Compact quantum subgroups and invariant C*-subalgebras. Introduction

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Outline of talks

1. Compact quantum subgroups and invariant C*-subalgebras. Introduction
2. Compact quantum subgroups of a co-amenable quantum group are co-amenable
3. Correspondence between compact quantum subgroups and invariant C*-subalgebras

C*-algebraic quantum groups

A (locally compact) quantum group is a C*-algebra A with

- co-multiplication Γ: a non-degenerate $*$-homomorphism $\Gamma\colon A \to \mathrm{M}(A \otimes A)$ such that

$$(\mathrm{id} \otimes \Gamma)\Gamma = (\Gamma \otimes \mathrm{id})\Gamma \qquad \text{(co-associativity)};$$

$$\overline{\mathrm{span}}\, \Gamma(A)(A \otimes 1) = \overline{\mathrm{span}}\, \Gamma(A)(1 \otimes A) = A \otimes A$$

- left Haar weight: a faithful, KMS-weight ϕ on A such that

$$\phi\big((\omega \otimes \mathrm{id})\Gamma(a)\big) = \omega(1)\phi(a) \quad (\omega \in A_+^*, a \in A_+, \phi(a) < \infty)$$

(left invariance); right Haar weight ψ on A.

Using the von Neumann algebraic definition with $\mathbb{G} = (\mathrm{L}^\infty(\mathbb{G}), \Gamma, \phi, \psi)$,

$$A = \overline{\{\, (\mathrm{id} \otimes \omega)W;\ \mathrm{B}(\mathrm{L}^2(\mathbb{G}))_* \,\}}^{\|\cdot\|}$$

where W is the left multiplicative unitary of $\mathbb{G}$.

Commutative case $\mathbb{G} = G$

Let G be a locally compact group.

- $A = \mathrm{C}_0(G)$
- For $f \in \mathrm{C}_0(G)$ and $s, t \in G$,

$$\Gamma(f)(s, t) = f(st).$$

  Note that $\Gamma(f) \in \mathrm{C}(G \times G) = \mathrm{M}(\mathrm{C}_0(G) \otimes \mathrm{C}_0(G))$.
- $\phi =$ integration w.r.t. the left Haar measure
- $\psi =$ integration w.r.t. the right Haar measure

Co-commutative case $\mathbb{G} = \widehat{G}$

Let λ be the *left regular representation* of G on $\mathrm{L}^2(G)$:

$$\lambda(s)\xi(t) = \xi(s^{-1}t) \qquad (\xi \in \mathrm{L}^2(G),\ s, t \in G).$$

Integration gives a representation of $\mathrm{L}^1(G)$:

$$\lambda(f) = \int_G f(s)\lambda(s)\, ds \qquad (f \in \mathrm{L}^1(G)).$$

- $A = \mathrm{C}^*_r(G) = \overline{\lambda(\mathrm{L}^1(G))}^{\text{norm}} \subseteq \mathrm{B}(\mathrm{L}^2(G))$
- The co-multiplication $\Gamma\colon \mathrm{C}^*_r(G) \to \mathrm{M}(\mathrm{C}^*_r(G) \otimes \mathrm{C}^*_r(G))$ is determined by

$$\Gamma(\lambda(s)) = \lambda(s) \otimes \lambda(s) \qquad (s \in G).$$

- $\phi = \psi$ is the Plancherel weight

Compactness

A quantum group (A, Γ) is compact if the C*-algebra A is unital.

A commutative quantum groups $C_0(G)$ is compact iff G is a compact group.

A co-commutative quantum group $C_r^*(G)$ is compact iff G is discrete.

Subgroups

Let H be a closed subgroup of a locally compact group G. Dualising $H \hookrightarrow G$ gives a surjection

$$\pi \colon \mathrm{C}_0(G) \to \mathrm{C}_0(H), \quad \pi(f) = f|_H \qquad (f \in \mathrm{C}_0(G)).$$

Now $\mathrm{C}_0(H)$ is a quantum group in its own right and

$$(\pi \otimes \pi)\Gamma_G = \Gamma_H \pi,$$

ie, $(\Gamma_G f)|_{H \times H} = \Gamma_H(f|_H)$ for every $f \in \mathrm{C}_0(G)$.

Definition of quantum subgroup

We say that a triple (B, Γ_B, π) is a quantum subgroup of (A, Γ_A) if (B, Γ_B) is a quantum group and

$$\pi \colon A \to B$$

is a surjective $*$-homomorphism such that

$$(\pi \otimes \pi)\Gamma_A = \Gamma_B \pi.$$

Examples in the co-commutative case

Let G be an amenable locally compact group and H be a closed normal subgroup of G. Define $Q\colon \mathrm{L}^1(G) \to \mathrm{L}^1(G/H)$ by

$$Qf(sH) = \int_H f(sh)\,dh \qquad (f \in \mathrm{L}^1(G),\ s \in G).$$

The map Q induces a surjective $*$-homomorphism $\rho\colon \mathrm{C}^*(G) \to \mathrm{C}^*(G/H)$.

The triple $(\mathrm{C}^*(G/H), \Gamma_{G/H}, \rho)$, where $\Gamma_{G/H}$ is the natural co-multiplication of $\mathrm{C}^*(G/H)$, is a quantum subgroup of $\mathrm{C}^*(G)$.

$U_q(2)$

Let $q \in [-1, 1]$, $q \neq 0$. $U_q(2)$ is the universal C*-algebra generated by elements a, c, v that satisfy

$$av = va \qquad cv = vc \qquad c^*c = cc^*$$
$$ac = qca \qquad ac^* = qc^*a \qquad v^*v = vv^* = 1$$
$$a^*a + c^*c = aa^* + q^2cc^* = 1.$$

The quantum group structure of $U_q(2)$ is determined by

$$\Gamma(a) = a \otimes a - qc^*v^* \otimes c$$
$$\Gamma(c) = c \otimes a + a^*v^* \otimes c \qquad \Gamma(v) = v \otimes v$$

Quantum subgroups of $U_q(2)$

The quantum group $U_q(2)$ may be represented as a twisted product $SU_q(2) \ltimes_\sigma U(1)$ [Wysoczański 04]

Quantum groups $SU_q(2) \ltimes_\sigma \mathbb{Z}_n$ are quantum subgroups of $U_q(2)$.

Also $\mathbb{T}^2$ is a quantum subgroup of $U_q(2)$: map $a \mapsto w_1$, $c \mapsto 0$, $v \mapsto w_2$ where w_1 and w_2 are commuting, unitary generators of $C(\mathbb{T}^2)$.

Taking in count the closed subgroups of $\mathbb{T}^2$, these are all the non-trivial quantum subgroups of $U_q(2)$ [Franz–Skalski–Tomatsu 09]

(Haar) idempotent states

A state μ of A is an idempotent state if

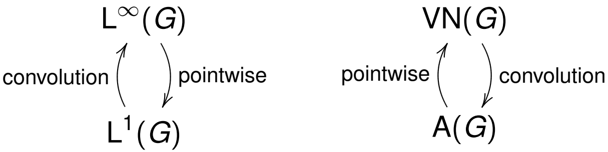
$$\mu * \mu \stackrel{\text{def}}{=} (\mu \otimes \mu)\Gamma = \mu.$$

If (B, Γ_B, π) is a compact quantum subgroup of A and h is the Haar state of B, then $h\pi$ is an idempotent state of A.

Unlike in the commutative case, not all idempotent states come from Haar states of compact quantum subgroups.

For more on idempotent states and their relation to subgroups see the work of Franz, Skalski, Tomatsu.

Takesaki–Tatsuuma ’71, Lau–Losert ’90



X is a submodule that is closed self-adjoint subalgebra

in $L^\infty(G)$	iff	$X = L^\infty(G/H)$,	H closed subgroup	[TT]
in $VN(G)$	iff	$X = VN(H)$,	H closed subgroup	[TT]
in $L^1(G)$	iff	$X = L^1(H)$,	H open subgroup	[TT]
in $A(G)$	iff	$X = A(G/H)$,	H compact subgroup	[TT]
in $C_0(G)$	iff	$X = C_0(G/H)$,	H compact subgroup	[LL]
in $C^*(G)$	iff	$X = C^*(H)$,	H open subgroup (G amenable)	

The result of Lau and Losert

Let G be a locally compact group. The left and the right translation of $f \in \mathrm{C}_0(G)$:

$$L_s f(t) = f(st), \qquad R_s f(t) = f(ts).$$

$X \subseteq \mathrm{C}_0(G)$ is left invariant if $L_s f \in X$ for every $f \in X$, $s \in G$.

Theorem (Lau–Losert '90)

There is a one-to-one correspondence between non-zero left invariant C-subalgebras X of $\mathrm{C}_0(G)$ and compact subgroups H of G:*

$$X = \mathrm{C}_0(G/H) = \{\, f \in \mathrm{C}_0(G);\ R_s f = f \quad \forall s \in H \,\},$$

$$H = \{\, s \in G;\ R_s f = f \quad \forall f \in X \,\}.$$

Moreover, H is normal iff X is also right invariant.

Related work

- de Leeuw and Mirkil (1960) considered abelian groups.
- The result of Lau and Losert gives a nice proof of the Kakutani–Kodaira theorem: every σ-compact group G has a compact normal subgroup H such that G/H is metrisable.
- Hu (2005) has shown even a more general version of Kakutani–Kodaira, using Lau–Losert.

Left invariant C*-subalgebras

To imitate the notion of left translation, define

$$L_\mu a = (\mu \otimes \mathrm{id})\Gamma(a) \qquad (\mu \in A^*, a \in A).$$

In the commutative case $A = C_0(G)$, L_{δ_s}, where δ_s is the point mass at s, is just the left translation operator L_s.
A subspace $X \subseteq A$ is said to be left invariant if X is invariant under operators L_μ, i.e.,

$$L_\mu x = (\mu \otimes \mathrm{id})\Gamma(x) \in X \qquad \text{for every } \mu \text{ in } A^* \text{ and } x \text{ in } X.$$

Action in the co-commutative case

Suppose that G is an *amenable* locally compact group.

$$A = \mathrm{C}_r^*(G) = \mathrm{C}^*(G)$$
$$A^* = \mathrm{B}(G) = \text{Fourier–Stieltjes algebra}$$
$$= \{ \langle \rho(\cdot)\xi | \zeta \rangle ;\ \rho \text{ representation of } G \}$$

$\mathrm{B}(G)$ acts on $\mathrm{C}^*(G)$:

$$ua := (u \otimes \mathrm{id})\Gamma(a) = (\mathrm{id} \otimes u)\Gamma(a) \quad (u \in \mathrm{B}(G), a \in \mathrm{C}^*(G)).$$

ua is just $L_u(a)$ from the definition of left invariance for general quantum groups.

For $f \in \mathrm{L}^1(G)$,

$$u\lambda(f) = \lambda(uf)$$

where uf is the pointwise product.

Dual version of Lau–Losert

Recall the notion of support (due to Eymard) for an element x in $\mathrm{C}^*(G)$:

$$s \in \operatorname{supp} x \iff \forall U \text{ nhood of } s, \exists u \in \mathrm{A}(G) \text{ with } \operatorname{supp} u \subseteq U \text{ such that } \langle x, u \rangle \neq 0.$$

Theorem

There is a one-to-one correspondence between non-zero invariant C-subalgebras X of $\mathrm{C}^*(G)$ and open subgroups H of G:*

$$X = \mathrm{C}^*(H) = \{ x \in \mathrm{C}^*(G); \operatorname{supp} x \subseteq H \},$$

$$H = \bigcup_{x \in X} \operatorname{supp} x.$$

Proof

$C^*(H)$ is easily seen invariant, and $\bigcup_{x \in C^*(H)} \operatorname{supp} x = H$.
Suppose that X is a non-zero, invariant C*-subalgebra of $C^*(G)$. Put

$$H = \{ s \in G;\ \lambda(s) \in X'' \}$$

and note that H is a closed subgroup of G. Define

$$Y = \{ x \in C^*(G);\ \operatorname{supp} x \subseteq H \}.$$

We show next that $X = Y$.
If $x \in X$ and $s \in \operatorname{supp} x$, then there is a net (u_α) in A(G) such that $u_\alpha x \to \lambda(s)$ in the weak* topology. Since X is invariant, each $u_\alpha x \in X$. It follows that $s \in H$, and so $\operatorname{supp} x \subseteq H$.
Therefore $X \subseteq Y$.

Proof continued

Let $y \in Y$. Since $\operatorname{supp} y \subseteq H$, $y \in \lambda(H)''$ by [Takesaki–Tatsuuma]. But $\lambda(H)'' \subseteq X''$, so $y \in X''$. Suppose that the support of y is compact.

Pick $u \in \mathrm{A}_c(G)$ such that $u = 1$ on a nhood of $\operatorname{supp} y$. Let $X \ni y_\alpha \to y$ weak*, so $uy_\alpha \to uy = y$. By invariance, $(uy_\alpha) \subseteq X$. Since $\operatorname{supp} u \subseteq K$ for some compact K, $\operatorname{supp} uy_\alpha \subseteq K$ for every α.

Let $v \in \mathrm{A}(G)$ such that $v = 1$ on a nhood of K. Since $\operatorname{supp} y, \operatorname{supp} uy_\alpha \subseteq K$, it follows that, for every w in $\mathrm{B}(G)$,

$$\langle uy_\alpha, w \rangle = \langle uy_\alpha, vw \rangle \to \langle y, vw \rangle = \langle y, w \rangle.$$

So $uy_\alpha \to y$ *weakly*. It follows that $y \in \overline{\mathrm{co}}\{uy_\alpha\}$. So $y \in X$.

The case of arbitrary $y \in Y$ reduces to the compactly supported case using a compactly supported b.a.i.

Proof continued

If H is not open, $|H| = 0$. Let $x \in X$ and $\epsilon > 0$. Choose $f \in C_c(G)$ such that $\|x - \lambda(f)\| < \epsilon$. There is open U such that

$$|U| < \epsilon/\|f\|_{\text{sup}} \quad \text{and} \quad \operatorname{supp} f \cap H \subseteq U.$$

Put $g = f 1_{G \setminus U}$. Then

$$\|x - \lambda(g)\| < \epsilon + \|\lambda(f) - \lambda(g)\| \leq \epsilon + \|f - g\|_1 \leq \epsilon + \|f\|_{\text{sup}} |U| < 2\epsilon.$$

So $x \approx \lambda(g)$ and $\operatorname{supp} g$ is separated from H.

Proof continued

Let

$$I(H) = \{ u \in \mathrm{A}(G);\ u = 0 \text{ on } H \}.$$

Now $I(H)^\perp = \lambda(H)'' = X''$. It follows from [Forrest–Kaniuth–Lau–Spronk] that there is a (completely) bounded projection $P\colon \mathrm{VN}(G) \to X''$ such that

$$P(ua) = uP(a) \qquad (u \in \mathrm{A}(G), a \in \mathrm{VN}(G)).$$

There is $u \in I(H)$ such that $u = 1$ on $\operatorname{supp} f \cap (G \setminus U)$. For every $v \in \mathrm{A}(G)$, $uv \in I(H)$, so

$$0 = \langle P(\lambda(g)), uv \rangle = \langle P(\lambda(ug)), v \rangle = \langle P(\lambda(g)), v \rangle.$$

Hence $P(\lambda(g)) = 0$, and so

$$\|x\| = \|P(x) - P(\lambda(g))\| \leq 2\|P\|\epsilon.$$

It follows that $x = 0$ and so $X = \{0\}$. We conclude that H is open. $\square$