Amenability of operator algebras on Banach spaces, II

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Let G be a finite subgroup of invertible elements of M_n such that $\operatorname{span} G = M_n$.
Set

$$\mathbf{d} := \frac{1}{|G|} \sum_{g \in G} g \otimes g^{-1}.$$

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Hence, $\mathcal{K}(E) = \mathcal{B}(E)$ is amenable.

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Theorem (N. Grønbæk & G. A. Willis, 1994)

Suppose that E has the approximation property. Then $\mathcal{K}(E)$ has a bounded approximate identity if and only if E^* has the bounded approximation property.

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is an algebra homomorphism.

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$$\sup_\lambda \max_{g \in G_\lambda} \|\theta_\lambda(g)\| < \infty.$$

Property ($\mathbb{A}$) and the amenability of $\mathcal{K}(E)$

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Suppose that E has property ($\mathbb{A}$). Then $\mathcal{K}(E)$ is amenable.

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Examples

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1. $L^p(\mu)$ has property $(\mathbb{A})$ for all $1 \leq p < \infty$ and all μ.

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Suppose that E^* has property $(\mathbb{A})$. Then so has E.

Corollary

There is an infinite-dimensional Banach space E such that $\mathcal{B}(E)$ is amenable.

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Theorem (G. A. Willis, unpublished)

Let $p, q \in (1, \infty)$ be such that $p \neq q$.

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, I

Theorem (G. A. Willis, unpublished)

Let $p, q \in (1, \infty)$ be such that $p \neq q$. Then $\mathcal{B}(\ell^p \oplus \ell^q)$ is not amenable.

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Ingredients

1. A quotient of an amenable Banach algebra is again amenable.
2. Every complemented closed ideal of an amenable Banach algebra is amenable.

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, I

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1. A quotient of an amenable Banach algebra is again amenable.
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1. A quotient of an amenable Banach algebra is again amenable.
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4. **Pitt's Theorem.**

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, I

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Let $p, q \in (1, \infty)$ be such that $p \neq q$. Then $\mathcal{B}(\ell^p \oplus \ell^q)$ is not amenable.

Ingredients

1. A quotient of an amenable Banach algebra is again amenable.
2. Every complemented closed ideal of an amenable Banach algebra is amenable.
3. Every amenable Banach algebra has a bounded approximate identity.
4. **Pitt's Theorem.** *If $p > q$, then $\mathcal{B}(\ell^p, \ell^q) = \mathcal{K}(\ell^p, \ell^q)$.*

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

Proof.

Non-amenability of $\mathcal{B}(\ell^p \oplus \ell^q)$ for $p \neq q$, II

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so that

$$\mathcal{C}(\ell^p \oplus \ell^q) = \begin{bmatrix} \mathcal{C}(\ell^p) & * \\ 0 & \mathcal{C}(\ell^q) \end{bmatrix}.$$

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Non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$

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Theorem (C. J. Read, <2006)

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Non-amenability of $\mathcal{B}(\ell^p)$ for $p = 1, 2, \infty$

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What if $\mathcal{B}(\ell^p)$ were amenable?

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Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$.

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Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then so are the Banach algebras $\ell^\infty(\mathcal{B}(\ell^p))$

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Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then so are the Banach algebras $\ell^\infty(\mathcal{B}(\ell^p))$ and $\ell^\infty(\mathcal{K}(\ell^p))$.

$\mathcal{L}^p$-spaces, I

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Definition (J. Lindenstrauss & A. Pełczyński, 1968)

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Examples

$\mathcal{L}^p$-spaces, I

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Examples

1. All Banach spaces isomorphic to an L^p-space are $\mathcal{L}^p$-spaces.

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Examples

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Examples

1. All Banach spaces isomorphic to an L^p-space are $\mathcal{L}^p$-spaces.
2. Let $p \in (1, \infty) \setminus \{2\}$. Then $\ell^p(\ell^2)$ and $\ell^2 \oplus \ell^p$ are $\mathcal{L}^p$-spaces,

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Examples

1. All Banach spaces isomorphic to an L^p-space are $\mathcal{L}^p$-spaces.
2. Let $p \in (1, \infty) \setminus \{2\}$. Then $\ell^p(\ell^2)$ and $\ell^2 \oplus \ell^p$ are $\mathcal{L}^p$-spaces, but not isomorphic to L^p-spaces.

$\mathcal{L}^p$-spaces, II

Amenability of operator algebras on Banach spaces, II

Volker Runde

Amenability of $\mathcal{K}(E)$

Amenability of $\mathcal{B}(E)$

A positive example

$\mathcal{B}(\ell^p \oplus \ell^q)$ with $p \neq p$

$\mathcal{B}(\ell^p)$

Theorem (M. Daws & VR, 2008)

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Let $p \in [1, \infty]$. Then one of the following is true:

$\mathcal{L}^p$-spaces, II

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Let $p \in [1, \infty]$. Then one of the following is true:

1. $\ell^\infty(\mathcal{K}(E))$ *is amenable for every* $\mathcal{L}^p$*-space* E *with* $\dim E = \infty$;

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Let $p \in [1, \infty]$. Then one of the following is true:

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Corollary

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Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$.

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Corollary

Suppose that $\mathcal{B}(\ell^p)$ is amenable for some $p \in [1, \infty)$. Then $\ell^\infty(\mathcal{K}(E))$ is amenable

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1. $\ell^\infty(\mathcal{K}(E))$ *is amenable for every* $\mathcal{L}^p$*-space* E *with* $\dim E = \infty$;
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Corollary

Suppose that $\mathcal{B}(\ell^p)$ *is amenable for some* $p \in [1, \infty)$. *Then* $\ell^\infty(\mathcal{K}(E))$ *is amenable for every* $\mathcal{L}^p$*-space* E *with* $\dim E = \infty$.

$\mathcal{L}^p$-spaces, II

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Question

Is $\ell^\infty(\mathcal{K}(\ell^2 \oplus \ell^p))$ amenable for any $p \in (1, \infty) \setminus \{2\}$?

Ozawa's proof revisited, I

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Definition

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A locally compact group G has Kazhdan's property (T)

Ozawa's proof revisited, I

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Ozawa's proof revisited, II

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Ozawa's proof revisited, III

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Ozawa's Lemma

It is impossible to find, for each $\epsilon > 0$, a number $r \in \mathbb{N}$ with the following property: for each $p \in \mathbb{P}$ there are $\xi_{1,p}, \eta_{1,p}, \ldots, \xi_{r,p}, \eta_{r,p} \in \ell^2(\Lambda_p)$ such that $\sum_{k=1}^r \xi_{k,p} \otimes \eta_{k,p} \neq 0$ and

$$\left\| \sum_{k=1}^r \xi_{j,p} \otimes \eta_{k,p} - (\pi_p(g_j) \otimes \pi_p(g_j))(\xi_{k,p} \otimes \eta_{k,p}) \right\|_{\ell^2(\Lambda_p)\hat{\otimes}\ell^2(\Lambda_p)} \leq \epsilon \left\| \sum_{k=1}^r \xi_{k,p} \otimes \eta_{k,p} \right\|_{\ell^2(\Lambda_p)\hat{\otimes}\ell^2(\Lambda_p)} \qquad (j = 1, \ldots, m+1).$$

Ozawa's proof revisited, IV

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(This estimate is no longer true for $p \in (1,\infty) \setminus \{2\}$.)

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, I

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A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, I

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It is easy to see that the following spaces satisfy the hypotheses of the theorem:

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It is easy to see that the following spaces satisfy the hypotheses of the theorem: c_0, ℓ^1, and ℓ^2.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, II

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Lemma

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A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, II

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and

$$\left\| \sum_{k=1}^{r} x a_k \otimes b_k - a_k \otimes b_k x \right\|_{\mathfrak{A}\hat{\otimes}\mathfrak{A}} < \epsilon \qquad (x \in F).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III

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Sketched proof of the Theorem

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Embed

$$\ell^\infty\text{-}\bigoplus_{p \in \mathbb{P}} \mathcal{B}(\ell^2(\Lambda_p))$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III

Sketched proof of the Theorem

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A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, III

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$$\ell^\infty\text{-}\bigoplus_{p\in\mathbb{P}} \mathcal{B}(\ell^2(\Lambda_p)) \subset \ell^\infty\text{-}\bigoplus_{p\in\mathbb{P}} \mathcal{K}(\ell^2 \oplus E) =: \mathfrak{A}$$

as "upper left corners". Let $\mathfrak{A}$ act on

$$\ell^2(\mathbb{P}, \ell^2 \oplus E) \cong \ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$,

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the p^{th} ℓ^2-summand

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the p^{th} ℓ^2-summand of

$$\ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the p^{th} ℓ^2-summand of

$$\ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$

Set $e = (P_p)_{p \in \mathbb{P}}$.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the p^{th} ℓ^2-summand of

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Set $e = (P_p)_{p \in \mathbb{P}}$. Then e is an idempotent in $\mathfrak{A}$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IV

Sketched proof of the Theorem (continued)

For $p \in \mathbb{P}$, let $P_p \in \mathcal{B}(\ell^2)$ be the canonical projection onto the first $|\Lambda_p|$ coordinates of the p^{th} ℓ^2-summand of

$$\ell^2(\mathbb{P}, \ell^2) \oplus \ell^2(\mathbb{P}, E).$$

Set $e = (P_p)_{p \in \mathbb{P}}$. Then e is an idempotent in $\mathfrak{A}$ with

$$e\mathfrak{A}e = \ell^\infty\text{-}\bigoplus_{p \in \mathbb{P}} \mathcal{B}(\ell^2(\Lambda_p)).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

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Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary. By the previous Lemma

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary. By the previous Lemma there are thus $a_1, b_1, \ldots, a_r, b_r \in \mathfrak{A}$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary. By the previous Lemma there are thus $a_1, b_1, \ldots, a_r, b_r \in \mathfrak{A}$ such that $\sum_{k=1}^r a_k b_k = e$

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

Let $\epsilon > 0$ be arbitrary. By the previous Lemma there are thus $a_1, b_1, \ldots, a_r, b_r \in \mathfrak{A}$ such that $\sum_{k=1}^r a_k b_k = e$ and

$$\left\| \sum_{k=1}^r x a_k \otimes b_k - a_k \otimes b_k x \right\| < \frac{\epsilon}{(C+1)(m+1)} \qquad (x \in F),$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, V

Sketched proof of the Theorem (continued)

Assume towards a contradiction that $\ell^\infty(\mathbb{P}, \mathcal{K}(\ell^2 \oplus E))$ is amenable.

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$$\left\| \sum_{k=1}^r x a_k \otimes b_k - a_k \otimes b_k x \right\| < \frac{\epsilon}{(C+1)(m+1)} \qquad (x \in F),$$

where

$$F := \{(\pi_p(g_j))_{p \in \mathbb{P}} : j = 1, \ldots, m+1\}.$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VI

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VI

Sketched proof of the Theorem (continued)

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VI

Sketched proof of the Theorem (continued)

For $p, q \in \mathbb{P}$ and $n \in \mathbb{N}$, define

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VI

Sketched proof of the Theorem (continued)

For $p, q \in \mathbb{P}$ and $n \in \mathbb{N}$, define

$$T_p(q,n) := \sum_{k=1}^{r} P_p a_k(e_q \otimes e_n) \otimes P_p^* b_k^*(e_q^* \otimes e_n^*) + P_p a_k(e_q \otimes x_n) \otimes P_p^* b_k^*(e_q^* \otimes x_n^*)$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VI

Sketched proof of the Theorem (continued)

For $p, q \in \mathbb{P}$ and $n \in \mathbb{N}$, define

$$\begin{aligned} T_p(q,n) := \sum_{k=1}^{r} & P_p a_k(e_q \otimes e_n) \otimes P_p^* b_k^*(e_q^* \otimes e_n^*) \\ & + P_p a_k(e_q \otimes x_n) \otimes P_p^* b_k^*(e_q^* \otimes x_n^*) \end{aligned}$$

Note that

$$T_p(q,n) \in \ell^2(\Lambda_p) \hat{\otimes} \ell^2(\Lambda_p).$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

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A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q,n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q,n) \| \leq \frac{\epsilon}{m+1} |\Lambda_p|$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q,n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q,n) \| \leq \frac{\epsilon}{m+1} |\Lambda_p|$$

for $j = 1, \ldots, m+1$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

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A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VII

Sketched proof of the Theorem (continued)

It follows that

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for $j = 1, \ldots, m+1$ and $p \in \mathbb{P}$ and thus

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \sum_{j=1}^{m+1} \| T_p(q,n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q,n) \| \leq \epsilon |\Lambda_p|.$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)

Sketched proof of the Theorem (continued)

On the other hand:

$$\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q,n) \|$$

$$\geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_{k,p} e_n, P_p^* b_{k,p}^* e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_{k,p} x_n, P_p^* b_{k,p}^* x_n^* \rangle \right|$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)

On the other hand:

$$
\begin{aligned}
&\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q,n) \| \\
&\geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_{k,p} e_n, P_p^* b_{k,p}^* e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_{k,p} x_n, P_p^* b_{k,p}^* x_n^* \rangle \right| \\
&= \operatorname{Tr} \sum_{k=1}^{r} b_{k,p} P_p a_{k,p}
\end{aligned}
$$

Sketched proof of the Theorem (continued)

On the other hand:

$$
\begin{aligned}
&\sum_{q\in\mathbb{P}} \sum_{n=1}^{\infty} \|T_p(q,n)\| \\
&\geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_{k,p} e_n, P_p^* b_{k,p}^* e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_{k,p} x_n, P_p^* b_{k,p}^* x_n^* \rangle \right| \\
&= \operatorname{Tr} \sum_{k=1}^{r} b_{k,p} P_p a_{k,p} \\
&= \operatorname{Tr} \sum_{k=1}^{r} P_p a_{k,p} b_{k,p}
\end{aligned}
$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, VIII

Sketched proof of the Theorem (continued)

On the other hand:

$$
\begin{aligned}
&\sum_{q \in \mathbb{P}} \sum_{n=1}^{\infty} \| T_p(q,n) \| \\
&\geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_{k,p} e_n, P_p^* b_{k,p}^* e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_{k,p} x_n, P_p^* b_{k,p}^* x_n^* \rangle \right| \\
&= \operatorname{Tr} \sum_{k=1}^{r} b_{k,p} P_p a_{k,p} \\
&= \operatorname{Tr} \sum_{k=1}^{r} P_p a_{k,p} b_{k,p} \\
&= \operatorname{Tr} P_p
\end{aligned}
$$

Sketched proof of the Theorem (continued)

On the other hand:

$$
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&\geq \sum_{n=1}^{\infty} \left| \sum_{k=1}^{r} \langle P_p a_{k,p} e_n, P_p^* b_{k,p}^* e_n^* \rangle + \sum_{k=1}^{r} \langle P_p a_{k,p} x_n, P_p^* b_{k,p}^* x_n^* \rangle \right| \\
&= \operatorname{Tr} \sum_{k=1}^{r} b_{k,p} P_p a_{k,p} \\
&= \operatorname{Tr} \sum_{k=1}^{r} P_p a_{k,p} b_{k,p} \\
&= \operatorname{Tr} P_p = |\Lambda_p|.
\end{aligned}
$$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that,

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

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It follows that, for each $p \in \mathbb{P}$,

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

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It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q, n) \neq 0$

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q,n) \neq 0$ and

$$\|T_p(q,n) - ((\pi_p(g_j) \otimes \pi_p(g_j)) T_p(q,n)\| \leq \epsilon \|T_p(q,n)\|$$

for $j = 1, \ldots, m+1$,

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q,n) \neq 0$ and

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for $j = 1, \ldots, m+1$, which violates Ozawa's Lemma.

A non-amenability result for $\ell^\infty(\mathcal{K}(\ell^2 \oplus E))$, IX

Sketched proof of the Theorem (conclusion)

It follows that, for each $p \in \mathbb{P}$, there are $q \in \mathbb{P}$ and $n \in \mathbb{N}$ with $T_p(q,n) \neq 0$ and

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p-summing operators

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Let $p \in [1, \infty)$,

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Let $p \in [1, \infty)$, and E and F be Banach spaces. A linear map $T : E \to F$ is called *p-summing* if the amplification $\mathrm{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$

p-summing operators

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Let $p \in [1, \infty)$, and E and F be Banach spaces. A linear map $T : E \to F$ is called *p-summing* if the amplification $\mathrm{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \check{\otimes} E$ to $\ell^p(F)$.

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p-summing operators

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Let $p \in [1, \infty)$, and E and F be Banach spaces. A linear map $T : E \to F$ is called p-summing if the amplification $\mathrm{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \check{\otimes} E$ to $\ell^p(F)$. The operator norm of $\mathrm{id}_{\ell^p \otimes T} : \ell^p \check{\otimes} E \to \ell^p(F)$ is called the p-summing norm of T and denoted by $\pi_p(T)$.

Theorem (Y. Gordon, 1969)

p-summing operators

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Let $p \in [1, \infty)$, and E and F be Banach spaces. A linear map $T : E \to F$ is called p-summing if the amplification $\mathrm{id}_{\ell^p} \otimes T : \ell^p \otimes E \to \ell^p \otimes F$ extends to a bounded map from $\ell^p \check{\otimes} E$ to $\ell^p(F)$. The operator norm of $\mathrm{id}_{\ell^p \otimes T} : \ell^p \check{\otimes} E \to \ell^p(F)$ is called the p-summing norm of T and denoted by $\pi_p(T)$.

Theorem (Y. Gordon, 1969)

$$\pi_p(\mathrm{id}_{\ell^2_N}) \sim N^{\frac{1}{2}}$$

p-summing operators

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Theorem (Y. Gordon, 1969)

$$\pi_p(\mathrm{id}_{\ell^2_N}) \sim N^{\frac{1}{2}}$$

for all $p \in [1, \infty)$.

A Lemma

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Let $p \in (1, \infty)$.

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Let $p \in (1, \infty)$. *Then there is* $C > 0$

A Lemma

Lemma

Let $p \in (1, \infty)$. Then there is $C > 0$ such that

$$\sum_{n=1}^{\infty} \|Se_n\|_{\ell^2_N} \|Te_n^*\|_{\ell^2_N} \leq C\, N \|S\| \|T\|$$

$$(N \in \mathbb{N},\ S \in \mathcal{B}(\ell^p, \ell^2_N),\ T \in \mathcal{B}(\ell^{p'}, \ell^2_N)).$$

Proof of the Lemma

Proof of the Lemma

Proof.

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Identify

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$$\mathcal{B}(\ell^p, \ell^2_N) = \ell^{p'} \check{\otimes} \ell^2_N$$

Proof of the Lemma

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Identify algebraically

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Note that

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Note that

$$\sum_{n=1}^{\infty} \|Se_n\|_{\ell^2_N} \|Te_n^*\|_{\ell^2_N} \leq \|S\|_{\ell^{p'}(\ell^2_N)} \|T\|_{\ell^p(\ell^2_N)}, \qquad \text{by Hölder,}$$

$\square$

Proof of the Lemma

Proof.

Identify algebraically

$$\mathcal{B}(\ell^p, \ell^2_N) = \ell^{p'} \check{\otimes} \ell^2_N = \ell^{p'} \otimes \ell^2_N = \ell^{p'}(\ell^2_N), \quad \text{and}$$
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Note that

$$\begin{aligned}\sum_{n=1}^{\infty} \|Se_n\|_{\ell^2_N} \|Te_n^*\|_{\ell^2_N} &\leq \|S\|_{\ell^{p'}(\ell^2_N)} \|T\|_{\ell^p(\ell^2_N)}, \qquad \text{by Hölder,} \\ &\leq \pi_{p'}(\mathrm{id}_{\ell^2_N}) \pi_p(\mathrm{id}_{\ell^2_N}) \|S\| \|T\|\end{aligned}$$

□

Proof of the Lemma

Proof.

Identify algebraically

$$\mathcal{B}(\ell^p, \ell^2_N) = \ell^{p'} \check{\otimes} \ell^2_N = \ell^{p'} \otimes \ell^2_N = \ell^{p'}(\ell^2_N), \quad \text{and}$$
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$$\begin{aligned}
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&\leq \pi_{p'}(\mathrm{id}_{\ell^2_N}) \pi_p(\mathrm{id}_{\ell^2_N}) \|S\| \|T\| \\
&\leq C\, N \|S\| \|T\|
\end{aligned}$$

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Identify algebraically

$$\mathcal{B}(\ell^p, \ell^2_N) = \ell^{p'} \check{\otimes} \ell^2_N = \ell^{p'} \otimes \ell^2_N = \ell^{p'}(\ell^2_N), \quad \text{and}$$
$$\mathcal{B}(\ell^{p'}, \ell^2_N) = \ell^{p} \check{\otimes} \ell^2_N = \ell^{p} \otimes \ell^2_N = \ell^{p}(\ell^2_N).$$

Note that

$$\begin{aligned}
\sum_{n=1}^{\infty} \|Se_n\|_{\ell^2_N} \|Te_n^*\|_{\ell^2_N} &\leq \|S\|_{\ell^{p'}(\ell^2_N)} \|T\|_{\ell^{p}(\ell^2_N)}, && \text{by Hölder,} \\
&\leq \pi_{p'}(\mathrm{id}_{\ell^2_N}) \pi_p(\mathrm{id}_{\ell^2_N}) \|S\| \|T\| \\
&\leq C\, N \|S\| \|T\|, && \text{by Gordon.} \quad \square
\end{aligned}$$

Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty)$

Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty)$

Corollary

Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1,\infty)$

Corollary

Let $p \in (1,\infty)$ and let E be an $\mathcal{L}^p$-space with $\dim E = \infty$.

Non-amenability of $\mathcal{B}(\ell^p)$ for $p \in (1, \infty)$

Corollary

Let $p \in (1, \infty)$ and let E be an $\mathcal{L}^p$-space with $\dim E = \infty$. Then $\ell^\infty(\mathcal{K}(E))$ is not amenable.

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Corollary

Let $p \in (1, \infty)$. Then $\mathcal{B}(\ell^p)$ and $\mathcal{B}(L^p[0, 1])$

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