Lecture 2: Operator Amenability of the Fourier Algebra

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Amenability of Groups

Let us first recall that a locally compact group G is amenable if there exists a left invariant mean on $L_\infty(G)$, i.e. there exists a positive linear functional

$$m : L_\infty(G) \to \mathbb{C}$$

such that $m(1) = 1$ and $m({}_sh) = m(h)$ for all $s \in G$ and $h \in L_\infty(G)$, where we define ${}_sh(t) = h(st)$.

Theorem: The following are equivalent:

1. G is amenable;

2. G satisfies the Følner condition: for every $\varepsilon > 0$ and compact subset $C \subseteq G$, there exists a compact subset $K \subseteq G$ such that

$$\frac{|K \Delta sK|}{\mu(K)} < \varepsilon \qquad \text{for all } s \in C;$$

3. $A(G)$ has a bounded (or contractive) approximate identity.

Amenability of Banach Algebras

In the early 1970's, B. Johnson introduced the bounded Hochschild cohomology and the amenability for Banach algebras.

A Banach algebra A is called amenable if for every bounded A-bimodule V, every bounded derivation $D : A \to V^*$ is inner, i.e. there exists $f \in V^*$ such that

$$D(a) = a \cdot f - f \cdot a.$$

A linear map $D : A \to V^*$ is a derivation if it satisfies

$$D(ab) = D(a) \cdot b + a \cdot D(b) \qquad \text{for all } a, b \in A.$$

Theorem [B. Johnson 1972] : A locally compact group G is amenable if and only if $L_1(G)$ is an amenable Banach algebra.

Amenability of $A(G)$

Question: Since $A(G)$ is the natural dual object of $L_1(G)$, it is natural to ask whether G is amenable if and only if $A(G)$ is amenable ??

B. Johnson studied this problem for compact groups in 1994. If G is a compact group, it is known that its irreducible representations are all finite dimensional, i.e.

$$\pi : G \to M_{d_\pi},$$

where we let d_π denote the dimension of the matrix. We let $\widehat{G}$ denote the set of all (non-equivalent class of) irreducible represetations of G.

Let t to be the characteristic function on the diagonal of $G \times G$. Johnson showed that if G is a finite group, then the diagonal element t has norm

$$\|t\|_{A(G)\otimes^\gamma A(G)} = \sum_{\pi \in \widehat{G}} d_\pi^3 / \sum_{\pi \in \widehat{G}} d_\pi^2$$

in $A(G) \otimes^\gamma A(G)$

This suggests that if G is an infinite group and $\{d_\pi : \pi \in \widehat{G}\}$ is unbounded, then there would be no bounded approximate diagonal in $A(G) \otimes^\gamma A(G)$.

Theorem [B. Johnson 1984]: If G is a compact group with $\sup\{d_\pi : \pi \in \widehat{G}\} < \infty$, then $A(G)$ is amenable.

Therefore, if a group G is almost abelian, i.e., it has an abelian subgroup H such that $|G/H| < \infty$, then $A(G)$ is amenable.

Theorem [B. Johnson 1984]: Let G be a compact non-discrete group for which $\{\pi \in \widehat{G} : d_\pi = n\}$ is finite for each $n \in \mathbb{N}$. Then Banach algebra $A(G)$ is not amenable.

As examples, $A(G)$ is not amenable if $G = SU(2, \mathbb{C})$

Recent Results on Amenability of $A(G)$

It is known that the check map $\check{}$: $f \in A(G) \to \check{f} \in A(G)$ is an isometric isomorphism. However, it is not necessarily completely bounded.

Theorem [F-R 2005]: Let G be a locally compact group.

- The check map $\check{}$ is cb on $A(G)$ if and only if G is almost abelian.
- The check map $\check{}$ is a c. isometry on $A(G)$ if and only if G is abelian

Then considering the anti-diagonal $\Gamma = \{(s, s^{-1}) : s \in G\}$ and the characteristic function χ_Γ on $G_d \times G_d$, they proved that

Theorem [F-R]: Let G be a locally compact group.

- $A(G)$ is amenable if and only if G is almost abelian if and only if $\chi_\Gamma \in B(G_d \times G_d)$
- $A(G)$ is 1-amenable if and only if G is abelian if and only if χ_Γ has norm 1 in $B(G_d \times G_d)$

We see that the amenability of $A(G)$ is closely related to the abelian, or alsmost abelian structure of G.

To consider the amenability of $A(G)$ for general groups G, we should study this in the category of operator spaces, replacing boundedness by completely boundedness, ...

Tensor Products

In Banach space theory, there are two very useful tensor products, the injective tensor product $\otimes^\lambda$ and the projective tensor product $\otimes^\gamma$, on Banach spaces.

Let us recall that for $u \in V \otimes W$, we define

$$\|u\|_\gamma = \inf\{\sum \|v_i\|\|w_i\| : u = \sum v_i \otimes w_i\}.$$

This defines a norm on $V \otimes W$ and we let $V \otimes^\gamma W$ denote the completion.

The projective tensor product satisfies some functorial properties such as

$$B(V \otimes^\gamma W, Z) = B(V, B(W, Z)).$$

In particular, if $Z = \mathbb{C}$, we have

$$(V \otimes^\gamma W)^* = B(V, W^*).$$

Projective tensor product is very useful in Banach algebra theory. For instance, the multiplication m on a Banach algebra A satisifes

$$\|xy\| \leq \|x\|\|y\|$$

if and only if it extends to a contractive linear map

$$m : x \otimes y \in A \otimes^{\gamma} A \to xy \in A.$$

Gven two locally compact groups G_1 and G_2, we have the isometric isomorphism

$$L_1(G_1) \otimes^{\gamma} L_1(G_2) = L_1(G_1 \times G_2).$$

However, in general, we can only have a contraction

$$A(G_1) \otimes^{\gamma} A(G_2) \to A(G_1 \times G_2).$$

Theorem [Losert 1984]: We have the linear isomorphism

$$A(G_1) \otimes^{\gamma} A(G_2) \cong A(G_1 \times G_2)$$

if and only if either G_1 or G_2 is almost abelian, i.e. there exists an abelian subgroup with finitely many distinct cosets.

Operator Space Projective Tensor Product

Correspondingly, we can define the the operator space projective tensor product $\hat{\otimes}$. Given $u \in V \otimes W$, we can write

$$u = [\alpha_{ik}]([x_{ij}] \otimes [y_{kl}])[\beta_{jl}] = \alpha(x \otimes y)\beta$$

with $\alpha = [\alpha_{ik}] \in M_{1,nm}$, $x = [x_{ij}] \in M_n(V)$, $y = [y_{kl}] \in M_m(W)$, and $\beta = [\beta_{jl}] \in M_{nm,1}$. So we let

$$\|u\|_{\wedge,1} = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\| : u = \alpha(x \otimes y)\beta\}.$$

This defines a norm on $V \otimes W$ we let $V \hat{\otimes} W$ denote the completion.

Now we can define an operator space matrix norm for $u \in M_n(V \otimes W)$ by letting

$$\|u\|_{\wedge,k} = \inf\{\|\alpha\|\|x\|\|y\|\|\beta\| : u = \alpha(x \otimes y)\beta\},$$

where the infimum is taken for $\alpha \in M_{k,nm}$, $x \in M_n(V)$, $y \in M_m(W)$, and $\beta \in M_{nm,k}$.

We also have some functorial properties such as

$$CB(V\hat{\otimes}W, Z) = CB(V, CB(W, Z)).$$

In particular, if $Z = \mathbb{C}$, we have

$$(V\hat{\otimes}W)^* = CB(V, W^*).$$

We also have $V\hat{\otimes}W = W\hat{\otimes}V$.

If V or W has the MAX matrix norm, then we have thsi isometry

$$V \otimes^\gamma W = V\hat{\otimes}W.$$

In particular, if W is an operator space, we have the isometric isomorphism

$$L_1(X, \mu) \otimes^\gamma W = L_1(X, \mu)\hat{\otimes}W.$$

Theorem [E-R 1990]: Let M and N are von Neumann algebras. Then we have the completely isometric isomorphism

$$(M\bar{\otimes}N)_* = M_*\hat{\otimes}N_*.$$

As a consequence, we get hte completely isometric isomorphisms

$$L_1(G_1 \times G_1) = L_1(G_1)\hat{\otimes}L_1(G_2)$$

and

$$A(G_1 \times G_2) = A(G_1)\hat{\otimes}A(G_2).$$

Remark: If G_1 is abelian, then $A(G_1) = L_1(\widehat{G}_1)$ has the MAX structure. In this case, we have

$$A(G_1 \otimes G_2) = A(G_1)\hat{\otimes}A(G_2) = A(G_1) \otimes^{\gamma} A(G_2).$$

Completely Contractive Banach Algebras

We say that A is a completely contractive Banach algebra or simply c.c Banach algebra if 1) A is a Banach algebra; 2) A has an operator space structure; 3) the multiplication extends to a complete contraction

$$m : x \otimes y \in A\widehat{\otimes}A \rightarrow xy \in A,$$

i.e. we have

$$\|[x_{ij}y_{kl}]\|_{nm} \leq \|[x_{ij}]\|_n \|[y_{kl}]\|_m$$

for all $[x_{ij}] \in M_n(A)$ and $[y_{kl}] \in M_m(A)$.

Remark 1: If the multiplication is completely bounded with $\|m\|_{cb} \leq K < \infty$, then an equivalent operator space matrix norm $|||[x_{ij}]|||_n = K\|[x_{ij}]\|_n$, we define a completely contractive Banach algebra structure.

Remark 2: Every C*-algebra, or operator algebra is a c.c. Banach algebra.

$L_1(G)$

Since $L_1(G)$ is equipped a natural MAX structure, we have

$$MAX(L_1(G))\hat{\otimes}MAX(L_1(G)) = MAX(L_1(G) \otimes^\gamma L_1(G)).$$

Since the multiplication m on $L_1(G)$ is contractive with respect to $\otimes^\gamma$, and thus completely contractive with respect to $\hat{\otimes}$. So $L_1(G)$ is a c.c. Banach algebra.

Co-multiplication on $L_\infty(G)$

Except the pointwise multiplication on $L_\infty(G)$, we have a natural co-multiplication

$$\Gamma_a : h \in L_\infty(G) \to \Gamma_a(h) \in L_\infty(G)\bar{\otimes}L_\infty(G) = L_\infty(G \times G)$$

which is given by $\Gamma_a(h)(s,t) = h(st)$.

• This co-multiplication Γ_a is a unital weak* continuous (normal) isometric *-homomorphism from $L_\infty(G)$ into $L_\infty(G \times G)$.

• Γ_a is associated with the multiplication of the group G. Since multiplication of G is associative, then Γ_a is co-associative in the sense that

$$(\Gamma_a \otimes \iota) \circ \Gamma_a = (\iota \otimes \Gamma_a) \circ \Gamma_a.$$

We call $(L_\infty(G), \Gamma_a)$ a commutative Hopf von Neumann algebra.

Fundamental Unitary

Moreover, there exists a unitary operator W on $L_2(G \times G) = L_2(G) \otimes L_2(G)$ defined by

$$W\xi(s,t) = \xi(s, s^{-1}t)$$

such that $\Gamma(h) = W^*(1 \otimes h)W$ for all $h \in L_\infty(G)$ since

$$W^*(1 \otimes h)W\xi(s,t) = (1 \otimes h)W\xi(s,st) = h(st)W\xi(s,st) = h(st)\xi(s,t).$$

The co-multiplication

$$\Gamma_a : h \in L_\infty(G) \to \Gamma_a(h) \in L_\infty(G)\bar{\otimes}L_\infty(G) = L_\infty(G \times G)$$

is a unital weak* continuous (completely) isometric *-homomorphism from $L_\infty(G)$ into $L_\infty(G \times G)$.

Taking the pre-adjoint, we get an associative (completely) contractive multiplication

$$\star = (\Gamma_a)_* : L_1(G)\hat{\otimes}L_1(G) = L_1(G) \otimes^\gamma L_1(G) \to L_1(G),$$

which is just the convolution on $L_1(G)$. This shows that $L_1(G)$ is a c.c. Banach algebra.

Co-multiplication on $L(G)$

We have a natural co-associative co-multiplication on $L(G)$ given by

$$\Gamma_G : \lambda_s \in L(G) \to \Gamma_G(\lambda_s) = \lambda_s \otimes \lambda_s \in L(G)\bar{\otimes}L(G) = L(G \times G).$$

In fact, we can write

$$\Gamma_G(\lambda_s) = W(\lambda_s \otimes 1)W^* = \Sigma W \Sigma(1 \otimes \lambda_s)\Sigma W^* \Sigma = \widehat{W}(1 \otimes \lambda_s)\widehat{W}^*.$$

Here Σ is the flip operator on $L_2(G) \otimes L_2(G)$, and we let $\widehat{W} = \Sigma W \Sigma$ to be the fundamental unitary operator for $L(G) =$ "$L_\infty(\widehat{G})$".

Therefore, $(L(G), \Gamma_G)$ is a co-commutative Hopf von Neumann algebra.

In fact, every co-commutative Hopf von Neumann algebra has such a form.

Taking the pre-adjoint, we obtain the completely contractive pointwise multiplication

$$m = (\Gamma_G)_* : u \otimes v \in A(G)\hat{\otimes}A(G) \to u \cdot v \in A(G).$$

Therefore, $A(G)$ is also a c.c. Banach algebra.

Operator A-bimodules

Let A be a c.c. Banach algebra and V be an operator space. We say that V is an operator A-bimodule if V is an A-bimodule such that the bimodule operation

$$\pi_l : a \otimes x \in A\widehat{\otimes}V \to a \cdot x \in\in V \text{ and } \pi_r : x \otimes a \in V\widehat{\otimes}A \to x \cdot a \in V$$

are completely bounded.

If V is an operator A-bimodule, then its operator dual V^* is also an operator A-bimodule with A-bimodule operation given by

$$\langle a \cdot f, x\rangle = \langle f, x \cdot a\rangle \text{ and } \langle f \cdot a, x\rangle = \langle f, a \cdot x\rangle.$$

Operator Amenability of C.C. Banach Algebras

Motivated by Johnson's definition for Banach algebras, a C.C. Banach algebra A is called operator amenable if for every operator A-bimodule V, every completely bounded derivation $D : A \to V^*$ is inner, i.e. there exists $f \in V^*$ such that

$$D(a) = a \cdot f - f \cdot a.$$

Theorem [R 1995]: Let A be a c.c. Banach algebra. Then A is operator amenable if and only if there exists a net of bounded elements $u_\alpha \in A \hat{\otimes} A$ such that for every $a \in A$,

1) $a \cdot u_\alpha - u_\alpha \cdot a \to 0$ in $A \hat{\otimes} A$

2) $m(u_\alpha)$ is a bounded approximate identity for A.

Theorem [R 1995]: Let G be a locally compact group. Then

(1) G is amenable if and only if $L_1(G)$ is operator amenable.

(2) G is amenable if and only if $A(G)$ is operator amenable.

Proof: Accoring to the previous theorem, if $A(G)$ is operator amenable, then $A(G)\hat{\otimes}A(G)$ has a bounded approximate diagonal, and thus $A(G)$ has a BAI. Therefore, G is amenable.

The difficult part is to show that the amenability of G implies the operator amenability of $A(G)$.

Look at Compact Group Case:

Suppose that G is a compact group. Then there exists a base of open nbhds $\{U_\alpha\}$ at e with compact closure such that $sU_\alpha s^{-1} = U_\alpha$.

Let $\xi_\alpha = \chi_{U_\alpha}/\mu(U_\alpha)^{\frac{1}{2}}$. Then $\{\xi_\alpha\}$ is a net of unit vectors in $L_2(G)$ such that

$$\xi_\alpha(st) = \xi_\alpha(ts) \qquad \text{for all } s,t \in G.$$

We note that

$$\pi_{\lambda,\rho} : (s,t) \in G \times G \to \lambda_s \rho_t \in B(L_2(G))$$

is a continuous unitary representation of G. Then the coefficient functions

$$u_\alpha(s,t) = \langle \lambda_s \rho_t \xi_\alpha | \xi_\alpha \rangle$$

are contractive elements in $B(G \times G) = A(G \times G) = A(G)\hat{\otimes}A(G)$.

Finally, we show that $\{u_\alpha\}$ is a contractive approximate diagonal in $A(G)\hat{\otimes}A(G)$. Therefore, $A(G)$ is operator amenable.

Comparing this with Johnson's Result

If G is a finite group, then

$$u_e(s,t) = \langle \lambda_s \rho_t \delta_e | \delta_e \rangle = \langle \lambda_s \delta | \rho(t^{-1}) \delta_e \rangle = \langle \delta_s | \delta_t \rangle$$

is just the characteristic function t on the diagonal of $G \times G$. It has norm

$$\|t\|_{A(G)\hat{\otimes}A(G)} = 1$$

in $A(G)\hat{\otimes}A(G)$.

Comparing Johnson's calculation

$$\|t\|_{A(G)\otimes^{\gamma}A(G)} = \sum_{\pi\in\widehat{G}} d_\pi^3 / \sum_{\pi\in\widehat{G}} d_\pi^2$$

in $A(G) \otimes^{\gamma} A(G)$

Weak Amenability

A Banach algebra A is called weakly amenable if every bounded derivation $D : A \to A^*$ is inner.

Theorem [Johnson 1991]: Let G be a locally compact group. Then $L_1(G)$ is weakly amenable.

However, Johnson observed that there exist some compact Lie group G such that $A(G)$ is not weakly amenable.

A c.c. Banach algebra A is called weakly operator amenable if every completely bounded derivation $D : A \to A^*$ is inner.

Since every bounded derivation $D : L_1(G) \to L_\infty(G)$ is completely bounded with $\|D\| = \|D\|_{cb}$, it is easy to see that $L_1(G)$ is weakly operator amenable.

Theorem [Spronk 2002]: Let G be a locally compact group. Then $A(G)$ is weakly operator amenable .

Question: Let $\mathbb{G}$ be a LCQG, can we prove that $L_1(\mathbb{G})$ is weakly operator amenable ?

Recommended References:

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