# UNIVERSAL QUANTUM GROUPS ACTING ON QUANTUM AND CLASSICAL SPACES

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Abstract. Groups first entered mathematics in their geometric guise, as collections of all symmetries of a given object, be it a finite set, a polygon, a metric space or a manifold. Original definitions of quantum groups (also in the topological context) had rather algebraic character. In these lectures we describe several examples of quantum symmetry groups of a given quantum (or classical) space. The theory is based on the notion of actions of (compact) quantum groups on $C^*$-algebras and viewing symmetry groups as universal objects acting on a given structure. Such approach was suggested by Woronowicz already in the late 1970s and later developed by Wang, Banica, Bichon, Goswami, Sołtan and others.

## Plan of lectures

Lecture 1 **Actions of compact quantum groups**: definition of actions, notion of continuity/nondegeneracy, invariant states, ergodicity, categories of quantum groups or semigroups acting on a given $C^*$-algebra and preserving some additional structure ([Wor], [Wan], [Pod], [So$_1$]).

Lecture 2 **Quantum symmetry groups of finite structures**: quantum permutation groups, quantum symmetry groups of graphs, Wang and Van Daele's universal compact quantum groups ([Wan], [BBC], [Bic], [VDW], [So$_2$]).

Lecture 3 **Quantum isometry groups of noncommutative manifolds**: quantum isometry groups of spectral triples, quantum symmetry groups of Bratteli diagrams and quantum isometry groups associated to group $C^*$-algebras ([Gos], [BG$_1$], [BG$_2$] [BGS], [BS]).

## References

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