Linear Analysis I: Example Sheet 1

If you want your answers marked (or "commented upon") please hand them in at the end of the lecture on Monday 5th October. We will discuss the sheet in detail on Thursday 8th October. *This are preliminary dates! We will discuss as a group how to handle the dates.*

Some of the questions below are marked as **[Revision]**. I do not mean to be pejorative by this, but I cannot think of a better word. In an ideal world this would really be "revision" for everyone, but you have all done different optional courses, so I am unsure as to what you know.

In any event, please make sure that you *really can* do these questions. I will provide model answers after tutorials, and if everyone is having difficulty, I will make time in lectures to discuss the material.

Question 1: The following will get you to prove lots of "algebra of limits" type results for normed vector spaces. These get rather tedious after a while, but we will repeatedly use them in the course, so it is important that you are happy with them.

Let V be a vector space with norm $\|\cdot\|$. Prove the following:

1. If (x_n) is a sequence in V tending to x, and μ is a scalar, then $\mu x_n \to \mu x$;
2. If (x_n) and (y_n) are sequences in V tending to x and y, respectively, then $x_n + y_n \to x + y$;
3. If (x_n) is a sequence in V tending to x, then $\|x_n\| \to \|x\|$.

Question 2: Recall that $\mathbb{K}$ is either the field of real numbers $\mathbb{R}$, or the field of complex numbers $\mathbb{C}$. We define $C_{\mathbb{K}}([0,1])$ to be the vector space of continuous functions from $[0,1]$ to $\mathbb{K}$. For $f \in C_{\mathbb{K}}([0,1])$, define

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|.$$

Prove *carefully* that $\|\cdot\|_\infty$ is a norm on $C_{\mathbb{K}}([0,1])$.

Question 3: We define $\mathbb{K}^n$ to be the vector space of length n column vectors with entries in $\mathbb{K}$. If we think of n as being the set $\{0, 1, \cdots, n-1\}$, then we can think of $\mathbb{K}^n$ as being the collection of all maps from n to $\mathbb{K}$. With "pointwise" operations, this becomes a vector space.

Analogously, for an abstract set I, we define $\mathbb{K}^I$ to be the vector space of all maps from I to $\mathbb{K}$. In particular, we have the vector space $\mathbb{K}^{[0,1]}$, which contains $C_{\mathbb{K}}([0,1])$ as a subspace. Do you think that the definition

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)| \qquad (f \in \mathbb{K}^{[0,1]}),$$

makes sense? Think about this...

What happens if we define $f \in \mathbb{K}^{[0,1]}$ by

$$f(t) = \begin{cases} 0 & : t = 0, \\ 1/t & : 0 < t \le 1. \end{cases}$$

We define $\ell^\infty([0,1])$ to be the set of all functions $f \in \mathbb{K}^{[0,1]}$ such that f is bounded. We can then define $\|\cdot\|_\infty$ on $\ell^\infty([0,1])$ as above. Prove that $\|\cdot\|_\infty$ is a norm on $\ell^\infty([0,1])$.

Question 4: [Revision] Recall that we define the norm $\|\cdot\|_2$ on $\mathbb{K}^n$ by

$$\|x\|_2 = \Big(\sum_{i=1}^n |x_i|^2 \Big)^{1/2} \qquad (x = (x_i) \in \mathbb{K}^n).$$

This just gives the Euclidean distance, which is the usual way of measuring distance in $\mathbb{R}^2$ or $\mathbb{R}^3$, for example. Prove that $(\mathbb{K}^n, \|\cdot\|_2)$ is complete.

Hint: We know that $\mathbb{K}$ is complete. So we can show that $\mathbb{K}^2$ is complete by working "co-ordinate wise". Now generalise. Also compare with the proof in lectures that ℓ^p is complete (for $\mathbb{K}^n$ you don't have to worry about convergence, but the principle is similar).

Question 5: Let $\mathbb{K}[X]$ be the space of polynomials over $\mathbb{K}$. In lectures, we defined the norm $\|\cdot\|_1$ on $\mathbb{K}[X]$ as follows. Let $p \in \mathbb{K}[X]$, so that $p(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_1 X + a_0$, say. We define

$$\|p\|_1 = \sum_{i=0}^n |a_i|.$$

Do you think that $\mathbb{K}[X]$ with this norm is complete? If you want a challenge, think about this before looking at the next line.

For $n \geq 1$, let p_n be the polynomial

$$p_n(X) = \frac{1}{2^n} X^n + \frac{1}{2^{n-1}} X^{n-1} + \cdots + \frac{1}{4} X^2 + \frac{1}{2} X.$$

Show that (p_n) is a Cauchy sequence. Does (p_n) converge to a limit in $\mathbb{K}[X]$?

Question 6: We define c_0 to be the collection of sequences in $\mathbb{K}$ which converge to 0. Check that you are happy that c_0 is a vector space. We define the norm $\|\cdot\|_\infty$ on c_0 by

$$\|(x_n)\|_\infty = \sup_n |x_n| \qquad ((x_n) \in c_0).$$

Why does this make sense? Show that c_0 is complete. *Hint:* Adapt the proof from lectures that ℓ^p is complete. This is quite a hard question!

Question 7: [Revision] As normed spaces are metric spaces, and Banach spaces are complete metric spaces, you should have a good grasp of metric space theory. The following is very useful.

Let (X, d) be a metric space, and let $Y \subseteq X$ be a subset. The restriction of d to Y turns Y into a metric space in its own right. What does it mean for Y to be *closed* in X? What does it mean for Y to be *open* in X? If X is complete, show that Y is closed in X if and only if Y is complete.

Question 8: [Revision] In the course, abstract compact topological spaces will play an important role. This question is preparation, and asks you to think about compact metric spaces.

The following are two common definitions of what it means for a metric space (X, d) to be *compact*.

1. Let I be some set and let $(U_i)_{i \in I}$ be a collection of open subsets of X such that $X = \bigcup_{i \in I} U_i$ (this is to say that (U_i) is an *open cover of* X). Then X is compact if we can always find $i_1, \cdots, i_n$ such that $X = U_{i_1} \cup \cdots \cup U_{i_n}$ (this is to say that (U_i) admits a *finite subcover*).

2. Let $(x_n)_{n=1}^\infty$ be some sequence in X. Then X is compact if we can always find a subsequence $n(1) < n(2) < \cdots$ such that $(x_{n(k)})_{k=1}^\infty$ is convergent.

Condition 1 will make sense for any topological space, but condition 2 is usually easier to work with in metric spaces. If you fancy a challenge, prove that 1 and 2 are equivalent in a metric space.

For example, working in $[0,1]$, if we define

$$x_n = \begin{cases} 0 & : n \text{ is even}, \\ 1 & : n \text{ is odd}, \end{cases}$$

then (x_n) does not converge, but the subsequences (x_{2n}) and (x_{2n+1}) both converge to, respectively, 0 and 1. In fact, $[0,1]$ is compact. Working in $\mathbb{R}$, if we let $x_n = n$ for all n, then clearly x_n has no convergent subsequence, and so $\mathbb{R}$ is not compact.

If (X,d) is a metric space, we say that a subset $Y \subseteq X$ is compact if Y is compact for the metric inherited from X. Show that if Y is compact, then Y is closed in X.

The Bolzano–Weierstraß theorem states that if (x_n) is a bounded sequence of real numbers, then (x_n) has a convergent subsequence. Use this result to prove that a subset $Y \subseteq \mathbb{R}$ is compact (for the usual metric on $\mathbb{R}$) if and only if Y is closed and bounded.

The Heine–Borel theorem tells us that a subset $Y \subseteq (\mathbb{R}^n, \|\cdot\|_2)$ is compact if and only if Y is closed and bounded. Prove this. *Hint:* Work co-ordinate wise. If (x_n) is a bounded sequence in $(\mathbb{R}^n, \|\cdot\|_2)$, then the Bolzano–Weierstraß theorem allows us to pick a subsequence such that the first co-ordinate convergences. Then pick a further subsequence such that the second co-ordinate convergences, and so forth. Compare with question 4 above.

Prove the same result for $(\mathbb{C}^n, \|\cdot\|_2)$. *Hint:* Is there really any difference between $(\mathbb{C}^n, \|\cdot\|_2)$ and $(\mathbb{R}^{2n}, \|\cdot\|_2)$?

Question 9: We shall now apply these ideas. Let (X,d) be a metric space, and let $C_{\mathbb{K}}(X)$ be the vector space of all continuous functions from X to $\mathbb{K}$. Check that you are happy that this is a vector space.

[**Revision**] We say that $f \in C_{\mathbb{K}}(X)$ is *uniformly continuous* if for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in X$ satisfy $d(x,y) \leq \delta$, we have that $|f(x) - f(y)| \leq \epsilon$. Show that as X is compact, every $f \in C_{\mathbb{K}}(X)$ is uniformly continuous.

Let (f_n) be a sequence in $C_{\mathbb{K}}(X)$, and let $f \in C_{\mathbb{K}}(X)$. We say that f_n *converges uniformly* to f if for each $\epsilon > 0$, there exists N such that $|f_n(x) - f(x)| \leq \epsilon$ for any $x \in X$ and $n \geq N$. Alternatively, define $\|\cdot\|_\infty$ on $C_{\mathbb{K}}(X)$ by

$$\|f\|_\infty = \sup_{x \in X} |f(x)| \qquad (f \in C_{\mathbb{K}}(X)).$$

Check that you are happy that $\|\cdot\|_\infty$ is a norm. Then f_n converges uniformly to f if and only if $\|f_n - f\|_\infty \to 0$. Part of the power of Linear Analysis is in developing a common language that allows lots of diverse ideas from analysis to be expressed in a unified way. This is one example.

[**Revision**] Show that, as X is compact, we can equivalently define

$$\|f\|_\infty = \max_{x \in X} |f(x)| \qquad (f \in C_{\mathbb{K}}(X)).$$

That is, show that any $f \in C_{\mathbb{K}}(X)$ attains its supremum.

[**Hard**] Show that $C_{\mathbb{K}}(X)$ is a Banach space for the norm $\|\cdot\|_\infty$. We will, eventually, prove a generalised version of this result in lectures.

Linear Analysis I: Example Sheet 2

To be handed in at the lecture on ??? October, if you want comments on your answers.

The final two questions are quite similar: please try to do one, but if you run short of time, leave the other.

Question 1: Let E and F be normed vector spaces, and let $T : E \to F$ be a bounded linear map. The first line of the following is the original definition of the norm of T. Prove carefully that the other expressions really are equal:

$$\begin{aligned}\|T\| &= \sup\Big\{\frac{\|T(x)\|}{\|x\|} : x \in E, x \neq 0\Big\} \\ &= \sup\big\{\|T(x)\| : x \in E, \|x\| \leq 1\big\} \\ &= \sup\big\{\|T(x)\| : x \in E, \|x\| = 1\big\}.\end{aligned}$$

Furthermore, show carefully that $\|T(x)\| \leq \|T\|\|x\|$ for $x \in E$.

Question 2: Let E be a normed vector space, and let $\phi : E \to \mathbb{K}$ be a linear map. Remember that ϕ is bounded if and only if ϕ is continuous. When ϕ is bounded, show that

$$\ker \phi = \{x \in E : \phi(x) = 0\} = \phi^{-1}\big(\{0\}\big)$$

is closed.

Now suppose that ϕ is linear, and we know that $\ker \phi$ is closed in E. We shall show that ϕ is bounded. Firstly, if $\ker \phi = E$, show that ϕ is bounded.

Now suppose that $\ker \phi \neq E$. Let $x_0 \in E \setminus \ker \phi$. Show that every vector $x \in E$ can be written as

$$x = \lambda x_0 + y$$

for some $\lambda \in K$ and $y \in \ker \phi$. (*Hint:* Show that $x - \phi(x_0)^{-1}\phi(x)x_0 \in \ker \phi$). Suppose, towards a contradiction, that ϕ is not bounded, so we can find a sequence (x_n) in E with $\|x_n\| \leq 1$ and $|\phi(x_n)| \geq n$ for each n. By writing each $x_n = \lambda_n x_0 + y_n$ for some $\lambda_n \in \mathbb{K}$ and $y_n \in \ker \phi$, derive a contradiction.

We conclude that a linear map $\phi : E \to \mathbb{K}$ is bounded if and only if $\ker \phi$ is closed.

Question 3: Let E be a normed vector space, let $\phi \in E^*$, and let $\psi : E \to \mathbb{K}$ be a linear map. Show that if $\ker \phi \subseteq \ker \psi$, then $\psi = \lambda\phi$ for some $\lambda \in \mathbb{K}$, and hence in particular, $\psi \in E^*$. *Hint:* As in the previous question, if $\phi(x_0) \neq 0$ for some $x_0 \in E$, then we can express every $x \in E$ as $x = \mu x_0 + y$ for some $\mu \in \mathbb{K}$ and $y \in \ker \phi$. What, then, is $\psi(x)$?

Question 4: This question will show that we cannot take $\theta = 1$ in the statement of Lemma 1.17. Let $E = c_0$ and let F be the subspace of all sequences $(x_n) \in c_0$ such that $\sum_{n=1}^{\infty} 2^{-n}x_n = 0$. Consider the linear map

$$f : c_0 \to \mathbb{K}, \qquad f\big((x_n)\big) = \sum_{n=1}^{\infty} 2^{-n}x_n \qquad ((x_n) \in c_0).$$

Show that f is bounded with $\|f\| \leq 1$. Hence conclude that F is closed.

If Lemma 1.17 held with $\theta = 1$, then strictly speaking, we would ask for a vector $x_0 \in E$ with $\|x_0\| \leq 1$ and such that $\|x_0 - y\| > 1$ for each $y \in F$. Setting $y = 0$, we get a contradiction. Instead, we will show that even if we only ask that $\|x_0 - y\| \geq 1$ for each $y \in F$, we get a contradiction.

Towards a contradiction, suppose that we have $x_0 \in E$ with $\|x_0 - y\| \geq 1$ for each $y \in F$. Show that $f(x_0) = 1$, and then use this to argue that $\|x_0\| > 1$, giving a contradiction. *Hint:* For each $\epsilon > 0$, show that there exists $z \in E$ with $\|z\| \leq 1$ and $|f(z)| > 1 - \epsilon$. Write $z = \lambda x_0 + y$ for some $\lambda \in \mathbb{K}$ and $y \in F$, and hence show that $|f(x_0)| > 1 - \epsilon$.

Question 5: We work in the Banach space c_0. Define subspaces

$$Y = \big\{(x_n)_{n=1}^{\infty} \in c_0 : x_{2k-1} = 0 \text{ for } k = 1, 2, 3, \cdots \big\}$$
$$Z = \big\{(x_n)_{n=1}^{\infty} \in c_0 : x_{2k} = k^2 x_{2k-1} \text{ for } k = 1, 2, 3, \cdots \big\}.$$

Show that Y and Z are closed subspaces.

Show that the vector $x = (1, 0, 1/4, 0, 1/9, 0, 1/16, 0, \cdots)$ is in the closure of the subspace $Y + Z$. That is, for each $\epsilon > 0$, you need to find $y \in Y$ and $z \in Z$ with $\|x - (y + z)\|_\infty < \epsilon$.

Show, however, that x is not in $Y + Z$. This shows that $Y + Z$ is not closed: so the sum of two closed subspaces need not be closed itself.

Question 6: Show that $c_0^* = \ell^1$. That is, for $a = (a_n) \in \ell^1$, define $\phi_a : c_0 \to \mathbb{K}$ by

$$\phi_a(x) = \sum_{n=1}^{\infty} a_n x_n \qquad (x = (x_n) \in c_0).$$

Show that ϕ_a is linear, bounded, and that $\|\phi_a\| \leq \|a\|_1$. Hence the map $\ell^1 \to c_0^*; a \mapsto \phi_a$ is linear and bounded. We wish to show that this is a bijection and an isometry.

So, let $\phi \in c_0^*$. Considering the proof in lectures that $(\ell^p)^* = \ell^q$, show that there exists $a \in \ell^1$ with $\phi = \phi_a$ and $\|a\|_1 = \|\phi\|$.

Question 7: Recall that ℓ^∞ is the space of all bounded scalar sequences (x_n) with the norm $\|\cdot\|_\infty$. Show, using a similar argument to Question 6, that $(\ell^1)^* = \ell^\infty$.

Linear Analysis I: Example Sheet 3

Question 1: Let E be a normed vector space, let F be a subspace of E, and let $x_0 \in E$. Define

$$d(x_0, F) = \inf \big\{ \|x_0 - y\| : y \in F \big\}.$$

Show that

$$d(x_0, F) \geq \sup \big\{ |\phi(x_0)| : \phi \in E^*, \|\phi\| \leq 1, \phi(y) = 0 \,\forall\, y \in F \big\}.$$

Define a map $\psi : \text{lin}\{F, x_0\} \to \mathbb{K}$ by

$$\psi(\lambda x_0 + y) = \lambda d(x_0, F) \qquad (\lambda \in \mathbb{K}, y \in F).$$

Show that this is well-defined and linear. Show that $\|\psi\| \leq 1$. By applying the Hahn-Banach theorem, conclude that

$$d(x_0, F) = \sup \big\{ |\phi(x_0)| : \phi \in E^*, \|\phi\| \leq 1, \phi(y) = 0 \,\forall\, y \in F \big\}.$$

Question 2: Let $1 \leq p < \infty$, and define a map $S : \ell^p \to \ell^p$ by setting $S(x) = y$ where, if $x = (x_n)$ and $y = (y_n)$ then $y_1 = 0$ and $y_n = x_{n-1}$ for $n \geq 2$. That is, if $x = (x_1, x_2, x_3, \cdots)$, then $y = (0, x_1, x_2, x_3, \cdots)$. For obvious reasons, S is called the "right shift".

Show that S is linear, bounded, and satisfies $\|S\| = 1$.

Show that there is a bounded linear map $T \in \mathcal{B}(\ell^p)$ such that $T \circ S$ is the identity on ℓ^p. Is $S \circ T$ the identity? Is S invertible in $\mathcal{B}(\ell^p)$?

Question 3: Let X be a compact topological space. Fix $f \in C_{\mathbb{K}}(X)$, and define $M_f : C_{\mathbb{K}}(X) \to C_{\mathbb{K}}(X)$ by setting $M_f(g) = gf$ for $g \in C_{\mathbb{K}}(X)$, where gf is the (pointwise) product of f and g.

Show that $M_f \in \mathcal{B}(C_{\mathbb{K}}(X))$, and calculate $\|M_f\|$.

Question 4: Use the notation as in Question 3. Show that if

$$\inf \big\{ |f(x)| : x \in X \big\} > 0,$$

then there exists $h \in C_{\mathbb{K}}(X)$ with $M_h M_f = M_f M_h$ being the identity on $C_{\mathbb{K}}(X)$. *Hint:* Try $h = f^{-1}$.

If $\inf \big\{ |f(x)| : x \in X \big\} = 0$, then is M_f invertible? *Hint:* Let $1 \in C_{\mathbb{K}}(X)$ be the constant function which is one everywhere. If M_f^{-1} exists, what is $M_f^{-1}(1)$?

Question 5: Let E and F be normed spaces, and let $T \in \mathcal{B}(E, F)$. Show that the following are equivalent:

1. T is invertible, that is, there exists $S \in \mathcal{B}(F, E)$ with ST and TS being the identities on E and F, respectively;

2. T is surjective, and there exists $M > 0$ such that, for all $x \in E$,

$$M^{-1}\|x\| \leq \|T(x)\| \leq M\|x\|.$$

Hint: If (2) holds, first show that T must be bijective, so that T^{-1} exists as a linear map. Then show that T^{-1} must be bounded.

Question 6: We define a *measure space* to be a triple $(X, \mathcal{R}, \mu)$ where X is a set, $\mathcal{R}$ is a σ-algebra on X and μ is a measure defined on $\mathcal{R}$. Let $Y \in \mathcal{R}$, and define $\mathcal{R}_Y$ by

$$\mathcal{R}_Y = \{S \cap Y : S \in \mathcal{R}\}.$$

Show that $\mathcal{R}_Y$ is a σ-algebra on Y. Define $\mu_Y : \mathcal{R}_Y \to [0, \infty]$ by $\mu_Y(S) = \mu(S \cap Y)$ for $S \in \mathcal{R}_Y$ (so μ_Y is simply the restriction of μ to Y). Show that μ_Y is a measure on $\mathcal{R}_Y$. Hence $(Y, \mathcal{R}_Y, \mu_Y)$ is a measure space.

Question 7: Let $(X, \mathcal{R}, \mu)$ be a measure space. Let $N \subseteq X$ be a set such that, for some $F \in \mathcal{R}$ with $\mu(F) = 0$, we have that $N \subseteq F$. We say that N is a *null set*. Define $\overline{\mathcal{R}}$ to be the collection of sets $E \cup N$ where $E \in \mathcal{R}$, and $N \subseteq X$ is a null set. Show that:

1. If (N_n) is a sequence of null sets, then $\bigcup_n N_n$ is null.
2. If $E \cup N \in \overline{\mathcal{R}}$, and M is null, then $(E \cup N) \setminus M \in \overline{\mathcal{R}}$.

Show that $\overline{\mathcal{R}}$ is a σ-algebra.

Define $\overline{\mu} : \overline{\mathcal{R}} \to [0, \infty]$ by $\overline{\mu}(E \cup N) = \mu(E)$ for $E \in \mathcal{R}$ and any null set N. Show that $\overline{\mu}$ is a measure on $\overline{\mathcal{R}}$.

We say that a measure μ on $\mathcal{R}$ is *complete* when every null set is in $\mathcal{R}$. Check quickly that $\overline{\mu}$ is complete: it is the *completion* of μ. You might like to check that the measure we construct in Theorem 2.5 is complete.

Bonus questions

Let E be a normed space. Let $E^{**} = (E^*)^*$, the *bidual* of E. We define a map $J : E \to E^{**}$ as follows. For $x \in E$, we want that $J(x) \in (E^*)^*$, so $J(x)$ should be a map $E^* \to \mathbb{K}$. Let this be the map

$$J(x) : E^* \to \mathbb{K}, \quad f \mapsto f(x).$$

More compactly, we might write $J(x)(f) = f(x)$.

Question 8: Show that $J : E \to E^{**}$ is linear. Show that J is an isometry, so that $\|J(x)\| = \|x\|$ for $x \in E$.

When J is surjective, we say that E is *reflexive*. Notice that any reflexive space is Banach. To make matters complicated, James proved in the 1950s that there are spaces E such that there is an isomorphism from E to E^{**}, but such that J is not surjective. So, informally, E and E^{**} can "be the same" without being reflexive!

Question 9: Let $1 < p < \infty$. Show *carefully* that ℓ^p is reflexive. (Informally, as $(\ell^p)^* = \ell^q$ and $(\ell^q)^* = \ell^p$, we are done. BUT you must show that actually the map J is surjective).

Linear Analysis I: Example Sheet 4

Question 1: The following is a really useful characterisation of a Banach space. Firstly, let E be a Banach space, and let $(x_n)_{n=1}^\infty$ be a sequence of vectors in E such that $\sum_{n=1}^\infty \|x_n\| < \infty$. Show that $\sum_{n=1}^\infty x_n$ converges. That is, you need to show that the partial sums

$$\Big(\sum_{n=1}^N x_n\Big)_{N\in\mathbb{N}}$$

form a Cauchy-sequence in E.

We now show the converse. That is, if E is a normed space such that, whenever (x_n) is a sequence in E with $\sum_n \|x_n\| < \infty$, we have that $\sum_n x_n$ converges, then actually E is a Banach space. To show this, let (z_n) be a Cauchy sequence in E. Show that we can find $1 = n(1) < n(2) < \cdots$ such that, if

$$x_1 = z_1, \quad x_k = z_{n(k)} - z_{n(k-1)} \qquad (k \geq 2),$$

then $\sum_n \|x_n\| < \infty$. What is $\sum_{n=1}^N x_n$? Conclude that if $z = \sum_n x_n$ (which exists by assumption) that z is the limit of the Cauchy sequence (z_n). So E is a Banach space.

Informally, we say that "E is a Banach space if and only if every absolutely convergent sum converges."

Question 2: Let X be a compact (Hausdorff) space. Let $\phi : X \to X$ be a continuous map. Show that we can define a linear map $T : C_\mathbb{R}(X) \to C_\mathbb{R}(X)$ by

$$T(f) = g \quad \text{where} \quad g(x) = f(\phi(x)) \qquad (x \in X).$$

Show that T is bounded, and find $\|T\|$.

We call T a *composition operator (with symbol ϕ).*

Bonus Question 3: With notation as in Question 2, now let $X = [0,1]$ and let ϕ be defined by

$$\phi(t) = \frac{1}{2} + \frac{t - \frac{1}{2}}{2} \qquad (0 \leq t \leq 1).$$

So $\phi(1/2) = 1/2$, $\phi(0) = 1/4$ and $\phi(1) = 3/4$. It might help you to sketch the graph of ϕ. Define T as in Question 2. Let $T^2 = TT, T^3 = TTT$ and so forth.

Show that for each $f \in C_\mathbb{R}([0,1])$,

$$\lim_{n\to\infty} T^n(f) = g$$

where $g(t) = f(1/2)$ for all $t \in [0,1]$. That is, g is a constant function. *Hint:* Remember (or look up on Wikipedia) the proof of the contraction mapping theorem. What are the iterates of ϕ?

Is it true that $(T^n)_{n=1}^\infty$ converges in the Banach space $\mathcal{B}(C_\mathbb{R}([0,1]))$? *Hint:* If the limit exists, we know what it must be by the previous paragraph. Then estimate norms.

Question 4: Let $(X, \mathcal{R}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a simple function (see the definition from the lectures). Show carefully that f is measurable, and that f takes finitely many values.

Conversely, show that if $f : X \to \mathbb{R}$ is measurable and takes finitely many values, then f is a simple function.

In particular, show that if $(A_k)_{k=1}^n$ is any collection of subsets of $\mathcal{R}$, and $(t_k)_{k=1}^n \subseteq \mathbb{R}$, then

$$f = \sum_{k=1}^n t_k \chi_{A_k}$$

is simple (even if the (A_k) are not pairwise disjoint).

Question 5: Let X be a set, let $\mathcal{R} = 2^X$, and let μ be the *counting measure* on $\mathcal{R}$, so $\mu(A)$ is the size of A, if A is finite, and is ∞ otherwise. Which functions $f : X \to \mathbb{R}$ are measurable?

Let $f : X \to [0, \infty)$ be a simple function. Show that f is integrable if and only if f is zero except at finitely many points of X. Conversely, show that if $f : X \to [0, \infty)$ is any function which is zero except at finitely many points, then f is an integrable, simple function.

Question 6: Let $(X, \mathcal{R}, \mu)$ be a measure space. A function $f : X \to \mathbb{R}$ is *measurable* if $f^{-1}(U) \in \mathcal{R}$ for any open set $U \subseteq \mathbb{R}$. Let $f : X \to \mathbb{R}$ be a function such that $f^{-1}((x, y)) \in \mathcal{R}$ for any $x, y \in \mathbb{R}$ with $x < y$. By thinking about the proof of Corollary 2.7, show that f is measurable.

Question 7: We work with notation as in Question 5. Which measurable functions $f : X \to [0, \infty)$ are integrable? What about functions $f : X \to \mathbb{R}$? You might find it easier to assume that $X = \mathbb{N}$ here.

Show that if $X = \mathbb{N}$, then we can identify ℓ^1 with the space of integrable functions $f : X \to \mathbb{R}$.

Bonus Question: Let $(X, \mathcal{R}, \mu)$ be a measure space. Let $f, g : X \to \mathbb{R}$ be measurable. Show that $f + g$ is measurable. To do this, you might like to first prove that

$$(f + g)^{-1}((a, \infty)) = \bigcup_{q \in \mathbb{Q}} \{x \in X : q < f(x) \text{ and } a - q < g(x)\}.$$

Use this to show that $(f+g)^{-1}((a, \infty)) \in \mathcal{R}$. Similarly, show that $(f+g)^{-1}((-\infty, a)) \in \mathcal{R}$. Use this to show that $(f + g)^{-1}((a, b)) \in \mathcal{R}$ for any open interval (a, b). Now follows the proof of Corollary 2.7 to show that $f + g$ is measurable.

Show that $\{x \in X : f(x) \geq g(x)\} \in \mathcal{R}$. *Hint:* It might help to show first that

$$\{x \in X : f(x) \geq g(x)\} = \bigcap_{q \in \mathbb{Q}, q > 0} \bigcup_{r \in \mathbb{Q}} \{x \in X : f(x) > r > g(x) - q\}.$$

Prove that also fg is measurable. This is a bit of a beast!

Linear Analysis I: Example Sheet 5

Question 1: Let (a_n) be a sequence in $\mathbb{R}$. Recall that (or define, if you have haven't seen these before)

$$\limsup_n a_n = \lim_{n\to\infty}\Big(\sup_{k\geq n} a_k\Big), \qquad \liminf_n a_n = \lim_{n\to\infty}\Big(\inf_{k\geq n} a_k\Big).$$

As $(\sup_{k\geq n} a_k)$ is a decreasing sequence in n, the limit exists (or is $-\infty$) and so $\limsup_n$ is well-defined. Similarly for $\liminf$.

Suppose that (a_n) converges. Prove that

$$\lim_n a_n = \limsup_n a_n = \liminf_n a_n.$$

Let (a_n) be any sequence of positive reals. Show that

$$\liminf_n a_n \leq \limsup_n a_n,$$

where these may be $\pm\infty$. Finally, show that if

$$\liminf_n a_n = \limsup_n a_n,$$

then (a_n) converges.

Question 2: Use the monotone convergence theorem to evaluate $\int_{\mathbb{R}} f(x)\ d\mu(x)$ for the following. You may assume that the Lebesgue integral of a continuous function on a bounded interval is the same as the Riemann integral, and use any standard results from Calculus to evaluate integrals.

1. $f(x) = e^{-|x|}$. *Hint:* Consider the sequence of functions $f_n(x) = e^{-|x|}\chi_{[-n,n]}$.
2. $f(x) = x^{-1/2}\chi_{(0,1]}$. *Hint:* Consider the sequence $f_n(x) = x^{-1/2}\chi_{[1/n,1]}$.

Similarly, establish that the following have finite integral (although you will find it difficult to evaluate to an exact value).

1. $f(x) = e^{-x^2}$. *Hint:* Consider $f_n(x) = e^{-x^2}\chi_{[-n,n]}$, and crudely estimate this.
2. $f(x) = x^{-2}\sin(x)\chi_{[\pi,\infty)}$. *Hint:* Consider $f_n(x) = x^{-2}\sin(x)\chi_{[\pi,\pi+n]}$, and also the functions $g_n(x) = x^{-2}\chi_{[\pi,\pi+n]}$. Be careful here, as f is not positive, so you need to deal with f_+ and f_-.

Finally, show that the following are not Lebesgue integrable (that is, they have infinite integrals).

1. $f(x) = x^{-1}\chi_{[1,\infty)}$.
2. $f(x) = \log(x)\chi_{[1,\infty)}$.

Question 3: Recall that $f(x) = \sin(x)/x$ is a continuous functions $\mathbb{R}\to\mathbb{R}$, as by L'Hopital's rule, $f(x)\to 1$ as $x\to 0$. The Riemann integral is defined as

$$\int_{-\infty}^{\infty} f(x)\ dx := \lim_{X\to\infty}\int_{-X}^{X} f(x)\ dx.$$

It is a result from complex analysis (using contour integration) that

$$\int_{-\infty}^{\infty} f(x)\ dx = \pi/2.$$

However, we saw in lectures that f is *not* Lebesgue integrable. This is because if f_+ and f_- are, respectively, the positive and negative parts of f, then f_+ and/or f_- do not have finite integral. Carefully prove this. *Hint:* We know that, say, for $x \geq 0$, we have that $\sin(x)/x \geq 0$ if and only if $\sin(x) \geq 0$, which is if and only if $2n\pi \leq x \leq (2n+1)\pi$, for some $n \in \mathbb{Z}$. So we can write down f_+ and f_- explicitly, and then we can use, say, the monotone convergence theorem to work out the integral of f_+ and f_-.

Question 4: For each n, let $f_n(x) = n^{3/2}x(1+n^2x^2)^{-1}$ for $x \in [0,1]$. By using the Dominated Convergence Theorem, find

$$\lim_n \int_0^1 f_n(x)\ dx.$$

Hint: To get something which bounds all the f_n, either guess, or try $g(x) = \sup_n f_n(x)$, and show that g has finite integral.

Question 5: Use the Dominated Convergence Theorem to show that $f : [0,4] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & : x = 0, \\ x^{-1/2}\sin(1/x) & : 0 < x \leq 4, \end{cases}$$

is integrable. *Hint:* Set $f_n(x) = x^{-1/2}\sin(1/x)\chi_{(1/n,4]}$, so $f_n \to f$ pointwise.

Question 6: Define $f_n : [0,1] \to [0,\infty)$ by

$$f_n(x) = \begin{cases} n & : 0 \leq x \leq 1/n, \\ 0 & : x > 1/n. \end{cases}$$

Show that $f_n(x) \to 0$ almost everywhere, but that

$$\int_0^1 f_n\ d\mu = 1,$$

for all n. Why can we not apply either the Monotone or the Dominated Convergence Theorems in this case?

Question 7: Let $(X, \mathcal{R}, \mu)$ be a measure space, and let $Y \in \mathcal{R}$. On a previous example sheet, we saw how to define the sub-measure space $(Y, \mathcal{R}_Y, \mu_Y)$. Let $f : X \to \mathbb{R}$ be measurable, and let f_Y be the restriction of f to Y. Show that f_Y is measurable with respect to $\mathcal{R}_Y$. Show that $f\chi_Y : X \to \mathbb{R}$ is measurable with respect to $\mathcal{R}$. Show that

$$\int_Y f_Y\ d\mu_Y = \int_X f\chi_Y\ d\mu.$$

(Hint: Check first for simple functions, and then use MCT.) Hence integrating with respect to a sub-measure space, or just multiplying by the characteristic function of a measurable subset, gives the same answer.

Linear Analysis I: Example Sheet 6

Question 1: Let $(X, \mathcal{R}, \mu)$ be a measure space. Let $f : X \to [0, \infty)$ be measurable. For each $A \in \mathcal{R}$, the indicator function χ_A is measurable, and so $f\chi_A$ is also measurable. So we can define a map $\mu_f : X \to [0, \infty]$ by

$$\mu_f(A) = \int_X f\chi_A \, d\mu \qquad (A \in \mathcal{R}).$$

The integral might be infinite, but as $f\chi_A \geq 0$, it is well-defined.

Show that μ_f is a measure: you need to prove countable additivity, for which the monotone convergence theorem should prove useful. That is, if (A_n) is a sequence of pairwise disjoint sets in $\mathcal{R}$, then show that $\mu_f(\bigcup_n A_n) = \sum_n \mu_f(A_n)$. *Hint:* Use $f_n = f\chi_{A_1 \cup \cdots \cup A_n}$.

Furthermore, show that if g is a simple function, then

$$\int_X g \, d\mu_f = \int_X gf \, d\mu.$$

Conclude (using Monotone convergence) that this holds for any integrable function $g : X \to \mathbb{R}$.

Later in the course, we will give an exact condition for when a measure on X can be written as μ_f.

Question 2: Let $(X, \mathcal{R}, \mu)$ be a measure space. A function $f : X \to \mathbb{R}$ is *essentially bounded* if there exists $K > 0$ such that $|f| \leq K$ almost everywhere. The inf of such K is called the *essential supremum* of f, and is denoted by

$$\text{ess-sup}_{x \in X} |f(x)| \quad \text{or simply} \quad \text{ess-sup}_X |f|.$$

Let f be essentially bounded, and suppose that $g : X \to \mathbb{R}$ is measurable and integrable. Show that fg is integrable, and that

$$\int_X |fg| \, d\mu \leq \Big(\text{ess-sup}_X |f|\Big) \int_X |g| \, d\mu.$$

Hint: If you have trouble dealing with the "almost everywhere" part, try first to do the question assuming that f is actually bounded.

Question 3: We define Lebesgue measure on $\mathbb{R}^3$ by identifying $\mathbb{R}^3$ with $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The *volume* of a measurable set $A \subseteq \mathbb{R}^3$ is then simply the integral of the characteristic function of A. Carefully apply Fubini's Theorem to find the volumes of the sets:

1. $\{(x, y, z) : 0 \leq z \leq 2 - x^2 - y^2\}$.
2. $\{(x, y, z) : x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.

Note: This is very computational, and maybe you won't learn much, so skip it if you're short of time.

Question 4: Let $(X, \mathcal{R}, \mu)$ and $(Y, \mathcal{S}, \lambda)$ be finite measure spaces. Let $f : X \to \mathbb{R}$ be $\mathcal{R}$-measurable, and let $g : Y \to \mathbb{R}$ be $\mathcal{S}$-measurable. Let $h : X \times Y \to \mathbb{R}$ be defined by $h(x, y) = f(x)g(y)$. You might like to think about why h is automatically $(\mathcal{R} \otimes \mathcal{S})$-measurable, but I think a formal proof of this is quite hard!

Suppose that f and g are integrable with respect to μ and λ, respectively. Use Fubini to show that

$$\int_{X \times Y} h \, d(\mu \otimes \lambda) = \int_X f \, d\mu \int_Y g \, d\lambda.$$

Question 5: Define $f : [0,1]^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{(x^2+y^2)^2} & : (x,y) \neq (0,0), \\ 0 & : \text{otherwise}. \end{cases}$$

Show by calculation that

$$\int_0^1 \int_0^1 f(x,y)\ dx\ dy \neq \int_0^1 \int_0^1 f(x,y)\ dy\ dx.$$

Why can we not apply Fubini's Theorem in this case?

Question 6: Let $(X, \mathcal{R}, \mu)$ and $(Y, \mathcal{S}, \lambda)$ be finite measure spaces, and let $E \in \mathcal{R} \otimes \mathcal{S}$. For each $x \in X$, let $E_x = \{y \in Y : (x,y) \in E\}$, a *cross-section* of E. Show that the following are equivalent:

1. $(\mu \otimes \lambda)(E) = 0$;
2. $\lambda(E_x) = 0$ for almost all x with respect to μ (that is, $\mu(\{x \in X : \lambda(E_x) \neq 0\}) = 0$).

Hint: The measure of E is simply the integral of the indicator function of E. So carefully apply Fubini.

Question 7: Let X be a set, and let $\mathcal{R}$ be a σ-algebra on X. For $x \in X$, define a map $\delta_x : \mathcal{R} \to [0, \infty)$ by

$$\delta_x(A) = \begin{cases} 1 & : x \in A, \\ 0 & : x \notin A. \end{cases}$$

Show that δ_x is a measure. It is often called the "Dirac Delta Measure at x". Why might it get this name?

Determine the completion of δ_x (that is, what are the null sets for δ_x?)

For a measurable function $f : X \to [0, \infty)$, what is $\int_X f\ d\delta_x$? Which functions $f : X \to \mathbb{R}$ are integrable for δ_x?

Such measures behave very differently from Lebesgue measure.

Question 8: Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable set with finite Lebesgue measure. Show that for $\epsilon > 0$, we can find an open set U with $A \subseteq U$ and $\mu(U) < \mu(A) + \epsilon$. *Hint:* Think about the definition of the Lebesgue measure and its relationship to Lebesgue outer measure.

Show that for $\epsilon > 0$, we can find a compact (that is, closed and bounded) set K with $K \subseteq A$ and $\mu(K) > \mu(A) - \epsilon$. *Hint:* Suppose first that A is bounded, say $A \subseteq [-n, n]$ for some n. Then let $A_0 = [-n, n] \setminus A$, so by the first part of the question, we can find an open U_0 with $A_0 \subseteq U_0$ and $\mu(U_0) < \mu(A_0) + \epsilon$. Then $K = [-n, n] \setminus U_0$ will work: why? Now try to handle the general case (if A has finite measure, show that $\mu(A) = \lim_n \mu(A \cap [-n, n])$).

Conclude that

$$\sup\{\mu(K) : K \subseteq A \text{ is compact }\} = \mu(A) = \inf\{\mu(U) : A \subseteq U \text{ is open }\}.$$

This shows that μ is a *regular* measure. We will learn more about regular measures later in the course.

Linear Analysis I: Example Sheet 7

Question 1: Consider the set $\mathbb{N}$ together with the trivial σ-algebra consisting of all subsets of $\mathbb{N}$. Let (ω_n) be a sequence of positive reals (that is, $\omega_n \geq 0$ for each n), with $(\omega_n) \in \ell^1$. Show that we may define a measure μ by

$$\mu(A) = \sum_{n \in A} \omega_n \qquad (A \subseteq \mathbb{N}).$$

We think of this as a "weighted" counting measure. What are the null sets for this measure?

Question 2: This follows on from Question 1. Determine when a function $f : \mathbb{N} \to \mathbb{C}$ is in $L^p(\mu)$. Describe, briefly, the space $\mathcal{L}^p(\mu)$. *Comment:* This might take you a long time if you are 100% rigourous. I'm more interested in seeing that you understand what's going on than in having everything perfect!

Question 3: Let $(X, \mathcal{R}, \mu)$ be a *finite* measure space. Show that if $1 \leq p < r < \infty$, then $L^r(\mu) \subseteq L^p(\mu)$. *Hint:* Given a function $f \in L^r(\mu)$, write

$$f = f\chi_{\{x:|f(x)|\leq 1\}} + f\chi_{\{x:|f(x)|>1\}},$$

then think about whether these two functions are in $L^p(\mu)$. *Alternative:* Try to use the Holder inequality!

Question 4: By considering $\mathbb{R}$ with Lebesgue measure, or otherwise, show that the conclusions of Question 3 no longer hold if we are not working with a finite measure space. *Hint:* Consider first the case when $p = 1$, so you want to find a function $f \in L^r(\mu)$ such that $\int_{\mathbb{R}} |f| \, d\mu = \infty$. A good place to look is at functions like $f(x) = x^{-1}$. Now generalise to other values of p.

Question 5: Let $(\mathbb{R}, \mathcal{R}, \mu)$ be Lebesgue measure on the real line. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Define a map $\lambda : \mathcal{R} \to \mathbb{R}$ by

$$\lambda(E) = \int_E f \, d\mu \qquad (E \in \mathcal{R}).$$

You saw something similar on a previous Question Sheet. Show quickly (which means, *don't* give me *all* the details) that λ is a signed measure. Let $A \cup B$ be a Hahn-Decomposition for λ. How can we relate the sets A and B to the function f?

Question 6: Let $(\mathbb{R}, \mathcal{R}, \mu)$ be Lebesgue measure on the real line. Show, quickly, that we can define a measure ν on $\mathbb{R}$ by

$$\nu(A) = \int_A |x| \, d\mu(x) \qquad (A \in \mathcal{R}).$$

Show that $\nu << \mu$. However, show that for any $\epsilon > 0$, there does not exist $\delta > 0$ such that $\mu(A) \leq \delta$ implies that $\nu(A) \leq \epsilon$. *Hint:* Consider $A = (t, t + \delta)$ for very large t. *Remark:* Naively, we might think of this as a naive notion of "absolute continuity"!

Question 7: Let $(X, \mathcal{R})$ be a set with a σ-algebra, and let μ, λ be *finite* measures on $\mathcal{R}$. Show that the following are equivalent:

1. $\mu << \lambda$ and $\lambda << \mu$;
2. $A \in \mathcal{R}$ is μ-null if and only if it is λ-null;

3. there exists a measurable function $f : X \to (0, \infty)$ (*note that I am not using* $[0, \infty)$ *or* $[0, \infty]$) such that $\lambda(A) = \int_A f \, d\mu$ for all $A \in \mathcal{R}$.

Hint: To show (3) from (1), apply the Radon-Nikodym Theorem, and then think if you can adjust the resulting function to satisfy (3).

Bonus Question 8: This gets you to construct a weird measure: this is fun, because we haven't seen many examples of measures. But it's also quite a long question.

Let $(\mathbb{R}, \mathcal{R}, \mu)$ be Lebesgue measure on the real line. Let (r_n) be an enumeration of the rationals. For each n, let

$$A_n = (r_n - 2^{-n}, r_n + 2^{-n}), \quad f_n = 2^n \chi_{A_n}.$$

Hence $f_n \geq 0$ and $\int_{\mathbb{R}} f_n \, d\mu = 2$.

Let B be the set of $x \in \mathbb{R}$ such that x is in infinitely many of the sets A_n. Show that

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Using Proposition 2.3, show that $\mu(B) = 0$, as

$$\mu\Big(\bigcup_{k=n}^{\infty} A_k \Big) \leq \sum_{k=n}^{\infty} \mu(A_k) = \sum_{k=n}^{\infty} 2^{1-k} = 2^{-n}.$$

Hence show that $\sum_n f_n < \infty$ almost everywhere.

Define a measure $\lambda : \mathcal{R} \to [0, \infty]$ by

$$\lambda(A) = \sum_{n=1}^{\infty} \int_A f_n \, d\mu.$$

For $a < b$, show that $\lambda((a, b)) = \infty$. *Hint:* There must be infinitely many rational numbers in the open set (a, b). Conclude that $\lambda(U) = \infty$ for any open set $U \subseteq \mathbb{R}$.

Show now that for $A \in \mathcal{R}$, we have that $\lambda(A) = 0$ if and only if $\mu(A \cap \bigcup_n A_n) = 0$. But $\mu(\bigcup_n A_n) = 2$, so $\bigcup_n A_n$ is much smaller than $\mathbb{R}$, and so there are lots of sets $D \in \mathcal{R}$ with $\mu(D)$ large but $\lambda(D) = 0$. (I am trying to justify that λ is quite complicated).

Conclude that $\lambda << \mu$. Hence absolutely continuous measures can be pretty nasty!

Linear Analysis I: Example Sheet 8

Question 1: Let $(\mathbb{R}, \mathcal{R}, \mu)$ be Lebesgue measure on the real line. Let X be the subset of $L^1(\mu)$ consisting of those $f \in L^1(\mu)$ such that, for some $K > 0$, we have that $|f| \leq K$ almost everywhere (loosely, we could write $f \in L^1(\mu) \cap L^\infty(\mu)$). Hence X is also a subspace of $\mathcal{L}^1(\mu)$.

Show that $f : \mathbb{R} \to [0, \infty)$ defined by

$$f(x) = \begin{cases} n^{1/2} & : (n+1)^{-1} < x \leq n^{-1} \text{ for some } n \in \mathbb{N}, \\ 0 & : \text{otherwise}, \end{cases}$$

is in $L^1(\mu)$. Hence, or otherwise, show *carefully* show that $X \neq \mathcal{L}^1(\mu)$.

Show, however, that X is dense in $\mathcal{L}^1(\mu)$. *Hint:* Try a proof by contradiction.

Question 2: This continues from Question 1. Show that the mapping

$$T(f) = g \quad \text{where} \quad g(t) = \int_{[0,t]} f \, d\mu \qquad (t \geq 0),$$

is a well-defined map $X \to C_{\mathbb{K}}([0, \infty))$. *Hint:* The tricky part is to show that g is continuous.

As usual, we give $C_{\mathbb{K}}([0, \infty))$ the $\|\cdot\|_\infty$ norm. Show that T is linear and bounded. What is $\|T\|$?

Does the definition of T make sense on $\mathcal{L}^1(\mu)$? *Hint:* This is tricky! Try to use the fact that X is dense in $\mathcal{L}^1(\mu)$.

Question 3: With notation as from Question 1: for $1 < p < \infty$, let $X_p \subseteq \mathcal{L}^p(\mu)$ have the same definition as X. Show quickly that X_p is a subspace. Use Question 1, and the fact that $\mathcal{L}^p(\mu)^* = \mathcal{L}^q(\mu)$, to show that X_p is dense in $\mathcal{L}^p(\mu)$. *Hint:* You need to show that if $g \in \mathcal{L}^q(\mu)$ is non-zero, then we can find $f \in X_p$ with $\int_{\mathbb{R}} fg \, d\mu \neq 0$.

Question 4: We show that $C([0, 1])$ is not dense in $\mathcal{L}^\infty([0, 1])$ (over either $\mathbb{R}$ or $\mathbb{C}$). Let $f : [0, 1] \to [-1, 1]$ be defined by

$$f(x) = \begin{cases} 0 & : x = 0, \\ \sin(1/x) & : 0 < x \leq 1. \end{cases}$$

As f is continuous, except at 0, it is measurable. Clearly f is bounded everywhere, so $f \in \mathcal{L}^\infty([0, 1])$. By considering what happens at zero, show that for any $g \in C([0, 1])$, we have that $\|f - g\|_\infty \geq 1$. *Hint:* This is relatively simple if you use the supremum norm. You'll have to work a bit harder to get the result in the essential-supremum norm. Try to show $\|f - g\|_\infty \geq 1/2$, say, first.

Question 5: Let $([0, 1], \mathcal{R}, \mu)$ be the restriction of the Lebesgue measure to $[0, 1]$. Let $f \in \mathcal{L}^\infty(\mu)$. Show that $f \in \mathcal{L}^p(\mu)$ for $1 \leq p < \infty$, and $\sup\{\|f\|_p : 1 \leq p < \infty\} < \infty$.

Conversely, suppose that $f : [0, 1] \to \mathbb{K}$ is measurable, that $f \in \mathcal{L}^p(\mu)$ for each $1 \leq p < \infty$, and that $\sup\{\|f\|_p : 1 \leq p < \infty\} < \infty$. Show that $f \in \mathcal{L}^\infty(\mu)$. *Hint:* Try a proof by contradiction. It might help to show first that if $0 < t \leq 1$, then $\sup_{p \geq 1} t^{1/p} = 1$.

Finally, show that if $f \in \mathcal{L}^\infty(\mu)$, then

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p.$$

Question 6: We know that $(\ell^1)^* = \ell^\infty$, so it might be tempting to believe that $(\ell^\infty)^* = \ell^1$. This is impossible, as ℓ^∞ is not separable[1], while ℓ^1 is. However, let us give a more direct argument.

Treat c_0 as a (closed) subspace of ℓ^∞. Let $A \subseteq \mathbb{N}$ be infinite, so $\chi_A \in \ell^\infty$, but $\chi_A \notin c_0$. Show that

$$d(\chi_A, c_0) := \inf \left\{ \|\chi_A - x\|_\infty : x \in c_0 \right\} = 1.$$

Show that the linear map defined by

$$\phi : c_0 + \mathbb{K}\chi_A = \{x + t\chi_A : x \in c_0, t \in \mathbb{K}\} \to \mathbb{K}, \quad \phi(x + t\chi_A) = t,$$

is well-defined, and that $\|\phi\| = 1$. Hence, by the Hahn-Banach Theorem, show that there exists $\psi \in (\ell^\infty)^*$ such that

$$\psi(\chi_A) = 1, \qquad \psi(x) = 0 \qquad (x \in c_0).$$

Show that there cannot exist $(a_n) \in \ell^1$ such that

$$\psi(x) = \sum_{n=1}^{\infty} a_n x_n \qquad (x = (x_n) \in \ell^\infty).$$

[1]Look this up if you don't know what it means

Linear Analysis I: Example Sheet 9

Question 1: Let K be a compact space. Let (f_n) be a sequence of positive functions in $C_{\mathbb{R}}(K)$, and let $f \in C_{\mathbb{R}}(K)$ be such that for each $x \in K$,

$$f_1(x) \leq f_2(x) \leq \cdots, \qquad f(x) = \lim_n f_n(x).$$

Show that

$$\lambda(f) = \lim_n \lambda(f_n) \qquad (\lambda \in C_{\mathbb{R}}(K)^*).$$

Hint: Use the Riesz Representation Theorem, Hahn-Decomposition, and Monotone Convergence.

Question 2: Let K be a compact space, let (f_n) be a sequence in $C_{\mathbb{C}}(K)$, let $f \in C_{\mathbb{C}}(K)$ and let $M > 0$ be such that

$$\|f_n\|_\infty \leq M \quad (n \in \mathbb{N}), \qquad f(x) = \lim_n f_n(x) \qquad (x \in K).$$

Show that

$$\lambda(f) = \lim_n \lambda(f_n) \qquad (\lambda \in C_{\mathbb{R}}(K)^*).$$

Hint: Use the Riesz Representation Theorem, Hahn-Decomposition, Dominated Convergence, and take positive and negative parts.

Question 3: Let $K = [0,1]$ and for each n, define $f_n \in C_{\mathbb{R}}(K)$ by

$$f_n(x) = \begin{cases} n^2 x & : 0 \leq x \leq 1/n, \\ 2n - n^2 x & : 1/n \leq x \leq 2/n, \\ 0 & : x > 2/n. \end{cases}$$

Show that $f_n(x) \to 0$ for each $x \in K$, but that there exists $\mu \in C_{\mathbb{R}}(K)^*$ such that $\mu(f_n) \not\to 0$. *Hint:* Sketch f_n.

Question 4: Let K be a topological space. We shall[1] define the *Borel σ-algebra* on K to be the σ-algebra generated by open sets in K; again we write $\mathcal{B}(K)$ for this. In particular, we get $\mathcal{B}(\mathbb{K})$.

Given two topological spaces K and L, we shall say that a map $f : K \to L$ is *Borel* if $f^{-1}(E) \in \mathcal{B}(K)$ for each $E \in \mathcal{B}(L)$.

Now let K be a compact space, and consider K with the Borel σ-algebra $\mathcal{B}(K)$. Show that $f : K \to \mathbb{K}$ is measurable if and only if f is Borel.

Question 5: Let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$. Show that there exists $S \in \mathcal{B}(F^*, E^*)$ with the following property: for $\phi \in F^*$, we have that $S(\phi) = \psi \in E^*$, where

$$\psi(x) = \phi\big(T(x)\big) \qquad (x \in E).$$

We call S the *adjoint* of T, and write $S = T^*$.

Question 6: Let $(X, \mathcal{R}, \mu)$ be a measure space. We say that $E \in \mathcal{R}$ is an *atom* if $\mu(E) \neq 0$, and if $F \in \mathcal{R}$ with $F \subseteq E$ then either $\mu(F) = \mu(E)$ or $\mu(F) = 0$.

Suppose that for some $x \in X$, we have that $\{x\} \in \mathcal{R}$. Show that $\{x\}$ is an atom if and only if $\mu(\{x\}) \neq 0$.

[1] There is a debate in the literature here: some authors use the *compact* sets as the generating sets. If K is itself compact, then these two definitions of course agree. If you want, check that the definitions agree on $\mathbb{K}$ as well.

Let $E \in \mathcal{R}$ be an atom. Let $(E_n)_{n=1}^{\infty}$ be a *partition* of E; that is, $E_n \in \mathcal{R}$ and $E_n \subseteq E$ for each n, for $n \neq m$ we have $E_n \cap E_m = \emptyset$, and finally $\bigcup_n E_n = E$. If μ is finite, show that there exists a unique n_0 with E_{n_0} being an atom.

Is this still true if μ is not finite? Or, an easier question, where (if at all) did you use that μ is finite in your proof?

Question 7: This follows on from Question 6. Let K be a compact Hausdorff space, and let μ be a finite, regular (positive) Borel measure. Let $E \in \mathcal{B}(K)$ be an atom. Show that there exists a closed set $F \subseteq E$ which is an atom.

Suppose, towards a contradiction, that $x \in F$ implies that $\{x\}$ is not an atom. Show that for each $x \in F$ there exists an open set U_x with $x \in U_x$ and $\mu(U_x) < \mu(F)$.

As F is compact, and $\{U_x : x \in F\}$ is an open cover, there exist $x_1, \cdots, x_n$ in F with $U_{x_1} \cup \cdots \cup U_{x_n} \supseteq F$. Let $A_j = U_{x_j} \cap F$ for $1 \leq j \leq n$, let $B_1 = A_1$ and $B_j = A_j \setminus (A_1 \cup \cdots \cup A_{j-1})$ for $j \geq 2$. Why is $(B_j)_{j=1}^{n}$ a partition of F? Show that $\mu(B_j) < \mu(F)$ for each j, and hence derive a contradiction (think about Question 6 here).

Hence show that if $E \in \mathcal{B}(K)$ is an atom, then there exists a unique $x \in E$ with $\{x\}$ being an atom, and $\mu(E \setminus \{x\}) = 0$.

The following two questions get you to explore something quite subtle; they are hence optional (but interesting!)

Question 8: Let K be a compact space. Given a Borel map $\psi : K \to K$ and $\mu \in M_{\mathbb{C}}(K)$, show that

$$\psi(\mu) : \mathcal{B}(K) \to \mathbb{C}, \quad A \mapsto \mu(\psi^{-1}(A)) \qquad (A \in \mathcal{B}(K))$$

defines a measure on $\mathcal{B}(K)$. Do you think that $\psi(\mu)$ need be regular? What if ψ is even continuous?

Question 9: This uses the notation of Question 5, and continued from Question 8. Let $\psi : K \to K$ be a *continuous* map. Show that we can define $S_\psi : C_{\mathbb{K}}(K) \to C_{\mathbb{K}}(K)$ by

$$S_\psi(f) = f \circ \psi \qquad (f \in C_{\mathbb{K}}(K)).$$

Show that S_ψ is bounded. What is $\|S_\psi\|$?

Calculate what S_ψ^* is: you will need to use the proof of the Riesz-representation theorem. (Actually, to give a 100% correct answer is very hard!)

Bonus question 1: This gives a proof of Urysohn's Lemma, and is very much an optional question. A topological space K is *normal* if, whenever E and F are disjoint closed sets, we can find disjoint open sets U and V such that $E \subseteq U$ and $F \subseteq V$.

First, we show that if K is compact (and Hausdorff) then K is normal. Let E and F be disjoint closed sets. Pick $x \in E$, and for each $y \in F$, as K is Hausdorff, we can find disjoint open sets U_y and V_y such that $x \in U_y$ and $y \in V_y$. Hence $\bigcup_{y \in F} V_y$ is an open cover of F. Conclude that we can find disjoint open sets U and V with $x \in U$ and $F \subseteq V$. Now apply a similar argument to show that K is normal.

Now we attack Urysohn's Lemma. Let E and F be disjoint closed subsets of K. We aim to find a continuous function $f : K \to [0,1]$ such that $f \equiv 0$ on E and $f \equiv 1$ on F.

Let $D = \{m2^{-n} : m, n \in \mathbb{N}, m < 2^n\} \subseteq (0,1)$. As K is normal, show that there exists an open set $U_{1/2}$ such that $E \subseteq U_{1/2}$, and if $\overline{U_{1/2}}$ is the closure of $U_{1/2}$, then $F \cap \overline{U_{1/2}} = \emptyset$.

Again using normality, show that there exist open sets $U_{1/4}$ and $U_{3/4}$ such that $E \subseteq U_{1/4} \subseteq \overline{U_{1/4}} \subseteq U_{1/2}$ and $\overline{U_{1/2}} \subseteq U_{3/4} \subseteq \overline{U_{3/4}} \subseteq K \setminus F$.

Continue inductively to find a family $\{U_r\}_{r \in D}$ of open sets, such that

$$E \subseteq U_r \subseteq \overline{U_r} \subseteq U_s \subseteq \overline{U_s} \subseteq K \setminus F$$

whenever $r, s \in D$ with $r < s$. *Hint:* At each stage n, choose $U_{m/2^n}$ for all m such that we haven't yet dealt with $m2^{-n}$.

We define $f : K \to [0,1]$ by

$$f(x) = \begin{cases} \inf\{r \in D : x \in U_r\} & : x \in \bigcup_r U_r, \\ 1 & : \text{otherwise.} \end{cases}$$

Show that f has the required properties.

Bonus question 2: I mentioned the *Borel hierarchy* in lectures. Let us defined it here. Fix a compact Hausdorff space K (although normally we define the following on a *Polish* space: as far as Borel sets are concerned, there is only one Polish space, which we can take to be the unit interval $[0,1]$: this is hard enough!) Define

$$\begin{aligned} \Sigma^0_1 &= \text{Open sets in } K, \\ \Pi^0_1 &= \{A \subseteq K : K \setminus A \in \Sigma^0_1\} = \text{Closed sets}, \\ A \in \Sigma^0_2 &\Leftrightarrow \exists (A_n) \subseteq \Pi^0_1,\ A = \bigcup_n A_n, \\ \Pi^0_2 &= \{A \subseteq K : K \setminus A \in \Sigma^0_2\}. \end{aligned}$$

We can continue for any *countable ordinal*[2] α:

$$\begin{aligned} A \in \Sigma^0_\alpha &\Leftrightarrow \exists (A_n) \subseteq \bigcup_{\beta<\alpha} \Pi^0_\beta,\ A = \bigcup_n A_n, \\ \Pi^0_\alpha &= \{A \subseteq K : K \setminus A \in \Sigma^0_\alpha\}. \end{aligned}$$

Prove that $A \in \Pi^0_\alpha$ if and only if we can find $(A_n) \subseteq \bigcup_{\beta<\alpha} \Sigma^0_\beta$ with $A = \bigcap_n A_n$.

Let μ be a finite (positive) measure defined on $\mathcal{B}(K)$. Suppose that μ is *outer regular*, which means that

$$\mu(E) = \inf\big\{\mu(U) : E \subseteq U \text{ open in } K\big\} \qquad (E \in \mathcal{B}(K)).$$

[2] If you don't know what this means, then just consider the natural numbers.

By using that μ is finite, show that μ must also be *inner regular*, that is,

$$\mu(E) = \sup\big\{\mu(A) : A \subseteq E \text{ closed in } K\big\} \qquad (E \in \mathcal{B}(K)).$$

So μ is regular if and only if it is outer regular.

We shall make a temporary definition: $E \in \mathcal{B}(K)$ is μ*-regular* if

$$\mu(E) = \inf\big\{\mu(U) : E \subseteq U \text{ open in } K\big\}.$$

Show that every open set is μ-regular.

Now suppose that K is a metric space (or just that $K = [0,1]$ with the usual topology). Then it is a fact that every closed set can be written as an intersection of a countable family of open sets.[3] Show that every member of Π^0_1 is μ-regular.

Claim 1: Let (A_n) be a sequence of μ-regular sets. Then $A = \bigcup_n A_n$ is μ-regular.

Prove this as follows: for $\epsilon > 0$, show that for each n, we can find U_n open with $A_n \subseteq U_n$ with $\mu(U_n \setminus A_n) < \epsilon 2^{-n}$. Let $U = \bigcup_n U_n$, so U is open. Show that $A \subseteq U$ and $\mu(U) < \mu(A) + \epsilon$. Conclude that A is μ-regular.

Claim 2: Let $A_1, \cdots, A_n$ be μ-regular sets. Then $A_1 \cap \cdots \cap A_n$ is μ-regular.

Prove this as follows: by induction, it is enough to prove the $n = 2$ case. For $i = 1, 2$, let U_i be open with $A_i \subseteq U_i$. Show that

$$(U_1 \cap U_2) \setminus (A_1 \cap A_2) \subseteq (U_1 \setminus A_1) \cup (U_2 \setminus A_2),$$

and hence show that

$$\mu\big((U_1 \cap U_2) \setminus (A_1 \cap A_2)\big) \leq \mu(U_1 \setminus A_1) + \mu(U_2 \setminus A_2).$$

Then prove the claim.

Claim 3: Let (A_n) be a sequence of μ-regular measures. Then $A = \bigcap_n A_n$ is μ-regular.

Prove this as follows: Let $B_1 = A_1$ and $B_n = A_1 \cap \cdots \cap A_n$ for $n \geq 2$. By claim 2, each B_n is μ-regular, we have that $B_1 \supseteq B_2 \supseteq \cdots$, and $A = \bigcap_n B_n$. For $\epsilon > 0$, show that we can find n with $\mu(B_n \setminus A) < \epsilon/2$ and that we can find an open set U with $\mu(U \setminus B_n) < \epsilon/2$. Show that $A \subseteq U$ and that $\mu(U \setminus A) < \epsilon$. Conclude that A is μ-regular.

So we have shown that each member of Σ^0_1 and Π^0_1 is μ-regular, and that countable intersections and unions of μ-regular sets are μ-regular. Conclude, by induction, that each member of Σ^0_α, or Π^0_α, is μ-regular for each $\alpha \in \mathbb{N}$.

If you know about ordinals, then use transfinite induction (up to ω_1) to show that each member of Σ^0_α, or Π^0_α, is μ-regular for each $\alpha < \omega_1$. It's a theorem that

$$\big\{\text{Borel sets}\big\} = \bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha,$$

as long as K is a separable, complete metric space (e.g. $K = [0,1]$).

We conclude that *if*[4] there exists a compact Hausdorff space K and a finite measure defined on $\mathcal{B}(K)$ which is *not* regular, then K cannot be a metric space! This explains why I will not give a counter-example!

[3] Prove this! Just use the metric.

[4] I have not found such an example in the literature, but maybe I haven't looked very hard!

Linear Analysis I: Example Sheet 10

Question 1: We used a similar argument to the following in lectures, and also on Example Sheet 8. Let E and G be Banach spaces, and let $F \subseteq E$ be a subspace which is dense. Let $T : F \to G$ be a bounded linear map. Show that we can extend T to give a bounded linear map $E \to G$. Show that such an extension must be unique. *Hint:* The model answers to Question 3, Sheet 8 sketched this in a special case. So try to do the general case, and check all the details!

Question 2: Define $f : [0, 1] \to \mathbb{C}$ by

$$f(t) = \begin{cases} \exp(t) & : 0 \leq t \leq 1/2, \\ \exp(1-t) & : 1/2 \leq t \leq 1. \end{cases}$$

Thus f is periodic. Calculate the Fourier transform of f.

By using Fejer's Theorem, and evaluating at $t = 0$ and $t = 1/2$, show that

$$\sum_{k=1}^{\infty} \frac{1}{1 + 16\pi^2 k^2} = \frac{1}{4(e^{1/2} - 1)} - \frac{3}{8}.$$

(It is at least plausible that I've messed this up, so don't assume that you are wrong if you get a different answer).

Question 3: Let $f(t) = e^t$ for $0 \leq t \leq 1$; show that $f \in \mathcal{L}^2([0, 1])$ and compute $\|f\|_2$. Find $\mathcal{F}(f)$, and hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1 + 4\pi^2 n^2} = \frac{3 - e}{4(e - 1)}.$$

Question 4: I used the following in lectures, but didn't really justify it. Recall that $\mathbb{T}$ is $[0, 1]$ with the end points identified. So we can think of functions in $C_{\mathbb{C}}(\mathbb{T})$ as being continuous, periodic functions (or continuous functions $f : [0, 1] \to \mathbb{C}$ such that $f(0) = f(1)$). As the set $\{0, 1\}$ has only two points, it has Lebesgue measure zero. It follows easily that $\mathcal{L}^p([0, 1])$ and $\mathcal{L}^p(\mathbb{T})$ can be identified. (Check this if you wish).

In lectures, I repeatedly used the fact that $C_{\mathbb{C}}(\mathbb{T})$ is dense in $\mathcal{L}^2(\mathbb{T}) = \mathcal{L}^2([0, 1])$. Prove this. *Hint:* We know that $C_{\mathbb{C}}([0, 1])$ is dense in $\mathcal{L}^2([0, 1])$. So for $f \in \mathcal{L}^2([0, 1])$ and $\epsilon > 0$, there exists $g \in C_{\mathbb{C}}([0, 1])$ with $\|f - g\|_2 < \epsilon$. Approximate g by something in $C_{\mathbb{C}}(\mathbb{T})$.

Question 5: Let (f_n) be a sequence in $C_{\mathbb{C}}([0, 1])$ converging to f with respect to the $\|\cdot\|_\infty$ norm. Suppose each f_n is differentiable (to be precise, on $(0, 1)$, or suppose each f_n is periodic) with a continuous derivative, and $f_n' \to g \in C_{\mathbb{C}}([0, 1])$ with respect to the $\|\cdot\|_\infty$ norm. Show that f is differentiable with derivative g.

Hint: Try using the Fundamental Theorem of Calculus: this tells us that if the integral of g is f, then f is differentiable with derivative g.

Question 6: I might have said a few words about this in lectures. We now dip our toes in the waters which you'll come to in MATH5016 next term. The *Open Mapping Theorem* states that for Banach spaces E and F, and a bounded linear map $T : E \to F$, if T is bijective, then T^{-1} (which always exists as a linear map) is bounded.

For $n \geq 1$ let $x_n = (x_m^{(n)})_{m \in \mathbb{Z}} \in c_0(\mathbb{Z})$ be defined by

$$x_m^{(n)} = \begin{cases} 1 & : |m| \leq n, \\ 0 & : |m| > n. \end{cases}$$

Then $x_n \in \ell^1(\mathbb{Z})$ so that $\mathcal{F}^{-1}(x_n)$ makes sense. Show that $\|\mathcal{F}^{-1}(x_n)\|_1$ is large (a *crude* estimate is all that's required!)

Hence, by using a result from lectures that $\mathcal{F}$ is injective, and assuming the Open Mapping Theorem, show that $\mathcal{F}$ does *not* map $\mathcal{L}^1([0,1])$ onto $c_0(\mathbb{Z})$.

Thinking more about Riesz Representation

Here are a couple of questions exploring the *complex* Riesz Representation Theorem: we didn't really prove everything in lectures (and so this won't be on the exam!) so it's interesting to see how to do so.

Question i: For a compact (Hausdorff) space K let $M_{\mathbb{C}}(K)$ be the space of finite, complex, regular Borel measures on K. For $\mu \in M_{\mathbb{C}}(K)$ define $\phi_\mu \in C_{\mathbb{C}}(K)^*$ by

$$\phi_\mu(f) = \int_K f \, d\mu \qquad (f \in C_{\mathbb{C}}(K)).$$

Let $g : K \to \mathbb{C}$ be a simple function (of course, not assumed continuous!) with $\|g\|_\infty \leq 1$. Show that

$$\Big| \int_K g \, d\mu \Big| \leq \|\mu\|.$$

Now let $f \in C_{\mathbb{C}}(K)$ with $\|f\|_\infty \leq 1$. Show that we can find a sequence (g_n) of simple functions with $g_n \to f$ pointwise, and with $|g_n| \leq |f|$ everywhere for each n. (*Hint:* Apply our "canonical" method for getting simple functions, but taking account of real and imaginary parts, etc.) Conclude, by using the Dominated Convergence Theorem, that $|\phi_\mu(f)| \leq \|\mu\|$. Conclude that $\|\phi_\mu\| \leq \|\mu\|$.

Question ii: Did anyone notice an omission from the lectures? I claimed that $M_{\mathbb{K}}(K)$ is a vector space. Certainly, if μ, λ are finite regular measures, then $\mu + \lambda$ makes sense and is finite. However, why is $\mu + \lambda$ regular? We prove this here!

Firstly, prove the following useful lemma. Let τ be a positive Borel measure. Show that τ is regular if and only if, for each $E \in \mathcal{B}(K)$ and $\epsilon > 0$, we can find an open set U and a closed set C with $C \subseteq E \subseteq U$ and with $\tau(U \setminus C) < \epsilon$.

Now prove a second useful lemma. For a signed measure τ, we defined $|\tau| = \tau_+ + \tau_-$, where τ_+ and τ_- are defined by way of a Hahn-Decomposition for τ. Show that

$$|\tau|(E) = \sup \big\{ \tau(U) - \tau(V) : U, V \in \mathcal{B}(K), U \cap V = \emptyset, U \cup V = E \big\} \qquad (E \in \mathcal{B}(K)).$$

So we don't actually need a Hahn-Decomposition to define $|\tau|$ (and this works for any measure on any σ-algebra).

Now prove a third useful lemma. Let $\tau \in M_{\mathbb{R}}(K)$. Show that τ is regular (defined to mean that τ_+ and τ_- are regular) if and only if $|\tau|$ is regular (hint: use the first lemma).

Let $\mu, \lambda \in M_{\mathbb{R}}(K)$, and let $\tau = \mu + \lambda$. Using the 2nd lemma, show that $|\tau| \leq |\mu| + |\lambda|$. Deduce, using the 3rd lemma, that τ is regular.

Show the same for complex measures: this is easier, as we can directly take real and imaginary parts.

If you want to see a self-contained proof of the (full version) of the Riesz representation theorem in the complex case, then you can look in Rudin, "Real and complex analysis". However, be warned that Rudin uses different (but equivalent!) notions for what "regular" means. The proof is very clever, but rather synthetic (and would have taken too long to cover in lectures, sadly). Here's half the proof:

Question iii: Let K be compact and Hausdorff, and let $\lambda \in C_{\mathbb{C}}(K)$ with $\|\lambda\| = 1$. It is possible[1] to construct a positive $\Phi \in C_{\mathbb{R}}(K)^*$ with the property that for any $f \in C_{\mathbb{C}}(K)$,

$$|\lambda(f)| = \Phi(|f|) \leq \|f\|_\infty,$$

where $|f|(x) = |f(x)|$ for each $x \in K$. Show that $\|\Phi\| = 1$.

We can then apply Riesz representation to find some a regular, positive Borel measure μ_0 with

$$\Phi(g) = \int_K g \; d\mu_0 \qquad (g \in C_{\mathbb{R}}(K)).$$

As $\|\Phi\| = 1$, we have that $\mu_0(K) = 1$.

We can hence form that space $\mathcal{L}^1(\mu_0)$. There is a natural map $C_{\mathbb{C}}(K) \to \mathcal{L}^1(\mu_0)$; let X be the image, so that X is a subspace[2] of $\mathcal{L}^1(\mu_0)$. Show that the map

$$\phi : X \to \mathbb{C}; \quad f \mapsto \lambda(f)$$

is linear and bounded. What is $\|\phi\|$? Using that $\mathcal{L}^1(\mu_0)^* \cong \mathcal{L}^\infty(\mu_0)$ (and Hahn-Banach), show that there exists $h \in \mathcal{L}^\infty(\mu_0)$ with

$$\lambda(f) = \int_K fh \; d\mu_0 \qquad (f \in C_{\mathbb{C}}(K)).$$

Let $\mu = h\mu_0$, so μ is the complex measure with

$$\mu(E) = \int_K \chi_E h \; d\mu_0.$$

This *is* regular: this isn't too hard to show, if you adopt the philosophy of question ii. We immediately see that

$$\lambda(f) = \int_K f \; d\mu \qquad (f \in C_{\mathbb{C}}(K)).$$

Finally, show that $\|h\|_\infty = 1$ (hint: what is $\|\phi\|$?) Deduce that $\|\mu\| = 1 = \|\lambda\|$ (hint: Use Question i).

More on Fourier analysis

The following questions are structured around things which you will find in the book "Fourier Analysis" by T. W. Körner (lots of copies available in the library!) These questions are not on the syllabus of the course, but are interesting, I think.

Question A: Let $(a_n) \in \ell^1(\mathbb{Z})$ be a sequence such that $(na_n) \in \ell^1(\mathbb{Z})$ as well (for example, $a_n = n^{-3}$ for each n). Let $f = \mathcal{F}^{-1}((a_n))$. Show that f is differentiable.

To do this, we use Question 5. Let

$$f_n(t) = \sum_{k=-n}^{n} a_k e^{-2\pi i k t}.$$

Hence $f_n \to f$ by Fejer's Theorem (or a corollary thereof). Calculate the derivative of f_n, and hence show that $f_n' \to g$ for some $g \in C_{\mathbb{C}}(\mathbb{T})$. Hence finish by applying Question 5.

[1] See Rudin's book; the construction is very similar to how we defined λ_+ given $\lambda \in C_{\mathbb{R}}(K)$.

[2] Actually, dense, but we don't need this.

Question B: The thinking behind Question 6 is that if (a_n) is a sufficiently "nice" sequence, then we can differentiate, term by term, the Fourier series of $f = \mathcal{F}^{-1}((a_n))$. Suppose for the moment we believe that we can always do this. Define a sequence (a_n) by

$$a_n = \begin{cases} (r!)^{-1} & : n = (r!)^2 \text{ for some } r \in \mathbb{N}, \\ 0 & : \text{otherwise.} \end{cases}$$

Show that $(a_n) \in \ell^1(\mathbb{Z})$, and hence that $f = \sum_n a_n e^{-2\pi int}$ converges in $C_{\mathbb{C}}([0,1])$. Formally, we see that

$$f'(t) = -2\pi i \sum_n n a_n e^{-2\pi int} = -2\pi i \sum_{r=1}^{\infty} (r!)^2 (r!)^{-1} e^{-2\pi it(r!)^2}.$$

However, this in no sense converges!

In fact, if you consult "Fourier Analysis", Chapter 11, you will find a proof that f is in fact differentiable nowhere! The initial discovery of such functions (remember, f is continuous!) was a shock to mathematicians, and lead, in part, to the formalism of modern analysis.

Question C: Let X be the subspace of $C_{\mathbb{C}}(\mathbb{T})$ spanned by functions of the form $t \mapsto e^{2\pi int}$, for $n \in \mathbb{Z}$. We saw in lectures that, because of Fejer's Theorem, X is dense in $C_{\mathbb{C}}(\mathbb{T})$.

Now let $f : [0,1] \to \mathbb{R}$ be continuous (but not necessarily periodic) and define $g \in C_{\mathbb{C}}(\mathbb{T})$ by

$$g(t) = \begin{cases} f(2t) & : 0 \le t \le 1/2, \\ f(2-2t) & : 1/2 \le t \le 0. \end{cases}$$

Fix $\epsilon > 0$. Then we can find $h \in X$ with $\|g - h\|_\infty < \epsilon$. We know that on the interval $[0,1]$ and for $n \in \mathbb{Z}$, we have that

$$\sum_{k=0}^{K} \frac{(2\pi int)^k}{k!}$$

converges uniformly to $e^{2\pi int}$, as $K \to \infty$. Use this to approximate h by a complex polynomial in t.

By taking real parts, and thinking about the definition of g, show that we have approximated f be a real polynomial.

This is the Weierstrauss Approximation Theorem, see "Fourier Analysis", Chapter 4.