Correspondence between compact quantum subgroups and invariant C*-subalgebras

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Outline

1. Compact quantum subgroups from left invariant C*-subalgebras
2. Left invariant C*-subalgebras from compact quantum subgroups
3. Correspondence

Set-up

Let (A, Γ) be a co-amenable quantum group, and let X be a non-zero, left invariant C*-subalgebra of A. So

$$(\mu \otimes \mathrm{id})\Gamma(x) \in X \qquad \text{for every } \mu \in A^*, x \in X.$$

Denote the state space of A by $\mathrm{S}(A)$. Define

$$F_0 = \{ \mu \in \mathrm{S}(A);\ (\mathrm{id} \otimes \mu)\Gamma(x) = x \text{ for every } x \in X \}$$
$$F = \operatorname{span} F_0$$

Lemma

The subspace F is a weak-closed subalgebra of A^*.*

Sketch of construction

If $F_\perp$ is a two-sided ideal, $B = A/F_\perp$ is a unital C*-algebra.
Then we can define a co-multiplication Γ_B on B by

$$\Gamma_B \pi = (\pi \otimes \pi)\Gamma$$

where $\pi\colon A \to B = A/F_\perp$ is the quotien map.

If X satisfies a further condition of co-normality, $F_\perp$ is an ideal.

Commutative and co-commutative cases

In the commutative case

$$A = \mathrm{C}_0(G) \quad \text{for locally compact group } G$$
$$X = \mathrm{C}_0(G/H) \quad \text{for compact subgroup } H \text{ of } G$$
$$F = \mathrm{M}(H)$$
$$X \text{ always co-normal}$$
$$A/F_\perp = \mathrm{C}(H)$$

In the co-commutative case

$$A = \mathrm{C}^*(G) \quad \text{for amenable locally compact group } G$$
$$X = \mathrm{C}^*(H) \quad \text{for open subgroup } H \text{ of } G$$
$$F = \mathrm{B}(G)\text{-functions constant on double cosets of } H$$
$$X \text{ co-normal} \quad \text{iff} \quad H \text{ normal}$$
$$A/F_\perp = \mathrm{C}^*(G/H) \quad \text{if } H \text{ normal}$$

Co-normality

We say that the left invariant C*-subalgebra X is co-normal if

$$W(x \otimes 1)W^* \in \mathrm{M}(X \otimes \mathrm{B}_0(H)) \qquad (x \in X).$$

- Vaes–Vainermann give a similar definition in the von Neumann algebraic context.
- In the commutative case, every left invariant C*-subalgebra is automatically co-normal.
- In the co-amenable, co-commutative case, an invariant C*-subalgebra $\mathrm{C}^*(H)$ of $\mathrm{C}^*(G)$ is co-normal if and only if the open subgroup H is normal.

Main technical lemma

Identify the C*-algebra A with its canonical image in the the universal enveloping von Neumann algebra A^{**}.

Lemma

Suppose that X is co-normal. For every $\mu \in F_0$ and $a \in A^{**}$ such that $\mu(a^*a) \neq 0$, the functional μ_a defined by

$$\mu_a(b) = \frac{\mu(a^*ba)}{\mu(a^*a)} \qquad (b \in A)$$

is in F_0.

Compact quantum subgroups from invariant C*-subalgebras

Theorem

Suppose that (A, Γ) *is a co-amenable quantum group and that* X *is non-zero, co-normal, left invariant C*-subalgebra of A. Then*

$$I = \{\, a \in A;\ \mu(a^* a) = 0 \textit{ for every } \mu \in F \,\}$$

is a closed two-sided ideal of A and coincides with

$$F_{\perp} = \{\, a \in A;\ \mu(a) = 0 \textit{ for every } \mu \in F \,\}.$$

The quotient map $\pi \colon A \to A/I$ *induces a quantum group structure on* $B = A/I$ *and* (B, Γ_B, π) *is a compact quantum subgroup of* (A, Γ)*.*

$I = F_\perp$ is two-sided ideal

Note that I is a closed left ideal and $F_\perp$ is self-adjoint. So if $I = F_\perp$ then I is a two-sided ideal.

$I \subseteq F_\perp$ follows from the Schwarz inequality:

$$\mu(a^* a) \geq \mu(a^*)\mu(a)$$

for $\mu \in \mathrm{S}(A)$ and $a \in A$. The converse can be shown by using the previous lemma.

A/I is a unital C*-algebra

Since $I = F_\perp$ is a two-sided ideal, $B = A/I$ is a C*-algebra and $B^* = (A/F_\perp)^* \cong F$.

Claim: B is unital.

Pick $x \in X \setminus \{0\}$ and let $\nu \in A^*$ such that $\nu(x) \neq 0$. Put

$$e = \frac{(\nu \otimes \mathrm{id})\Gamma(x)}{\nu(x)}.$$

Since $e \in X$ by left invariance,

$$\mu(ae) = \frac{\mu(a)\mu\big((\nu \otimes \mathrm{id})\Gamma(x)\big)}{\nu(x)} = \mu(a)$$

for every a in A and μ in F_0. (X is contained in the multiplicative domain of every μ in F_0.) Therefore $e + I$ is a right identity in A/I. The left side is similar.

$B = A/I$ is a compact quantum subgroup

Define

$$\Gamma_B \pi(a) = (\pi \otimes \pi)\Gamma(a) \qquad (a \in A).$$

The map Γ_B is well defined because $F \cong B^*$ is a subalgebra of A^*. Properties of Γ imply that Γ_B is a co-multiplication that satisfies the "cancellation laws"

$$\overline{\text{span}}\, \Gamma_B(B)(B \otimes 1) = \overline{\text{span}}\, \Gamma_B(B)(1 \otimes B) = B \otimes B.$$

Since B is unital, there is a state ϕ_B of B that is both left and right invariant.
It is not clear that ϕ_B is faithful, but the kernel of ϕ_B can be quotient out so that we obtain compact quantum subgroup B_r of A:

$$A \to B \to B_r.$$

Since A is co-amenable, so is B_r. Therefore, in fact, $B_r = B$. $\square$

Set-up for the converse

Let (B, Γ_B, π) be a compact quantum subgroup of a co-amenable quantum group (A, Γ).
Let $F = (\ker \pi)^\perp \subseteq A^*$ and $F_0 = F \cap \mathrm{S}(A)$. Then

$$X = \{\, x \in A;\ (\mathrm{id} \otimes \mu)\Gamma(x) = x \text{ for every } \mu \in F_0 \,\}.$$

is a closed self-adjoint subspace of A that is left invariant.
Let ϕ_B be the Haar state of B. Put $\theta = \phi_B \pi \in F_0$. Define $P\colon A \to A$ by

$$P(a) = (\mathrm{id} \otimes \theta)\Gamma(a) \qquad (a \in A).$$

Remark: θ is an idempotent state of A.

Invariant C*-subalgebras from compact quantum subgroups

Recall that a map from a C*-algebra onto its C*-subalgebra is called a conditional expectation if it is a projection of norm 1.

Theorem

Suppose that (B, Γ_B, π) is a compact quantum subgroup of a co-amenable quantum group (A, Γ).

- *The subspace X associated with B is a non-zero, co-normal, left invariant C*-subalgebra of A.*
- *$P\colon A \to X$ is a conditional expectation such that $(\mathrm{id} \otimes P)\Gamma = \Gamma P$.*

Uniqueness results

$$X \rightsquigarrow (B_X, \Gamma_X, \pi_X)$$
$$(B, \Gamma_B, \pi) \rightsquigarrow X_B$$

Theorem

If there is a conditional expectation $P\colon A \to X$ such that $(\mathrm{id} \otimes P)\Gamma = \Gamma P$, then $X_{B_X} = X$.

Theorem

$(B_{X_B}, \Gamma_{X_B}, \pi_{X_B}) \cong (B, \Gamma_B, \pi)$.

Corollary

$\{$*compact quantum subgroup of* $(A, \Gamma)\} \longleftrightarrow$
$\{$*non-zero, co-normal, left invariant C*-subalgebra X of A with conditional expectation $P\colon A \to X$ such that* $(\mathrm{id} \otimes P)\Gamma = \Gamma P\}$