Lecture 3: Approximation Properties for Group C*-Algebras

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Amenability of Groups

Let us first recall that a locally compact group G is amenable if there exists a left invariant mean on $L_\infty(G)$, i.e. there exists a positive linear functional

$$m : L_\infty(G) \to \mathbb{C}$$

such that $m(1) = 1$ and $m({}_sh) = m(h)$ for all $s \in G$ and $h \in L_\infty(G)$, where we define ${}_sh(t) = h(st)$.

Theorem: The following are equivalent:

1. G is amenable;
2. G satisfies the Følner condition: for every $\varepsilon > 0$ and compact subset $C \subseteq G$, there exists a compact subset $K \subseteq G$ such that
$$\frac{|K \Delta sK|}{\mu(K)} < \varepsilon \qquad \text{for all } s \in C;$$
3. $A(G)$ has a bounded (or contractive) approximate identity.

Amenability of G is closely related to the Nuclearity of $C_\lambda^*(G)$!

Nuclear C*-algebras

A C*-algebra A is said to be nuclear if there exists two nets of completely positive and contractive maps $S_\alpha : A \to M_{n(\alpha)}$ and $T_\alpha : M_{n(\alpha)} \to A$ such that

$$T_\alpha \circ S_\alpha \to id_A$$

in the point-norm topology.

A C*-algebra A is said to have the CPAP if there exists a net of comletely positive and contractive finite rank maps $T_\alpha : A \to A$ such that $T_\alpha \to id_A$ in the point-norm topology.

It is known that f. d. C*-algebras $A = M_{n_1} \oplus \cdots \oplus M_{n_k}$, $C(\Omega)$, $K(H)$, inductive limit, c_0-direct sum, ... are nulcear C*-algebras.

Group C*-algebras

Let G be a discrete group. In this case, $C^*_\lambda(G) = \overline{span\{\lambda_s : s \in G\}}^{\|\cdot\|}$.

Theorem: Let G be a discrete group. Then the following are equivalent:

1. G is amenable;
2. $C^*_\lambda(G)$ is nuclear;
3. $C^*_\lambda(G)$ has the CPAP;
4. there exists a net of positive and contractive elements $f_\alpha \in A(G)$ such that

   $$f_\alpha(s) \to 1$$

   for all $s \in G$.

Idea of Proof: 1. $\Rightarrow$ 2. Suppose that G is discrete and amenable. It is known from the Følner condition that for any finite set E in G and $\varepsilon > 0$, there exists a finite subset $F_\alpha = F_{(E,\varepsilon)}$ in G such that

$$\frac{|F_\alpha \Delta s \cdot F_\alpha|}{|F_\alpha|} < \varepsilon$$

for all $s \in E$.

Let ι_α be the isometric inclusion of $\ell_2(F_\alpha)$ into $\ell_2(G)$ and P_α be the contractive projection from $\ell_2(G)$ onto $\ell_2(F_\alpha)$. We can obtain a complete contraction

$$S_\alpha : x \in C^*_\lambda(G) \to P_\alpha x \iota_\alpha \in B(\ell_2(F_\alpha)) = M_{n(\alpha)},$$

where $n(\alpha) = |F_\alpha|$ is the cardinality of F_α.

Let $\{e^{\alpha}_{s,t}\}_{s,t\in F_\alpha}$ be the matrix unit of $B(\ell_2(F_\alpha))$. We can define a linear map

$$T_\alpha : e^{\alpha}_{s,t} \in B(\ell_2(F_\alpha)) = M_{n(\alpha)} \to \frac{\lambda_{st^{-1}}}{n(\alpha)} \in C^*_\lambda(G).$$

Now it is easy to verify that

$$e^{\alpha}_{s,s}\lambda_p(g)e^{\alpha}_{t,t} = \begin{cases} e^{\alpha}_{s,t} & \text{if } g = st^{-1} \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, for any $g \in G$, we have

$$S_\alpha(\lambda_g) = P_\alpha \lambda_g \iota_\alpha = \sum_{s,t\in F_\alpha} e^{\alpha}_{s,s}\lambda_g e^{\alpha}_{t,t} = \sum_{s\in F_\alpha \cap gF_\alpha} e^{\alpha}_{s,g^{-1}s},$$

and thus

$$T_\alpha \circ S_\alpha(\lambda_g) = \frac{|F_\alpha \cap gF_\alpha|}{n(\alpha)}\lambda_g.$$

It follows that

$$\|T_\alpha \circ S_\alpha(\lambda_g) - \lambda_g\| \le \frac{|F_\alpha \Delta gF_\alpha|}{n(\alpha)}\|\lambda_g\| < \varepsilon \qquad \text{for all } g \in E.$$

Therefore, we have $\|T_\alpha \circ S_\alpha(x) - x\| \to 0$ for every $x \in C^*_\lambda(G)$.

2. $\Rightarrow$ 3. is obvious.

3. $\Rightarrow$ 4. Suppose that there is a net of c.p. finite rank contractions $T_\alpha : C^*_\lambda(G) \to C^*_\lambda(G)$ such that $T_\alpha \to id_{C^*_\lambda(G)}$ in the point-norm topology.

Then we consider a net of functions $\{u_\alpha\}$ on G defined by

$$u_\alpha(s) = \langle \lambda_s^* T_\alpha(\lambda_s)\delta_e | \delta_e \rangle = \langle T_\alpha(\lambda_s)\delta_e | \lambda_s \delta_e \rangle.$$

Since T_α are completely positive maps, each u_α is a positive definite function contained in $B(G)$ and we have

$$\|u_\alpha\|_{B(G)} = u_\alpha(e) = \langle T_\alpha(1)\delta_e | \delta_e \rangle \leq \|T_\alpha(1)\| \leq 1.$$

Moreover, it is known by Haagerup that since each T_α is finite rank, then $u_\alpha \in \ell_2(G) \subseteq A(G)$ with $\|u_\alpha\|_{A(G)} = \|u_\alpha\|_{B(G)} \leq 1$.

Finally, we see that for each $s \in G$, $T_\alpha(\lambda_s) \to \lambda_s$ in norm-topology implies that

$$u_\alpha(s) = \langle T_\alpha(\lambda_s)\delta_e | \lambda_s \delta_e \rangle \to \langle \lambda_s \delta_e | \lambda_s \delta_e \rangle = 1.$$

4. $\Rightarrow$ 1. If we have $\{u_\alpha\}$ in $A(G)$ such that $\|u_\alpha\|_{A(G)} \leq 1$ and $u_\alpha(s) \to 1$ for every $s \in G$. Then for each $\delta_s = \omega_{\delta_e,\delta_s} \in A(G)$, we have

$$\|u_\alpha\delta_s - \delta_s\|_{A(G)} = \|u_\alpha(s)\delta_s - \delta_s\|_{A(G)} = |u_\alpha(s) - 1|\|\delta_s\|_{A(G)} \to 0.$$

This implies that $\{u_\alpha\}$ is a contractive approximate identity of $A(G)$.

cb-Fourier Algebra $A_{cb}(G)$

Let $M_{cb}A(G)$ be the space of all cb-multipliers of $A(G)$. It is clear that $A(G)$ is a subalgebra of $M_{cb}A(G)$. Since for every $u \in A(G)$ and $[\omega_{ij}] \in M_n(A(G))$,

$$\|[m_u(\omega_{ij})]\|_{M_n(A(G))} = \|[u\omega_{ij}]\|_{M_n(A(G))} \leq \|u\|_{A(G)}\|[\omega_{ij}]\|_{M_n(A(G))},$$

we get

$$\|u\|_{cb} := \|m_u\|_{cb} \leq \|u\|_{A(G)}.$$

With this new cb-norm on $A(G)$, we can let $A_{cb}(G)$ to be the cb-norm closure

$$A_{cb}(G) = \overline{A(G)}^{\|\cdot\|} \subseteq M_{cb}A(G).$$

Then $A_{cb}(G)$ is a c.c. Banach subalgebra of $M_{cb}A(G)$.

Weak Amenability of Groups

A locally compact group G is weakly amenable if there exists a net of elements $\{u_\alpha\}$ in $A_c(G) = A(G) \cap C_c(G)$ such that

$$\|u_\alpha\|_{cb} := \|m_{u_\alpha}\|_{cb} \leq 1 \text{ (or } \leq k < \infty)$$

and

$$\|u_\alpha \omega - \omega\|_{A(G)} \to 0$$

for all $\omega \in A(G)$. This is equivalent to saying that $A_{cb}(G)$ has a CAI (or has a BAI).

We let $\Lambda(G) = \inf\{k : \|u_\alpha\|_{cb} \leq k\}$.

Amenability of $A_{cb}(G)$

It is natural to consider the

amenability (or operator amenability) of $A_{cb}(G)$.

Theorem [FVS 2007]: The cb-Fourier algebra $A_{cb}(\mathbb{F}_2)$ is operator amenable.

Mainly they prove that if G is a weakly amenable discrete group such that $C^*(G)$ is residually finite, then $A_{cb}(G)$ is operator amenable.

CCAP and CBAP

An operator space V is to have the CBAP (resp. CCAP) if there exists a net of comletely bounded (resp. completely contractive) finite rank maps $T_\alpha : V \to V$ such that $T_\alpha \to id_V$ in the point-norm topology.

Theorem [Haagerup]: Let G be a discrete group. Then the following are equivalent:

1. G is weakly amenable with $\Lambda(G) \leq k$;
2. $C^*_\lambda(G)$ has the CBAP with cb-norm $\leq k$;
3. there exists a net of $\{u_\alpha\}$ in $A_c(G)$ such that $\|u_\alpha\|_{cb} \leq k$ and
$$u_\alpha(s) \to 1$$
for all $s \in G$.

Proof of 2. $\Rightarrow$ 3. Consider $u_\alpha = \langle \lambda_s^* T_\alpha(\lambda_s)\delta_e | \delta_e \rangle$ with $\|u_\alpha\|_{cb} \leq \|T_\alpha\|_{cb}$.

Predual of $M_{cb}A(G)$

Let us recall that

$$M_{cb}A(G) = \{u : G \to \mathbb{C} \text{ continuous such that } m_u \text{ is cb on } A(G)\}$$

with

$$\|u\|_{cb} := \|m_u\|_{cb} \geq \|u\|_{L_\infty(G)}.$$

Then each $f \in L_1(G)$ defines a bounded linear functional

$$\tau_f(u) = \int_G f(s)u(s)ds$$

on $M_{cb}A(G)$ with $\|\tau_f\| \leq \|f\|_{L_1(G)}$. Since $\tau : L_1(G) \to M_{cb}A(G)^*$ is an injection, we define

$$Q_{cb}(G) = \overline{\tau(L_1(G))}^{\|\cdot\|} \subseteq M_{cb}A(G)^*.$$

It is known by H-K that we have the isometric isomorphism

$$Q_{cb}(G)^* = M_{cb}A(G).$$

Remark: It was shown by K-R in 1996 (for Kac algebras), and by H-N-R in 2009 and Daws in 2010 (for LCQG) that $M_{cb}A(G)$ is a dual Banach algebra, i.e. multiplication is weak* continuous in each component.

Elements in $Q_{cb}(G)$

Let G be a discrete group and let $u \in M_{cb}A(G)$. Then

$$m_u : \omega \in A(G) \to u\omega \in A(G)$$

is a cb-map. Its adjoint map

$$M_u = (m_u)^* : \lambda_s \in L(G) \to \psi(s)\lambda_s \in L(G)$$

is completely bounded and weak* continuous on the group von Neumann algebra $L(G)$. The restriction of M_u to $C^*_\lambda(G)$ defines a cb map

$$\bar{M}_\psi = M_\psi|_{C^*_\lambda(G)} : \lambda_s \in C^*_\lambda(G) \to u(s)\lambda_s \in C^*_\lambda(G)$$

on $C^*_\lambda(G)$ and we have

$$\|m_u\|_{cb} = \|M_u\|_{cb} = \|\bar{M}_u\|_{cb}.$$

Given $a = [a_{ij}] \in K_\infty(C^*_\lambda(G)) \subseteq B(\ell_2)\check{\otimes} C^*_\lambda(G)$ and $u = [u_{ij}] \in K_\infty(C^*_\lambda(G))^*$ (or $u = [u_{ij}] \in (B(\ell_2)\check{\otimes} C^*_\lambda(G))^*$), we obtain a linear functional $\omega_{a,\varphi}$ on $M_{cb}A(G)$ given by

$$\langle \omega_{a,\varphi}, u\rangle = \langle [\bar{M}_u(a_{ij})], [\varphi_{ij}]\rangle = \sum_{ij} \langle \bar{M}_u(a_{ij}), \varphi_{ij}\rangle.$$

It is easy to see that $\omega_{a,\varphi}$ is bounded on $M_{cb}A(G)$ with

$$\|\omega_{a,\varphi}\| \le \|[a_{ij}]\|_{K_\infty(C^*_\lambda(G))} \|[\varphi_{ij}]\|_{K_\infty(C^*_\lambda(G))^*}.$$

Theorem [H-K 1996]: Let G be a discrete group. We have

$$\begin{aligned} Q_{cb}(G) &= \{\omega_{a,\varphi} : a \in K_\infty(C^*_\lambda(G)) \text{ and } \varphi \in (K_\infty(C^*_\lambda(G)))^*\} \\ &= \{\omega_{a,\varphi} : a \in B(\ell_2)\check{\otimes} C^*_\lambda(G), \text{ and } \varphi \in (B(\ell_2)\check{\otimes} C^*_\lambda(G)^*)\}. \end{aligned}$$

Groups with the AP

We say that a locally compat group has the AP if there exists a net of (not necessarily bounded in $A(G)$-norm or cb-norm) elements $\{u_\alpha\}$ in $A_c(G)$ such that

$$u_\alpha \to 1 \subseteq M_{cb}A(G)$$

in $\sigma(M_{cb}A(G), Q_{cb}(G))$-topology.

Examples

- It is clear that every amenable groups and weakly amenable groups have the AP.

- If G has the AP, then closed subgroups has the AP.

- If H and K are locally compact groups with the AP, then their direct product (or semi-direct product if action is continuous) has the AP.

- In particular, $\mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ has the AP, but not weakly amenable.

Grothendick's Approximation Property

A Banach space is said to have Grothendicks' AP if there exists a net of bounded finite rank maps $T_\alpha : V \to V$ such that $T_\alpha \to id_V$ uniformly on compact subsets of V.

We note that a subset $K \subseteq V$ is compact if and only if there exists a sequence $(x_n) \in c_0(V)$ such that

$$K \subseteq \overline{conv\{x_n\}}^{\|\cdot\|} \subseteq V.$$

Therefore, V has Grothendick's AP if and only if there exists a net of finite rank bounded maps T_α on V such that

$$\|(T_\alpha(x_n)) - (x_n)\|_{c_0(V)} \to 0$$

for all $(x_n) \in c_0(V)$.

Operator Space Approximation Property

An operator space V is said to have the operator space approximation property (or simply, OAP) if there exists a net of finite rank bounded maps T_α on V such that

$$\|[T_\alpha(x_{ij})] - [x_{ij}]\|_{K_\infty(V)} \to 0$$

for all $[x_{ij}] \in K_\infty(V)$, where we let $K_\infty(V) = \overline{\cup_{n=1}^{\infty} M_n(V)}$. In this case, we say that $T_\alpha \to id_V$ in the stable point-norm topology.

We say that $V \subseteq B(H)$ has the strong OAP if we can replace $K_\infty(V)$ by $B(\ell_2)\check{\otimes}V$, which is the norm closure of $B(\ell_2) \otimes V$ in $B(\ell_2 \otimes \ell_2(G))$.

Remark: For C*-algebras,

Nuclearity $\Rightarrow$ CBAP $\Rightarrow$ strong OAP $\Rightarrow$ OAP $\Rightarrow$ Grothendieck AP.

Theorem [H-K 1994]: Let G be a discrete group. Then the following are equivalent:

1. G has the AP;

2. $C_\lambda^*(G)$ has the OAP;

3. $C_\lambda^*(G)$ has the strong OAP.

Remark: For a discrete group C*-algebra $C_\lambda^*(G)$, it is an open question whether Grothendick AP implies OAP.

It is also an open question whether there exists any discrete group such that $C_\lambda^*(G)$ does not have Grothendick's AP.

Proposition [J-R 2003]: A discrete group G has the AP if and only if $A(G)$ has the OAP (respectively, the strong OAP).

Proof of Theorem:

1. $\Rightarrow$ 2. Suppose that G has the AP and suppose that $\{u_\alpha\}$ is a net of elements in $A_c(G)$ such that

$$u_\alpha \to 1 \subseteq M_{cb}A(G)$$

in $\sigma(M_{cb}A(G), Q_{cb}(G))$-topology. In this case, each m_{u_α} is a finite rank cb-map on $A(G)$, and thus $\bar{M}_{u_\alpha}$ is a finite rank map on $C^*_\lambda(G)$ such that

$$\langle [\bar{M}_{u_\alpha}(a_{ij})] - [a_{ij}], [\varphi_{ij}] \rangle = \langle u_\alpha - 1, \omega_{a,\varphi} \rangle \to 0$$

for all $a = [a_{ij}] \in B(\ell_2)\check{\otimes} C^*_\lambda(G)$ and $\varphi = [\varphi_{ij}] \in (B(\ell_2)\check{\otimes} C^*_\lambda(G))^*$.

Therefore, $[\bar{M}_{\psi_\alpha}(a_{ij})] \to [a_{ij}]$ in the weak topology on $B(\ell_2)\check{\otimes} C^*_\lambda(G)$. Then by a convex argument, we can show that $C^*_\lambda(G)$ has the strong OAP.

3. $\Rightarrow$ 2. It is clear strong OAP implies OAP.

2. $\Rightarrow$ 1. We need to show that if $C^*_\lambda(G)$ has the OAP, then G has the AP.

Suppose that T_α is a net of bounded finite rank maps on $C^*_\lambda(G)$ such that $T_\alpha \to id$ in the stable-point-norm toplogy on $C^*_\lambda(G)$. Then

$$u_\alpha(s) = \langle \lambda_s^* T_\alpha(\lambda_s)\delta_e | \delta_e \rangle = \langle T_\alpha(\lambda_s)\delta_e | \lambda_s \delta_e \rangle$$

is a net of functions on G such that each u_α is contained in $\ell_2(G) \subseteq A(G)$ and we have

$$\bar{M}_{u_\alpha}(\lambda_s) = P(\iota \otimes T_\alpha) \circ \Gamma_G(\lambda_s) = P(\lambda_s \otimes T_\alpha(\lambda_s)) \to \lambda_s$$

in the stable-point-norm topology on $C^*_\lambda(G)$, where $P : C^*_\lambda(G \times G) \to \Gamma_G(C^*_\lambda(G))$ is a canonical c.c. projection. It follows that $u_\alpha \to 1$ in the $\sigma(M_{cb}A(G), Q_{cb}(G))$ topology, i.e. we have

$$\langle u_\alpha - 1, \omega_{a,\varphi} \rangle = \langle [\bar{M}_{u_\alpha}(a_{ij})] - [a_{ij}], [\varphi_{ij}] \rangle \to 0$$

for all $a = [a_{ij}] \in K(\ell_2) \check{\otimes} C^*_\lambda(G)$ and $\varphi = [\varphi_{ij}] \in (K(\ell_2) \check{\otimes} C^*_\lambda(G))^*$.