mathcal{H}_\pi) \, d\mu(\pi)$

$A(G) \qquad \int_{\hat{G}}^{\oplus} N_1(\mathcal{H}_\pi) \, d\mu(\pi)$

trace class operators

for $u \in L^1 \cap A(G)$ $\qquad$ non-commutative L^1-spaces

$$u(e) = \int_{\hat{G}} \operatorname{tr}(\pi(u)) \, d\mu(\pi) \qquad \text{(inversion formula)}$$

Examples: G conn. Lie group, either semisimple or nilpotent

For G discrete: G type I $\Leftrightarrow$ $\exists H$ abelian subgroup of finite index (Thoma)

Properties of $A(G)$:

$A(G) \subseteq C_0(G)$ subalgebra (proper for G infinite)

Riemann-Lebesgue lemma

functions with compact support are dense in $A(G)$

translation invariant

separates pts. of G $\Rightarrow$ dense in $C_0(G)$ w.r.t. $\|\ \|_\infty$

Gelfand spectrum of $A(G)$: G

$\|u\|_A$ difficult to compute (in general)

special case: u positive definite ($\Leftrightarrow u = u_{f,f}$

then $\|u\|_A = u(e)$

$A(G)$ is generated by pos. def. functions

Conditions on G: amenability

G amenable $\iff$ $\exists$ left invariant mean on $L^\infty(G)$

Ex: amenable: abelian, compact ...

non-amenable: F_2 (free groups, discrete)

$SL(n,\mathbb{R})$ $(n \geq 2)$ semisimple Lie groups

$A(G)$ has unit $\iff$ G compact

TFAE: (i) G amenable

(ii) $A(G)$ has bounded approx. unit

(iii) $A(G)$ factorizes

(i.e. $A(G) = \{u \cdot v : u, v \in A(G)\}$)

growth properties:

For G amenable, non-compact decrease to 0 of $u \in A(G)$ can be arbitrarily slow

$w \in C_0(G) \Rightarrow \exists u \in A(G): u(x) \geq |w(x)| \; \forall x \in G$

For G non-amenable the elements of $A(G)$ satisfy growth conditions:

$\exists \rho$ σ-finite, non-neg. measure on G with $\rho(G) = \infty$ such that $A(G) \subseteq L^1(G, \rho)$

for $G = F_2$: $\sum_{x \in F_2} \frac{|u(x)|}{3^{\frac{n}{2}} \cdot n^3} < \infty$ (Haagerup)

($n = |x|$ word length)

for semisimple Lie groups G with finite centre:
Kunze-Stein phenomenon (Cowling)

$$A(G) \subseteq L^p(G) \quad \forall p > 2$$

irred. repr. such that $u_{f,g} \in L^2(G)$: square-integr. repr.

this holds iff π is a subrepres. of λ

($A(G) \subseteq L^2(G) \underset{\text{Rickert}}{\Leftrightarrow} G$ comp.)

Arens products

A Banach algebra, $\quad A \subseteq A''$ (bidual space)

For $u \in A$ $\quad \omega \mapsto u \odot \omega$ bidual of $v \mapsto u \cdot v$
$\qquad\qquad A'' \to A'' \qquad\qquad A \to A$
(multiplication operator)

For $T \in A'$ $\quad \langle \omega \cdot T, u \rangle = \langle u \odot \omega, T \rangle$ defines $\omega \cdot T \in A'$
(A' A''-module)

For $\eta \in A''$ $\quad \langle \eta \odot \omega, T \rangle = \langle \eta, \omega \cdot T \rangle$ defines $\eta \odot \omega \in A''$
first Arens product

(if $\omega = \lim u_i$, $\eta = \lim v_j$ $\quad$ w*-limits $(u_i), (v_j) \subseteq A$ $\quad \eta \odot \omega = \lim_j \lim_i v_j \cdot u_i$)

$\Rightarrow A''$ Banach algebra, A subalgebra

For $A = A(G)$ commutative Banach algebra

$A(G)''$ <u>not</u> commutative for "most" infinite G

if there exists an infinite G such that

$A(G)''$ commut. ("$A(G)$ is Arens regular")

then G must be discrete

and must not contain any infinite amenable subgroup ($\Rightarrow$ G contains no infinite abelian subgroup)

$A(G) \subseteq Z(A(G)'')$ (centre)

for many amenable groups $Z(A(G)'') = A(G)$

("$A(G)$ strongly Arens irregular")

for example: G discrete amenable

or metrizable solvable

for many non-amenable groups $Z(A(G)'') \supsetneq A(G)$

for example: G discrete, $G \supseteq F_2$

or conn. Lie group semisimple finite centre

$A(G)' = VN(G)$ von Neumann algebra,
in particular a C^*-algebra

abstract: functionals on a C^*-algebra can always be represented as coefficients of a representation on a Hilbert space
(GNS-construction)

more explicitly: limits of representations can be realized using <u>ultraproducts</u>

π_n repr. of $VN(G)$ on $\mathcal{H}_n$ $(n \in \mathbb{N})$

p ultrafilter on $\mathbb{N}$ (non-trivial)

consider sequences (h_n) with $h_n \in \mathcal{H}_n$, $\sup \|h_n\| < \infty$
equivalence: $(h_n) \sim (h_n') \iff \lim_{n \to p} \|h_n - h_n'\| = 0$

$\mathcal{H}_p$: equivalence classes $\rightarrow$ Hilbert space

inner product: $\left((h_n) | (h_n')\right) = \lim_{n \to p} (h_n | h_n')$

$T \in VN(G)$ $\qquad \pi_p(T)((h_n)) = (\pi_n(T) h_n)$
defines repres. π_p of $VN(G)$ on $\mathcal{H}_p$

$f = (f_n)^\sim,\ g = (g_n)^\sim \in \mathcal{H}_p \implies (\pi_p(T) f | g) = \lim_{n \to p} (T, u_{f_n, g_n})$

represents $w^*\text{-}\lim_{n \to p} u_{f_n, g_n} \in A(G)''$.

in general: π_p <u>not</u> strongly continuous
"singular representation"