Similarity for C*-algebras an introduction by a non-expert.

Thanks to Gilles Pisier, Erik Christensen, Stuart White and Roger Smith for discussions

The definition of length and all results are due to Gilles Pisier.

Leeds 19 May 2010

Groups

Dixmier and Day [1950 independently] showed that a bounded representation of an amenable group on a Hilbert space can be unitized.

A representation

$$\pi : G \to \big(\text{invertibles on } H\big)$$

is "strongly unitizable" if there is an invertible $T \in \big(\pi(G),\, \pi(G)^*\big)''$ such that $g \mapsto T\pi(g)T^{-1}$ is a unitary representation.

Theorem *[Pisier, Simultaneous similarity, bounded generation and length, Archive 2005]*
Every bounded representation of a discrete group $G \to$ Invertibles on H is strongly unitizable if, and only if, G is amenable.

Kadison similarity conjecture [1955]

Let $\mathcal{A}$ be a unital C*-algebra and let θ be unital bounded homomorphism from $\mathcal{A}$ into $B(H)$. Show that there is an invertible $T \in B(H)$ such that $x \mapsto T\theta(x)T^{-1}$ is a *-homomorphism.

There are results due to Christensen, Haagerup and others on C*-algebras and Paulsen [1984] on operator algebras and complete boundedness.

A unital operator algebra $\mathcal{A}$ has the *similarity property* if, and only if, each bounded homomorphism $\pi : \mathcal{A} \to B(H)$ is completely bounded.

Theorem *[Pisier, St Petersburg M J '99]* A unital operator algebra $\mathcal{A}$ has the similarity property if, and only if, it has finite length. The similarity degree and length are equal.

Gilles intuition on similarity and length: We call this [generation by diagonals and similarity] the "dual" view point because it is reminiscent of the fact that the closed convex hull C of a subset $B \subset E$ of a Banach space E is characterized by the implication

$$\sup_{b \in B} f(b) \leq 1 \implies \sup_{s \in C} f(s) \leq 1$$

for all continuous real linear forms f. Although this is a wild analogy, we feel that our results on length are a kind of "nonlinear" analog of the very classical duality principle of convex hulls.

All integer values of length are attained for general operator algebras [Pisier] but the only current known values for C*-algebras are 1, 2 and 3.

Allan's intuition on length: Every matrix over $\mathcal{A}$ can be factorized in a good metric way with the length of the factors tending to infinity by the Blecher-Paulsen Theorem or in a good algebraic way with length one; in general when the metric version is good, the algebraic one is poor and vice versa. Finite length encapsulates the opposing tensions of these two properties, metric/algebra, which lie at the core of operator algebras.

Idea

Scalar matrices and diagonal matrices over $\mathcal{A}$ are good.

Notation

$\mathcal{A}$ is subsequently a unital C*-algebra
$\mathbb{M}_{n,N} = n \times N$ matrices over $\mathbb{C}$
$\mathbb{M}_n = n \times n$ matrices over $\mathbb{C}$
$\mathbb{M}_n(\mathcal{A}) = n \times n$ matrices over $\mathcal{A}$
$\mathbb{D}_n(\mathcal{A}) = n \times n$ diagonal matrices over $\mathcal{A}$

If $(x_{ij}) \in \mathbb{M}_n(\mathcal{A})$, then $(x_{ij}) = VDW$, where

$$V = row_n\big(1\big) \otimes I_n$$
$$= \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 & \ddots & 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \dots & 0 & 1 & \cdots & 1 \end{pmatrix}$$
$$\in \mathbb{M}_{n,n^2},$$
$$W = \big(I_n \ I_n \ \cdots \ I_n\big)^{\mathbf{T}}$$
$$\in \mathbb{M}_{n^2,n} \quad \text{and}$$
$$D$$
$$= diag_{n^2}\big(x_{11}, x_{12}, \cdots, x_{1n}, x_{21}, x_{22}, \cdots, x_{nn}\big)$$
$$\in \mathbb{D}_{n^2}(\mathcal{A}).$$

This factorization is algebraically good, analytically poor as

$$\|V\|\|D\|\|W\| \leq n\|X\|.$$

If $d, n \in \mathbb{N}$, define $\|\cdot\|_{(d)}$ on $\mathbb{M}(\mathcal{A})$ by

$$
\begin{aligned}
\|X\|_{(d)} &= \inf\{\prod_{j=0}^{d} \|V_j\| \prod_{j=1}^{d} \|D_j\| : \\
&\quad \text{where } X = V_0 D_1 V_1 \cdots D_d V_d \text{ with} \\
&\quad V_0, V_d^* \in \mathbb{M}_{n,N} \\
&\quad V_j \in \mathbb{M}_N \ (1 \le j \le d-1) \quad \text{and} \\
&\quad D_j \in \mathbb{D}_N(\mathcal{A}) \ (1 \le j \le d)\}
\end{aligned}
$$

Lemma

(1) $\|\cdot\|_{(d)}$ is an operator space norm,

(2) $\|X\| \le \|X\|_{(d+1)} \le \|X\|_{(d)} \le \|X\|_{(1)} \le n\|X\|$,

(3) $\|XY\|_{(d+r)} \le \|X\|_{(d)} \|Y\|_{(r)}$

(4) $\|\cdot\|_{(1)} = \|\cdot\|_{\mathrm{MAX}}$ is the maximal operator space norm.

Theorem
[Blecher + Paulsen,PAMS, 1991]

If $\mathcal{A}$ is a unital operator algebra, then

$$\lim_{d\to\infty} \|X\|_{(d)} = \|X\|$$

for all $X \in \mathbb{M}_n(\mathcal{A})$ and all $n \in \mathbb{N}$.

Good analytically, poor algebraically.

Gilles Pisier's definition of length asks for efficiency both algebraically and analytically

Definition of length [Pisier, 1999]

The algebra $\mathcal{A}$ has length $\leq d$ if, and only if, there is a constant K such that $\|X\|_{(d)} \leq K\|X\|$ for all $X \in \mathbb{M}_n(\mathcal{A})$ and all $n \in \mathbb{N}$. The length $l(\mathcal{A})$ is the minimum of d such that $\mathcal{A}$ has length $\leq d$.

Length can be calculated via similarity and direct calculation of length.

Generally

Similarity calculation of degree(= length) $\leq$ Direct calculation of length

Definition

If $d, n \in \mathbb{N}$, let

$$K_{(d)}(n) = K_{(d)}(n, \mathcal{A})$$
$$= \sup\{\|X\|_{(d)} \colon X \in \mathbb{M}_n(\mathcal{A}), \|X\| \le 1\}.$$

If $K \ge 1$, let

$$N_{(d)}(n, K)$$
$$= \min\{N_0 \colon X \in \mathbb{M}_n(\mathcal{A}), \|X\|_{(d)} < K\|X\| \text{ with } N \le N_0 \text{ in factorization.}\}.$$

Then $1 \le K_{(d)}(n) \le n$.

Lemma (Pisier) If $\mathcal{A}$ is a unital C*-algebra and $p_1, p_2 \cdots p_n$ are projections in $\mathcal{A}$ with $\sum_1^n p_j = 1$, then
$\|\big(p_1, \cdots, p_n\big)\|_{(1)} = 1 = \|\big(p_1, \cdots, p_n\big)\|.$

Proof Here $row.row^* = 1$ gives the second equality. Let $W = \big(w_{ij}\big)$ be a unitary matrix in $\mathbb{M}_n$ with $|w_{ij}| = n^{-1/2}$ for $1 \leq i, j \leq n$. Let

$$V = \big(1, 1, \cdots, 1\big) \in \mathbb{M}_{1,n} \quad \text{and}$$
$$D = diag\Big(\sum_{j=1}^{n} \overline{w_{ji}} p_j\Big) \in \mathbb{D}(\mathcal{A}).$$

Then

$\big(p_1, \cdots, p_n\big) = VDW$ and
$\|V\| = n^{1/2},\ \|D\| = n^{-1/2},\ \|W\| = 1.$

Examples

1. $\mathcal{A} = \mathbb{C}^k = l_\infty^k$ has
$$(k/2)^{1/2} \leq K_{(1)} \leq (k-1)^{1/2}$$
using duality, Clifford algebras and $C_r^*(\mathbb{F}_{k-1})$. Here $K_{(2)} = 1$.

2. $\mathcal{A} = \mathbb{M}_k$ has
$$K_{(1)}(n) = \min\{n, k^{3/2}\}, \quad K_{(2)} \leq k$$
$$K_{(3)} \leq k^{1/2} \quad \text{and}$$
$$K_{(4)} = 1 \quad \text{with } N_{(4)}(n,1) \leq k^2 n.$$

3. $\mathcal{A} = \mathcal{M}$ is a II_1 factor with property Γ, then
$$3 \leq l(\mathcal{M}) \leq 5 \quad \text{with } K_{(5)} = 1 \qquad \text{[Pisier]}$$
$$l(\mathcal{M}) = 3 \quad \text{[Christensen].}$$

4. $\mathcal{A} = N$ is a properly infinite von Neumann algebra, then $l(N) = 3$, $K_{(3)} = 1$ and $N_{(3)}(n,1) = n$.

Corollary of [Pisier] and [Christensen, Smith, S] using Popa's constructive methods

Let $\mathcal{M}$ be separable II_1 factor with property Γ. There is a hyperfinite subfactor R in $\mathcal{M}$ such that each continuous R-bimodule map ϕ from $\mathcal{M}$ is completely bounded with $\|\phi\|_{cb} = \|\phi\|$.

Proposition [Pisier] A unital C*-algebra has length 1 if, and only if, it is finite dimensional.

Theorem [Pisier] A unital C*-algebra has length 2 if, and only if, $\mathcal{A}$ is amenable.

Theorem [Pisier] For each $d \in \mathbb{N}$ there is an operator algebra $\mathcal{A}$ with length d.

Theorem [Pisier] Every C*-algebra has finite length if, and only if, there are $d, K \in \mathbb{N}$ such that $K_{(d)}(n, \mathcal{A}) \leq K$ for all $n \in \mathbb{N}$ and all (unital) C*-algebras $\mathcal{A}$.

Pisier's conjecture

$$l^\infty\big(l^\infty(\mathbb{M}_k \colon k \in \mathbb{N})\big)$$

has infinite length.

Table of lengths of various algebras calculated by similarity or by length arguments.

? = currently calculable

?? = unknown

$\mathcal{S} =$ Estimate by similarity.

$\mathcal{L} =$ Estimate by length.

Algebra	Length	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{L}$	$\mathcal{L}$	$\mathcal{L}$
$\mathcal{A}$	$l(\mathcal{A})$	d	$K_{(d)}$	d	$K_{(d)}$	$N_{(d)}(n, K)$
Abelian $\mathbb{C}^k$	1	?	?	1	?	?
Matrix $\mathbb{M}_k$	1	?	?	1	$K_{(4)} = 1$	nk^2
Amenable $\mathcal{A}$	2	2	1 ?	??	??	??
I_∞,II_∞,III	3	3	1?	3	1	n
II_1 R	3	3	1?	4	1	∞
Γ-factor $\mathcal{M}$	3	3	?	5	1	n^2