Lecture 4: Local Properties of Group C*-Algebras

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Local Property of Banach Spaces

It is known from the Hahn-Banach theorem that given any Banach space V, there exists an index set I such that we have the isometric inclusion

$$V \hookrightarrow \ell_\infty(I).$$

Usually I is an infinite index (even V is finite dimensional).

Question: If V is finite dimensinal, can we

“approximately embed” V into a finite dimensional $\ell_\infty(n)$

for some positive integer $n \in \mathbb{N}$?

Finite Representability in $\{\ell_\infty(n)\}$

Theorem: Let E be a f.d. Banach space. For any $\varepsilon > 0$, there exsit $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq \ell^\infty(n(\varepsilon))$ such that

$$E \overset{1+\varepsilon}{\cong} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\| \, \|T^{-1}\| < 1 + \varepsilon.$$

Therefore, we say that

- every f.d. Banach space E is representable in $\{\ell_\infty(n)\}$;
- every Banach space V is finitely representable in $\{\ell_\infty(n)\}$.

Proof: Since E^* is finite dim, the closed unit ball E_1^* is totally bounded. For arbitrary $1 > \varepsilon > 0$, there exists finitely many functionals $f_1, \cdots, f_n \in E_1^*$ such that for every $f \in E_1^*$, there exists some f_j such that

$$\|f - f_j\| < \frac{\varepsilon}{1+\varepsilon}.$$

Then we obtain a linear contraction

$$T : x \in E \to (f_1(x), \cdots, f_n(x_n)) \in \ell_n^\infty.$$

For any $f \in E_1^*$, we let f_j such that $\|f - f_j\| < \frac{\varepsilon}{1+\varepsilon}$. Then we get

$$\|T(x)\| \geq |f_j(x)| \geq |f(x)| - |f(x) - f_j(x)| \geq |f(x)| - \frac{\varepsilon\|x\|}{1+\varepsilon}.$$

This shows that

$$\|T(x)\| \geq \|x\| - \frac{\varepsilon\|x\|}{1+\varepsilon} = \frac{\|x\|}{1+\varepsilon}.$$

Therefore, $\|T^{-1}\| < 1 + \varepsilon$.

Finite Representatility of Operator Spaces in $\{M_n\}$

An operator space V is called finitely representable in $\{M_n\}$ if for every f.d. subspace E and $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq M_{n(\varepsilon)}$ such that

$$E \overset{1+\varepsilon}{\cong}_{cb} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon.$$

It is natural to ask whether every finite dim operator space is representable in $\{M_n\}$, or whether every operator space is finitely representable in $\{M_n\}$?

Theorem: Let $A \subseteq B(H)$ be a C*-algebra. Then A is finitely representable in $\{M_n\}$ if and only if there exists two nets of completely contractive maps

$$S_\alpha : A \to M_{n(\alpha)} \text{ and } T_\alpha : M_{n(\alpha)} \to B(H)$$

such that $\|T_\alpha \circ S_\alpha(x) - x\| \to 0$ for all $x \in A$.

Exact C*-algebras

We recall from Kirchberg that a C*-algebra A is an exact C*-algebra if we have the short exact sequence

$$0 \to K(\ell_2)\check{\otimes}A \hookrightarrow B(\ell_2)\check{\otimes}A \to Q(\ell_2)\check{\otimes}A \to 0,$$

where $Q(H) = B(\ell_2)/K(\ell_2)$.

Theorem [Kirchberg (Pisier) 1995]: A C*-algebra A is exact if and only if there exists two nets of completely positive and contractive maps (complete contractions)

$$S_\alpha : A \to M_{n(\alpha)} \text{ and } T_\alpha : M_{n(\alpha)} \to B(H)$$

such that $\|T_\alpha \circ S_\alpha(x) - x\| \to 0$ for all $x \in A$.

Therefore, A is finitely representable in $\{M_n\}$ iff A is exact.

Theorem (Pisier 1995): Let $\ell_1(n)$ be the operator dual of $\ell_\infty(n)$. If

$$T : \ell_1(n) \to F \subseteq M_k$$

is a linear isomorphism, then for $n \geq 3$

$$\|T\|_{cb}\|T^{-1}\|_{cb} \geq n/2\sqrt{n-1}.$$

Hence for $n \geq 3$,

$$\ell_1(n) \hookrightarrow C^*(\mathbb{F}_{n-1}) \subseteq B(H_\pi)$$

are note finitely represenetable in $\{M_n\}$.

So $C^*(\mathbb{F}_{n-1})$ and $B(H_\pi)$ are examples of non-exact C*-algebras.

Examples of Exact C^*-algebras

- For C*-algebras, we have

Nulcearity $\Rightarrow$ CBAP $\Rightarrow$ Strong OAP $\Rightarrow$ Exactness

- For any discrete group G, we have

Nuclearity $\Rightarrow$ Weakly Amenable $\Rightarrow$ AP $\Rightarrow$ Exact, i.e. $C^*_\lambda(G)$ is exact

- Groups like $G = \mathbb{F}_n, \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}), G = SL(3, \mathbb{Z})$ are exact.

Some Interesting Theorems

It is easy to see that if A is an exact C*-algebra, then any C*-subalgebra or subspace of A is also exact. Therefore, every C*-subalgebra of nuclear C*-algebra is exact.

Theorem [Kirchberg and Phillips 2000]: If A is a separable exact C*-algebra, then A is *-isomorphic to a C*-subalgebra of O_2.

How about group C*-algebras ?

Now let G be a discrete group. Then

$$\mathcal{UC}(G) = \overline{span\{f\lambda_s : f \in \ell_\infty(G), s \in G\}}^{\|\cdot\|} \subseteq B(\ell_2(G))$$

is a unital C*-algebra, which is called uniform algebra, or uniform Roe algebra. In fact, $\mathcal{UC}(G) = \ell_\infty(G) \rtimes G$.

The following theorem was first observed by Guentner and Kaminker, but was finally proved by Ozawa.

Theorem [Ozawa]: Let G be a discrete group. Then TFAE:

1. G is exact;
2. for any finite subset $E \subseteq G$ and $\varepsilon > 0$, there exists a subset $F \subseteq G$ and a positive definite kernel $u : G \times G \to \mathbb{C}$ such that
$$|u(s,t) - 1| < \varepsilon \text{ if } \; st^{-1} \in E \text{ and } u(s,t) = 0 \quad \text{if} \quad st^{-1} \notin F.$$
3. $\mathcal{UC}(G) = \ell_\infty(G) \rtimes C^*_\lambda(G)$ is nuclear.

Finite Representability in $\{\ell_n^1\}$

In Banach space theory it is known that a Banach space V is finitely represebtable in $\{\ell_n^1\}$ if and only

$$V \hookrightarrow L_1(\mu)$$

is isometric to a closed subspace of some $L_1(\mu)$ space.

Finite Representability in $\{T_n\}$

An operator space V is finitely representable in $\{T_n\}$ if for any f.d. subspace E and $\varepsilon > 0$, there exist $n(\varepsilon) \in \mathbb{N}$ and $F \subseteq T_{n(\varepsilon)}$ such that

$$E \overset{1+\varepsilon}{\cong}_{cb} F,$$

i.e., there exists a linear isomorphism $T : E \to F$ such that

$$\|T\|_{cb} \|T^{-1}\|_{cb} < 1 + \varepsilon.$$

- If A is a nuclear C*-algebra, then A^* and A^{***} are finitely representable in $\{T_n\}$. For example

$$C(X)^*, \quad T(\ell_2), \quad , B(\ell_2)^*.$$

- $C^*_\lambda(\mathbb{F}_2)^*$ is finitely representable in $\{T_n\}$.

Question: Is the predual M_* of a von Neumann algebra is finitely representable in $\{T_n\}$?

Theorem [E-J-R 2000]: Let M be a von Neumann algebra. Then M_* is finitely representable in $\{T_n\}$ if and only if M has the QWEP, i.e. M is a quotient of a C*-algebra with Lance's weak expectation proeprty.

A C*-algebra has the WEP if for the universal representation $\pi : A \to B(H)$, there exists a completely positive and contraction $P : B(H) \to A^{**}$ such that $P \circ \pi = id_A$.

A. Connes' conjecture 1976: Every finite von Neumann algebra with separable predual is $*$-isomorphic to a von Neumann subalgebra of the ultrapower of the hyperfinite II_1 factor

$$M \hookrightarrow \prod_{\mathcal{U}} R_0.$$

E. Kirchberg 's conjecture 1993: Every C^*-algebra has QWEP.

Residually Finite Groups

Let G be a discrete group. We say that G is residually finite if for any finitely many distinct elements $s_1, \cdots, s_n$ in G there exists a group homomorphism θ from G into a finite group H such that $\theta(s_1), \cdots, \theta(s_n)$ are distinct in H.

Theorem [Kirchberg 1993, Wassermann 1994]: If a discrete group G is residually finite, then G has property (F) and thus $L(G)$ has the QWEP.

More Examples

For the following groups G, $C^*_\lambda(G)$ are exact C*-algebras, i.e. finitely representable in $\{M_n\}$, and $A(G) = L(G)_*$ and $B_\lambda(G) = (C^*_\lambda(G))^*$ are finitely representable in $\{T_n\}$.

- For $n \geq 2$, $G = SL(n, \mathbb{Z})$ is residually finite since for any distinct $s_1, \cdots s_n$ in G, we can find a suffiently large prime numbers p such that the homomophism

$$\theta_p : SL(n, \mathbb{Z}) \to SL(n, \mathbb{Z}_p)$$

with disctinct image $\theta_p(s_1), \cdots, \theta_p(s_n)$ in finite group $SL(n, \mathbb{Z}_p)$.

- $G = \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z})$ is residually. We can consider

$$\theta_p : \mathbb{Z}^2 \rtimes SL(2, \mathbb{Z}) \to \mathbb{Z}_p^2 \rtimes SL(2, \mathbb{Z}_p).$$