Linear Analysis I: Worked Solutions 1

I do not intend to give worked solutions to every question. However, I *will* give full solutions to any [**Revision**] questions.

Question 1: For a normed vector space $(V, \|\cdot\|)$, show that if (x_n) is a sequence in V tending to x, and μ is a scalar, then $\mu x_n \to \mu x$.
Answer: If $\mu = 0$, then the result is obvious, so we may suppose that $|\mu| > 0$. Let $\epsilon > 0$, so as $x_n \to x$, there exists an $N > 0$ such that $\|x_n - x\| < \epsilon|\mu|^{-1}$ whenever $n \geq N$. Then $\|\mu x_n - \mu x\| = |\mu|\|x_n - x\| < \epsilon$. As $\epsilon > 0$ was arbitrary, we conclude that $\mu x_n \to \mu x$, as required.

The other answers are similar.

Question 3: Do you think that the definition

$$\|f\|_\infty = \sup_{t\in[0,1]} |f(t)| \qquad (f \in \mathbb{K}^{[0,1]}),$$

makes sense???
Answer: No, because we have said nothing about f. For example, we could have that

$$f(t) = \begin{cases} 0 & : t = 0, \\ 1/t & : 0 < t \leq 1. \end{cases}$$

This is a function $[0,1] \to \mathbb{R}$, and the set $\{|f(t)| : t \in [0,1]\}$ is simply $\{0\} \cup [1,\infty)$, so the supremum is ∞, that is, it doesn't really exist.

We define $\ell^\infty([0,1])$ to be the *bounded* functions $[0,1] \to \mathbb{K}$. Then the supremum does exist, and it is not too hard to check that it is a norm.

Question 4: [**Revision**] Recall that we define the norm $\|\cdot\|_2$ on $\mathbb{K}^n$ by

$$\|x\|_2 = \Big(\sum_{i=1}^n |x_i|^2\Big)^{1/2} \qquad (x = (x_i) \in \mathbb{K}^n).$$

Prove that $(\mathbb{K}^n, \|\cdot\|_2)$ is complete.
Answer: Let (x_k) be a Cauchy-sequence in $(\mathbb{K}^n, \|\cdot\|_2)$. For each k, x_k is a vector in $\mathbb{K}^n$, say that

$$x_k = \begin{pmatrix} x_{k,1} \\ x_{k,2} \\ \vdots \\ x_{k,n} \end{pmatrix}.$$

For $\epsilon > 0$, there exists $N > 0$ such that $\|x_j - x_k\|_2 \leq \epsilon$ for $j,k \geq N$. That is,

$$\Big(\sum_{i=1}^n |x_{j,i} - x_{k,i}|^2\Big)^{1/2} \leq \epsilon \qquad (j,k \geq N).$$

Fix t between 1 and n, so that

$$|x_{j,t} - x_{k,t}| \leq \Big(\sum_{i=1}^n |x_{j,i} - x_{k,i}|^2\Big)^{1/2} \leq \epsilon \qquad (j,k \geq N).$$

Hence $(x_{k,t})_{k=1}^\infty$ is a Cauchy-sequence in $\mathbb{K}$, and hence converges to, say, a_t. Let

$$x = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \in \mathbb{K}^n.$$

Then

$$\lim_k \|x - x_k\|_2 = \lim_k \Big(\sum_{i=1}^n |a_i - x_{k,i}|^2 \Big)^{1/2} = \Big(\lim_k \sum_{i=1}^n |a_i - x_{k,i}|^2 \Big)^{1/2}$$
$$= \Big(\sum_{i=1}^n \lim_k |a_i - x_{k,i}|^2 \Big)^{1/2} = 0,$$

as required.

Question 5: Let $\mathbb{K}[X]$ be the space of polynomials over $\mathbb{K}$. For $p(X) = a_n X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 \in \mathbb{K}[X]$, we define

$$\|p\|_1 = \sum_{i=0}^n |a_i|.$$

For $n \geq 1$, let p_n be the polynomial

$$p_n(X) = \frac{1}{2^n} X^n + \frac{1}{2^{n-1}} X^{n-1} + \cdots + \frac{1}{4} X^2 + \frac{1}{2} X.$$

Show that (p_n) is a Cauchy sequence. Does (p_n) converge to a limit in $\mathbb{K}[X]$?
Answer: For $n > m$, we calculate that

$$\|p_n - p_m\|_1 = \sum_{i=m+1}^n \frac{1}{2^i} = 2^{-m} \sum_{i=1}^{n-m} 2^{-i} \leq 2^{-m}.$$

Hence (p_n) is a Cauchy sequence in $(\mathbb{K}[X], \|\cdot\|_1)$.

Let $p(X) = a_k X^k + a_{k-1} X^{k-1} + \cdots + a_0 \in \mathbb{K}[X]$ be some polynomial. Then, for large n,

$$\|p - p_n\|_1 = |a_0| + \sum_{i=1}^k |a_i - 2^{-i}| + \sum_{i=k+1}^n 2^{-i} \geq \sum_{i=k+1}^n 2^{-i} \geq 2^{-k-1}.$$

Thus we see that $p_n \not\to p$. As p was arbitrary, we conclude that (p_n) does not converge to any member of $\mathbb{K}[X]$. So $(\mathbb{K}[X], \|\cdot\|_1)$ is not complete.

Actually, we have used nothing about the structure of the polynomials here. The *completion* would be simply the Banach space ℓ^1.

Question 6: We define c_0 to be the collection of sequences in $\mathbb{K}$ which converge to 0, with the norm

$$\|(x_n)\|_\infty = \sup_n |x_n| \qquad ((x_n) \in c_0).$$

Show that c_0 is complete.
Answer: Let (x_n) be a Cauchy-sequence in c_0. Hence, for each n, $x_n \in c_0$, say that $x_n = (x_k^{(n)})_{k=1}^\infty$, so that $\lim_k x_k^{(n)} = 0$. For $\epsilon > 0$, there exists $N > 0$ such that $\|x_n - x_m\|_\infty \leq \epsilon$ for $n, m \geq N$. For k fixed, we see that

$$|x_k^{(n)} - x_k^{(m)}| \leq \sup_j |x_j^{(n)} - x_j^{(m)}| = \|x_n - x_m\|_\infty \leq \epsilon,$$

so we see that $(x_k^{(n)})_{n=1}^\infty$ is a Cauchy-sequence in $\mathbb{K}$, and so converges to a_k say.

We first check that $\lim_k a_k = 0$, so that $(a_k) \in c_0$. Let $\epsilon > 0$, so for some $N > 0$, we have that $\|x_n - x_m\|_\infty \leq \epsilon$ for $n, m \geq N$. Then $\lim_k x_k^{(N)} = 0$, so there exists $M > 0$ such that $|x_k^{(N)}| \leq \epsilon$ for $k \geq M$. For $k \geq M$, we see that

$$|a_k - x_k^{(N)}| = \lim_n |x_k^{(n)} - x_k^{(N)}| \leq \lim_n \|x_n - x_N\|_\infty \leq \epsilon.$$

We conclude that

$$|a_k| \leq |a_k - x_k^{(N)}| + |x_k^{(N)}| \leq 2\epsilon \qquad (k \geq M).$$

As $\epsilon > 0$ was arbitrary, we conclude that $\lim_k a_k = 0$, as required.

Finally, we check that $\lim_n \|x_n - (a_k)\| = 0$. Let $\epsilon > 0$, so, again, there exists $N > 0$ such that $\|x_n - x_m\|_\infty \leq \epsilon$ for $n, m \geq N$. Let $k \geq 1$, and let $n \geq N$, so that

$$|x_k^{(n)} - a_k| = \lim_m |x_k^{(n)} - x_k^{(m)}| \leq \lim_m \|x_n - x_m\|_\infty \leq \epsilon.$$

As k was arbitrary, we see that

$$\|x_n - (a_k)\|_\infty = \sup_k |x_k^{(n)} - a_k| \leq \epsilon.$$

As $n \geq N$ was arbitrary, we conclude that $\lim_n \|x_n - (a_k)\| = 0$, as required.

Question 7: Let (X, d) be a metric space, and let $Y \subseteq X$ be a subset. The restriction of d to Y turns Y into a metric space in its own right. What does it mean for Y to be *closed* in X? What does it mean for Y to be *open* in X? If X is complete, show that Y is closed in X if and only if Y is complete.

Answer: Y is *closed* in X if whenever (y_n) is a sequence in Y converging to $x \in X$, then actually $x \in Y$.

Y is *open* in X if for each $y \in Y$, there exists $\epsilon > 0$ such that

$$B(y, \epsilon) = \{x \in X : d(x, y) < \epsilon\} \subseteq Y.$$

Let X be complete. Suppose that Y is closed in X. If (y_n) is Cauchy in Y, then (y_n) is Cauchy in X, and so converges to $x \in X$. As Y is closed, $x \in Y$, so we see that every Cauchy sequence in Y converges in Y. Hence Y is complete.

Conversely, suppose that Y is complete, and let (y_n) be a sequence in Y converging to $x \in X$. Then (y_n) is Cauchy, so as Y is complete, (y_n) converges to $y \in Y$. Then $d(x, y) = \lim_n d(x, y_n) = 0$, so that $x = y$, and hence Y is closed.

Question 8: A metric space (X, d) is *compact* if whenever $(x_n)_{n=1}^\infty$ is a sequence in X, we can find a subsequence $n(1) < n(2) < \cdots$ such that $(x_{n(k)})_{k=1}^\infty$ is convergent.

If (X, d) is a metric space, we say that a subset $Y \subseteq X$ is compact if Y is compact for the metric inherited from X. Show that if Y is compact, then Y is closed in X.

Answer: Let (y_n) be a sequence in Y converging to $x \in X$. As Y is compact, there exists $n(1) < n(2) < \cdots$ such that $(y_{n(k)})$ is convergent in Y, say to $y \in Y$. Clearly $(y_{n(k)})$ also converges to x, so as above, $x = y$. Hence Y is closed.

Question 8 cont.: The Bolzano–Weierstraß theorem states that if (x_n) is a bounded sequence of real numbers, then (x_n) has a convergent subsequence. Use this result to prove that a subset $Y \subseteq \mathbb{R}$ is compact (for the usual metric on $\mathbb{R}$) if and only if Y is closed and bounded.

Answer: If Y is compact, then by the above, it is closed. If Y is not bounded, then for every n, we can find $y_n \in Y$ with $|y_n| > n$. Then (y_n) can not have any convergent subsequences, so Y cannot be compact, a contradiction. Hence Y is bounded.

Conversely, let Y be closed and bounded. Let (y_n) be a sequence in Y. As Y is bounded, so is (y_n), and hence the Bolzano–Weierstraß theorem tells us that a subsequence $(y_{n(k)})$ converges, say to $y \in \mathbb{R}$. As Y is closed, $y \in Y$, and so we may conclude that Y is compact.

Question 8 cont.: The Heine–Borel theorem tells us that a subset $Y \subseteq (\mathbb{R}^n, \|\cdot\|_2)$ is compact if and only if Y is closed and bounded. Prove this.
Answer: If Y is compact, then much the same argument as above shows that Y is closed and bounded. Conversely, let Y be closed and bounded, say that for $M > 0$, each $y \in Y$ satisfies $\|y\|_2 \leq M$. Let (y_k) be a sequence in Y. For each k, let $y_k = (y_{k,1}, \cdots, y_{k,n})$ (here I am using row vectors, instead of column vectors, for space reasons). For each k,

$$|y_{k,1}| \leq \|y_{k,1}\|_2 \leq M,$$

so we see that $(y_{k,1})$ is a bounded sequence in $\mathbb{R}$. By Bolzano–Weierstraß, we can find a subsequence $k_1(1) < k_1(2) < \cdots$ such that $(y_{k_1(j),1})_{j=1}^\infty$ converges.

Similarly, $(y_{k_1(j)}, 2)$ is a bounded sequence in $\mathbb{R}$, and so we can find a subsequence $k_2(1) < k_2(2) < \cdots$ of $(k_1(j))$, such that $(y_{k_2(j),2})_{j=1}^\infty$ converges. As $(k_2(j))$ is a subsequence of $(k_1(j))$, we also have that $(y_{k_2(j),1})_{j=1}^\infty$ converges.

Continuing, we can ultimately find a subsequence $k_n(1) < k_n(2) < \cdots$ such that $(y_{k_n(j),i})_{j=1}^\infty$ converges for each i, say $\lim_j y_{k_n(j),i} = z_i$. Let $z = (z_1, \cdots, z_n) \in \mathbb{R}^n$, so as in Question 4 above, we thus have that $\lim_j (y_{k_n(j)}) = z$. Thus Y is compact, as required.

Question 8 cont.: Prove the same result for $(\mathbb{C}^n, \|\cdot\|_2)$.
Answer: Define a map $\theta : \mathbb{C}^n \to \mathbb{R}^{2n}$ as follows. Let $x = (x_1, \cdots, x_n) \in \mathbb{C}^n$, so for each i, we have that $x_i = y_i + \imath z_i$ say, where $\imath^2 = -1$. Let

$$\theta(x) = (y_1, z_1, y_2, z_2, \cdots, y_n, z_n) \in \mathbb{R}^{2n}.$$

The θ is a bijection. Furthermore, θ is a distance preserving map for the metrics induced by $\|\cdot\|_2$. Hence $Y \subseteq \mathbb{C}^n$ is compact, or closed and bounded, if and only if $\theta(Y) \subseteq \mathbb{R}^{2n}$ is compact, or closed and bounded, respectively. The claim in the question follows at once.

Question 9: We shall now apply these ideas. Let (X, d) be a metric space, and let $C_\mathbb{K}(X)$ be the vector space of all continuous functions from X to $\mathbb{K}$.

We say that $f \in C_\mathbb{K}(X)$ is *uniformly continuous* if for each $\epsilon > 0$ there exists $\delta > 0$ such that whenever $x, y \in X$ satisfy $d(x, y) \leq \delta$, we have that $|f(x) - f(y)| \leq \epsilon$. Show that as X is compact, every $f \in C_\mathbb{K}(X)$ is uniformly continuous.
Answer: Suppose not, so that some $f \in C_\mathbb{K}(X)$ is not uniformly continuous. That is, there exists some $\epsilon > 0$ such that for each $\delta > 0$, we can find $x, y \in X$ with $d(x, y) \leq \delta$, but $|f(x) - f(y)| > \epsilon$. Hence for each n, we can find $x_n, y_n \in X$ with $d(x_n, y_n) \leq 1/n$ and $|f(x_n) - f(y_n)| > \epsilon$. As X is compact, we can find a subsequence $n(1) < n(2) < \cdots$ such that $(x_{n(k)})_{k=1}^\infty$ converges in X. Similarly, we can find a subsequence $(m(k))$ or $(n(k))$ such that $(y_{m(k)})_{k=1}^\infty$ converges. Let

$$x = \lim_k x_{m(k)} = \lim_k x_{n(k)}, \quad y = \lim_k x_{m(k)}.$$

Notice that

$$d(x, y) = \lim_k d(x_{m(k)}, y_{m(k)}) \leq \lim_k 1/m(k) = 0,$$

so that $x = y$. As f is continuous, we have that

$$f(x) = \lim_k f(x_{m(k)}), \quad f(y) = \lim_k f(y_{m(k)}),$$

and so we have that

$$0 = |f(x) - f(y)| = \lim_k |f(x_{m(k)}) - f(y_{m(k)})| \geq \epsilon,$$

giving us our required contradiction.

Question 9 cont.: Show that any $f \in C_{\mathbb{K}}(X)$ attains its supremum.
Answer: By the definition of the supremum, for each n, we can find $x_n \in X$ with

$$|f(x_n)| > \sup_{x \in X} |f(x)| - \frac{1}{n}.$$

We can find a subsequence $(x_{n(k)})$ which converges to, say, $y \in X$. As f is continuous,

$$|f(y)| = \lim_k |f(x_{n(k)}| \geq \lim_k \sup_{x \in X} |f(x)| - \frac{1}{n(k)} = \sup_{x \in X} |f(x)|,$$

as required.

Linear Analysis I: Worked Solutions 2

Question 1: Let E and F be normed vector spaces, and let $T : E \to F$ be a bounded linear map. The first line of the following is the original definition of the norm of T. Prove carefully that the other expressions really are equal:

$$\begin{aligned}\|T\| &= \sup\Big\{\frac{\|T(x)\|}{\|x\|} : x \in E, x \neq 0\Big\}\\ &= \sup\big\{\|T(x)\| : x \in E, \|x\| \leq 1\big\}\\ &= \sup\big\{\|T(x)\| : x \in E, \|x\| = 1\big\}.\end{aligned}$$

Answer: We show that the 2nd and 3rd expressions are equal. Set

$$K_2 = \sup\big\{\|T(x)\| : x \in E, \|x\| \leq 1\big\}, \quad K_3 = \sup\big\{\|T(x)\| : x \in E, \|x\| = 1\big\}.$$

As we are taking the supremum over a smaller set, clearly $K_3 \leq K_2$. For $x \in E$ with $\|x\| \leq 1$, let $y = x/\|x\|$, so that $\|y\| = 1$. Then $\|T(y)\| = \|T(x)\|/\|x\| \geq \|T(x)\|$ as $1/\|x\| \geq 1$. This shows that $K_3 \geq K_2$, so that actually $K_2 = K_3$.

Question 2: Let E be a normed vector space, and let $\phi : E \to \mathbb{K}$ be a linear map. When ϕ is bounded, show that

$$\ker \phi = \{x \in E : \phi(x) = 0\} = \phi^{-1}\big(\{0\}\big)$$

is closed.

Answer: Quick proof: as ϕ is continuous, and $\{0\}$ is closed, we have that $\phi^{-1}(\{0\})$ is closed. Longer proof: Let $(x_n) \subseteq \ker \phi$ with $x_n \to x$. Then $\|x_n - x\| \to 0$, so $|\phi(x_n - x)| \leq \|\phi\|\|x_n - x\| \to 0$. However, $\phi(x_n) = 0$ for each n, so $|\phi(x)| = 0$, so $x \in \ker \phi$.

Question continued: Now suppose that ϕ is linear, and we know that $\ker \phi$ is closed in E. We shall show that ϕ is bounded. Firstly, if $\ker \phi = E$, show that ϕ is bounded.

Answer: If $\ker \phi = E$ then $\phi = 0$, and so ϕ is obviously bounded!

Question continued: Now suppose that $\ker \phi \neq E$. Let $x_0 \in E \setminus \ker \phi$. Show that every vector $x \in E$ can be written as

$$x = \lambda x_0 + y$$

for some $\lambda \in K$ and $y \in \ker \phi$. Suppose, towards a contradiction, that ϕ is not bounded, so we can find a sequence (x_n) in E with $\|x_n\| \leq 1$ and $|\phi(x_n)| \geq n$ for each n. By writing each $x_n = \lambda_n x_0 + y_n$ for some $\lambda_n \in \mathbb{K}$ and $y_n \in \ker \phi$, derive a contradiction.

Answer: Following the hint, we calculate that

$$\phi\big(x - \phi(x_0)^{-1}\phi(x)x_0\big) = \phi(x) - \phi(x_0)^{-1}\phi(x)\phi(x_0) = 0.$$

So $y = x - \phi(x_0)^{-1}\phi(x)x_0 \in \ker \phi$, and then

$$x = \frac{\phi(x)}{\phi(x_0)}x_0 + y,$$

as claimed.

We write $x_n = \lambda_n x_0 + y_n$ as suggested. Then $\|\lambda_n x_0 + y_n\| \leq 1$ for each n, and $|\phi(x_n)| = |\lambda_n||\phi(x_0)| \geq n$ for each n. All we know is that $\ker \phi$ is closed. So lets look at

$$z_n = \lambda_n^{-1}x_n = x_0 + \lambda_n^{-1}y_n.$$

Then

$$\|z_n\| = |\lambda_n|^{-1}\|x_n\| \leq |\lambda_n|^{-1} \leq \frac{|\phi(x_0)|}{n} \to 0.$$

However, then $\|x_0 + \lambda_n^{-1}y_n\| \to 0$, so as each vector $(-\lambda_n^{-1}y_n) \in \ker \phi$, and this is a closed subspace, we conclude that $x_0 \in \ker \phi$. This is a contradiction.

Question 3: Let E be a normed vector space, let $\phi \in E^*$, and let $\psi : E \to \mathbb{K}$ be a linear map. Show that if $\ker\phi \subseteq \ker\psi$, then $\psi = \lambda\phi$ for some $\lambda \in \mathbb{K}$, and hence in particular, $\psi \in E^*$.
Answer: If $\ker\phi = E$ then $\phi = 0$, and $E \subseteq \ker\psi$, so $\ker\psi = E$ and hence $\psi = 0 = 0\phi$, as required.

If $\phi \neq 0$ then pick $x_0 \in E$ with $\phi(x_0) \neq 0$. For $x \in E$, notice that $x - \phi(x_0)^{-1}\phi(x)x_0 \in \ker\phi \subseteq \ker\psi$, and so

$$0 = \psi\big(x - \phi(x_0)^{-1}\phi(x)x_0\big) = \psi(x) - \frac{\psi(x_0)}{\phi(x_0)}\phi(x).$$

As x was arbitrary, we conclude that $\psi = \psi(x_0)\phi(x_0)^{-1}\phi$ as required.

Question 4: Let $E = c_0$ and let F be the subspace of all sequences $(x_n) \in c_0$ such that $\sum_{n=1}^\infty 2^{-n}x_n = 0$. Consider the linear map

$$f : c_0 \to \mathbb{K}, \qquad f\big((x_n)\big) = \sum_{n=1}^\infty 2^{-n}x_n \qquad ((x_n) \in c_0).$$

Show that f is bounded with $\|f\| \leq 1$, and hence that F is closed.
Answer: We have that

$$|f((x_n))| \leq \sum_n 2^{-n}|x_n| \leq \|(x_n)\|_\infty \sum_n 2^{-n} = \|(x_n)\|_\infty,$$

so $\|f\| \leq 1$, and hence $F = \ker f$ is closed.
Question continued: Suppose that there exists $x_0 \in E$ with $\|x_0\| \leq 1$ and $\|x_0 - y\| \geq 1$ for each $y \in F$. Show that $f(x_0) = 1$, and hence derive a contradiction.
Answer: Let $\epsilon > 0$, and pick N such that $\sum_{n=1}^N 2^{-n} > 1-\epsilon$. Define $y = (y_n)$ by setting $y_n = 1$ if $n \leq N$, and $y_n = 0$ otherwise. Then $\lim_n y_n = 0$, so that $y \in c_0$. Then $f(y) = \sum_{n=1}^N 2^{-n} > 1-\epsilon$, and $\|y\|_\infty = 1$. As in question 2 above, observe that $z = x_0 - f(y)^{-1}f(x_0)y \in F$, so by the hypthosis,

$$1 \leq \|x_0 - z\| = \big\|x_0 - x_0 + f(y)^{-1}f(x_0)y\big\| = |f(y)|^{-1}|f(x_0)|\|y\| < \frac{|f(x_0)|}{1-\epsilon}.$$

Hence $|f(x_0)| > 1 - \epsilon$, so as $\epsilon > 0$ was arbitrary, we conclude that $|f(x_0)| \geq 1$.

But, now let $x_0 = (x_n) \in c_0$, so $\lim_n x_n = 0$. Hence, for some M, we have that $|x_n| < \frac{1}{2}$ for $n > M$. As $\|x_0\| \leq 1$, we have that $|x_n| \leq 1$ for every n. Hence

$$1 \leq |f(x_0)| = \Big|\sum_{n=1}^M 2^{-n}x_n + \sum_{n>M} 2^{-n}x_n\Big| \leq \sum_{n=1}^M 2^{-n} + \frac{1}{2}\sum_{n>M} 2^{-n} < 1,$$

a contradiction.

Question 5: We work in the Banach space c_0. Define subspaces

$$Y = \big\{(x_n)_{n=1}^\infty \in c_0 : x_{2k-1} = 0 \text{ for } k = 1, 2, 3, \cdots\big\}$$
$$Z = \big\{(x_n)_{n=1}^\infty \in c_0 : x_{2k} = k^2 x_{2k-1} \text{ for } k = 1, 2, 3, \cdots\big\}.$$

Show that Y and Z are closed subspaces.
Answer: For each k, the map

$$\phi_k : c_0 \to \mathbb{K}, \quad (x_n) \mapsto x_{2k-1}$$

is linear and bounded (as $\|\phi_k\| = 1$). Then Y is the intersection $\bigcap_{k\geq 1}\ker\phi_k$, which is closed, as each $\ker\phi_k$ is closed.

Similarly, define

$$\psi_k : c_0 \to \mathbb{K}, \quad (x_n) \mapsto x_{2k} - k^2 x_{2k-1}.$$

Clearly ψ_k is linear, and $|\psi_k((x_n))| \leq |x_{2k}| + k^2|x_{2k-1}| \leq (k^2+1)\|(x_n)\|_\infty$, so ψ_k is bounded (and actually, $\|\psi_k\| = k^2+1$). Then $Z = \bigcap_{k\geq 1} \ker \psi_k$ is also closed.

Question continued: Show that the vector $x = (1, 0, 1/4, 0, 1/9, 0, 1/16, 0, \cdots)$ is in the closure of the subspace $Y + Z$. That is, for each $\epsilon > 0$, you need to find $y \in Y$ and $z \in Z$ with $\|x - (y+z)\|_\infty < \epsilon$.

Answer: Let $x = (x_n)$, so that $x_{2k} = 0$ for each k, and $x_{2k-1} = 1/k^2$, for each k. Pick $\epsilon > 0$, and pick K with $1/K^2 < \epsilon$.

We have little choice but to set $z = (1, 1, 1/4, 1, 1/9, 1, \cdots, 1/K^2, 1, 0, 0, \cdots)$, that is,

$$z_{2k-1} = 1/k^2, \quad z_{2k} = 1 \qquad (1 \leq k \leq K),$$

and $z_{2k-1} = z_{2k} = 0$ for $k > K$. Thus $z \in Z$. Then we set $y = (0, 1, 0, 1, \cdots, 1, 0, 0, \cdots)$, that is, $y_{2k-1} = 0$ for all k, and $y_{2k} = 1$ for $1 \leq k \leq K$, while $y_{2k} = 0$ for $k > K$. Thus $y \in Y$. Then $y + z = (1, 0, 1/4, 0, 1/9, 0, \cdots, 1/K^2, 0, 0, \cdots)$, so $\|x - (y+z)\|_\infty = 1/(K+1)^2 < \epsilon$, as required.

Question continued: Show, however, that x is not in $Y + Z$.

Answer: Suppose that we can find $y \in Y$ and $z \in Z$ with $x = y + z$. As $y_{2k-1} = 0$ for all k, we must have that $z_{2k-1} = x_{2k-1} = 1/k^2$ for all k. As $z \in Z$, we have that $z_{2k} = k^2 z_{2k-1} = 1$ for all k. However, $z \in c_0$, so $z_k \to 0$ as $k \to \infty$, a contradiction.

Question 6: Show that $c_0^* = \ell^1$. That is, for $a = (a_n) \in \ell^1$, define $\phi_a : c_0 \to \mathbb{K}$ by

$$\phi_a(x) = \sum_{n=1}^\infty a_n x_n \qquad (x = (x_n) \in c_0).$$

Show that ϕ_a is linear, bounded, and that $\|\phi_a\| \leq \|a\|_1$.

Answer: Notice that ϕ_a is defined, as

$$\Big|\sum_{n=1}^\infty a_n x_n\Big| \leq \sum_{n=1}^\infty |a_n||x_n| \leq \sum_{n=1}^\infty \|x\|_\infty |a_n| = \|a\|_1 \|x\|_\infty.$$

Thus also ϕ_a is bounded, with $\|\phi_a\| \leq \|a\|_1$. Also, ϕ_a is linear, for given $x = (x_n), y = (y_n) \in c_0$ and $t \in \mathbb{K}$,

$$\phi_a(x + ty) = \sum_{n=1}^\infty a_n(x_n + ty_n) = \sum_{n=1}^\infty a_n x_n + t\sum_{n=1}^\infty a_n y_n = \phi_a(x) + t\phi_a(y).$$

Question continued: Hence the map $\ell^1 \to c_0^*; a \mapsto \phi_a$ is linear and bounded. We wish to show that this is a bijection and an isometry.

Answer: Let $\phi \in c_0^*$. For each n, let $e_n \in c_0$ be the sequence which is zero, except that in the nth place, we have 1. Let $a_n = \phi(e_n)$ for all n.

Fix some large $N \in \mathbb{N}$. For each n, define

$$x_n = \begin{cases} 0 & : a_n = 0 \text{ or } n > N, \\ \overline{a_n}/a_n & : a_n \neq 0. \end{cases}$$

Thus $\lim_n x_n = 0$, so $x = (x_n) \in c_0$. Notice also that $|x_n| = 1$ or 0 for all n, so $\|x\|_\infty \leq 1$. Finally, notice that

$$x = \sum_{n=1}^N x_n e_n.$$

Thus

$$\phi(x) = \sum_{n=1}^{N} \phi(x_n e_n) = \sum_{n=1}^{N} x_n \phi(e_n) = \sum_{n=1}^{N} x_n a_n = \sum_{n=1}^{N} a_n \overline{a_n}/a_n = \sum_{n=1}^{N} |a_n|.$$

But $|\phi(x)| \leq \|\phi\| \|x\|_\infty \leq \|\phi\|$. By letting N tend to infinity, we conclude that

$$\sum_{n=1}^{\infty} |a_n| \leq \|\phi\|.$$

So $a = (a_n) \in \ell^1$ with $\|a\|_1 \leq \|\phi\|$.

For any $y = (y_n) \in c_0$, we observe that

$$\Big\| y - \sum_{n=1}^{N} y_n e_n \Big\|_\infty = \sup_{n>N} |y_n|,$$

which converges to 0 as $N \to \infty$, because $\lim_n y_n = 0$. So

$$y = \sum_{n=1}^{\infty} y_n e_n$$

which convergence in norm. As ϕ is bounded and hence continuous,

$$\phi(y) = \sum_{n=1}^{\infty} \phi(y_n e_n) = \sum_{n=1}^{\infty} y_n a_n = \phi_a(y).$$

So $\phi_a = \phi$, and so the map $\ell^1 \to c_0^*$ is surjective. Notice also that $\|\phi\| = \|\phi_a\| \leq \|a\|_1 \leq \|\phi\|$, so we have equality throughout. Hence our map $\ell^1 \to c_0^*$ is an isometry, and hence injective, and so bijective.

Question 7: Recall that ℓ^∞ is the space of all bounded scalar sequences (x_n) with the norm $\|\cdot\|_\infty$. Show that $(\ell^1)^* = \ell^\infty$.

Answer: For $u = (u_n) \in \ell^\infty$ define $\phi_u : \ell^1 \to \mathbb{K}$ by

$$\phi_u(x) = \sum_{n=1}^{\infty} x_n u_n \qquad (x = (x_n) \in \ell^1).$$

This is well-defined, as

$$\Big| \sum_{n=1}^{\infty} x_n u_n \Big| \leq \|u\|_\infty \|x\|_1,$$

and so we see that $\phi_u \in (\ell^1)^*$ with $\|\phi_u\| \leq \|u\|_\infty$.

Let $\phi \in (\ell^1)^*$, let $e_n \in \ell^1$ be the usual sequence which is zero, apart from a one in the nth place. Let $u_n = \phi(e_n)$, so that $|u_n| \leq \|\phi\| \|e_n\| = \|\phi\|$. Hence $u = (u_n) \in \ell^\infty$ with $\|u\|_\infty \leq \|\phi\|$. Let $x = (x_n) \in \ell^1$ and observe that

$$\lim_N \Big\| x - \sum_{n=1}^{N} x_n e_n \Big\|_1 = \lim_N \sum_{n=N+1}^{\infty} |x_n| = 0,$$

as $\sum_n |x_n|$ converges. Thus

$$\phi(x) = \lim_N \sum_{n=1}^{N} \phi(x_n e_n) = \lim_N \sum_{n=1}^{N} x_n u_n = \phi_u(x).$$

As $x \in \ell^1$ was arbitrary, we conclude that $\phi = \phi_u$ and that

$$\|\phi_u\| \leq \|u\|_\infty \leq \|\phi\| = \|\phi_u\|.$$

Hence the map $\ell^\infty \to (\ell^1)^*; u \mapsto \phi_u$ is an isometric isomorphism of Banach spaces, as required.

Linear Analysis I: Worked Solutions 3

Answer 1: Let $\phi \in E^*$ with $\|\phi\| \leq 1$ and $\phi(y) = 0$ for all $y \in F$. Then, for $y \in F$,

$$|\phi(x_0)| = |\phi(x_0 - y)| \leq \|\phi\|\|x_0 - y\| \leq \|x_0 - y\|.$$

Hence taking the infimum, we conclude that

$$|\phi(x_0)| \leq d(x_0, F),$$

as required. We define $\psi : \text{lin}\{F, x_0\} \to \mathbb{K}$ by

$$\psi(\lambda x_0 + y) = \lambda d(x_0, F) \qquad (\lambda \in \mathbb{K}, y \in F).$$

If $x_0 \in F$, then $d(x_0, F) = 0$, so $\psi = 0$. Otherwise, if $\lambda x_0 + y = \mu x_0 + z$ then $(\lambda - \mu)x_0 = z - y \in F$, and so $\lambda = \mu$, so we can conclude that ψ is well-defined. Obviously ψ is linear. Let $\lambda \in \mathbb{K}$ and $y \in F$. If $\lambda = 0$ then $\psi(\lambda x_0 + y) = 0 \leq \|\lambda x_0 + y\|$. Otherwise, we have that

$$d(x_0, F) \leq \|x_0 + \lambda^{-1} y\| = |\lambda|^{-1}\|\lambda x_0 + y\|,$$

and so $|\lambda| d(x_0, F) = |\psi(\lambda x_0 + y)| \leq \|\lambda x_0 + y\|$. Hence $\|\psi\| \leq 1$. By the Hahn-Banach theorem, there exists $\phi \in E^*$ extending ψ with $\|\phi\| \leq \|\psi\| \leq 1$. As $\phi(x_0) = \psi(x_0) = d(x_0, F)$, we are done.

Question 2: Let $1 \leq p < \infty$, and define a map $S : \ell^p \to \ell^p$ by setting $S(x) = y$ where, if $x = (x_1, x_2, x_3, \cdots)$, then $y = (0, x_1, x_2, x_3, \cdots)$. Show that S is linear, bounded, and satisfies $\|S\| = 1$.

Show that there is a bounded linear map $T \in \mathcal{B}(\ell^p)$ such that $T \circ S$ is the identity on ℓ^p. Is $S \circ T$ the identity? Is S invertible in $\mathcal{B}(\ell^p)$?

Answer: Clearly S is linear, and observe that

$$\|S(x)\|_p = \Big(0^p + \sum_{n=1}^{\infty} |x_n|^p\Big)^{1/p} = \|x\|_p \qquad (x \in \ell^p),$$

so that S is even an isometry.

Let T be the "left-shift", that is, $T(x) = y$ where is $x = (x_1, x_2, x_3, \cdots)$ then $y = (x_2, x_3, x_4, \cdots)$. Similarly T is linear, bounded and satisfies $\|T\| \leq 1$. Clearly $TS = I_{\ell^p}$ the identity on ℓ^p. However, $ST(x_1, x_2, \cdots) = (0, x_2, x_3, \cdots)$, so that ST is not the identity.

Suppose that S is invertible with inverse S^{-1}. Then $S^{-1} = I_{\ell^p} S^{-1} = TSS^{-1} = TI_{\ell^p} = T$, but then $ST = SS^{-1} = I_{\ell^p}$, a contradiction.

Question 3: Let X be a compact topological space (remember that we always assume the Hausdorff condition). Fix $f \in C_{\mathbb{K}}(X)$, and define $M_f : C_{\mathbb{K}}(X) \to C_{\mathbb{K}}(X)$ by setting $M_f(g) = gf$ for $g \in C_{\mathbb{K}}(X)$. Show that $M_f \in \mathcal{B}(C_{\mathbb{K}}(X))$, and calculate $\|M_f\|$.

Answer: Clearly M_f is linear, and for $g \in C_{\mathbb{K}}(X)$,

$$\|M_f(g)\|_\infty = \|fg\|_\infty = \sup_{x \in X} |f(x)||g(x)| \leq \|f\|_\infty \sup_{x \in X} |g(x)| = \|f\|_\infty \|g\|_\infty.$$

Hence $\|M_f\| \leq \|f\|_\infty$. The constant function 1 is in $C_{\mathbb{K}}(X)$, with $\|1\|_\infty = 1$, and we see that $M_f(1) = f$, so that $\|M_f\| \geq \|M_f(1)\|_\infty = \|f\|_\infty$. Hence $\|M_f\| = \|f\|_\infty$.

Question 4: Show that if

$$\inf \big\{ |f(x)| : x \in X \big\} > 0,$$

then there exists $h \in C_{\mathbb{K}}(X)$ with $M_h M_f = M_f M_h$ being the identity on $C_{\mathbb{K}}(X)$. If $\inf\big\{|f(x)| : x \in X\big\} = 0$, then is M_f invertible?
Answer: Let $h = f^{-1}$, so for $g \in C_{\mathbb{K}}(X)$,

$$M_h M_f(g) = M_h(fg) = fhg = M_f(hg) = M_f M_h(g).$$

Hence $M_h M_f = M_f M_h$ and as $fhg = ff^{-1}g = g$, $M_h M_f$ is the identity on $C_{\mathbb{K}}(X)$.

Suppose now that $\inf\big\{|f(x)| : x \in X\big\} = 0$, and yet $T = M_f^{-1}$ exists. Let $h = T(1)$, so that

$$fh = M_f(h) = M_f T(1) = M_f M_f^{-1}(1) = 1,$$

and so $f(x)h(x) = 1$ for all $x \in X$, that is, $h(x) = f(x)^{-1}$. Hence

$$\inf\big\{|f(x)| : x \in X\big\} = \sup\big\{|f(x)|^{-1} : x \in X\big\}^{-1} = \|h\|_\infty^{-1} > 0,$$

a contradiction.

Aside: Upon re-reading this, I realise that I have not used that X is compact. As X is compact, $|f|$ attains its minimum, so either f is bounded below, or there exists $x \in X$ with $f(x) = 0$. This would make the proof a little easier.

Question 5: Let E and F be normed spaces, and let $T \in \mathcal{B}(E, F)$. Show that the following are equivalent:

1. T is invertible;
2. T is surjective, and there exists $M > 0$ such that, for all $x \in E$,

$$M^{-1}\|x\| \leq \|T(x)\| \leq M\|x\|.$$

Answer: If (1) holds, then first note that for $y \in F$, then $T(T^{-1}(y)) = y$, so that T is surjective. For $x \in E$,

$$\|x\| = \|T^{-1}T(x)\| \leq \|T^{-1}\|\|T(x)\| \leq \|T^{-1}\|\|T\|\|x\|,$$

so that (2) holds with $M = \max\{\|T\|, \|T^{-1}\|\}$.

If (2) holds, then suppose that $T(x) = T(y)$ for $x, y \in E$. Then $T(x - y) = 0$, so that $M^{-1}\|x - y\| \leq \|T(x - y)\| = 0$, so that $\|x - y\| = 0$, that is, $x = y$. Hence T is injective, and surjective, and so T^{-1} exists. By basic linear algebra, T^{-1} is linear. Then, for $y \in F$, let $x \in E$ be such that $T(x) = y$, so that

$$\|T^{-1}(y)\| = \|T^{-1}(T(x))\| = \|x\| \leq M\|T(x)\| = M\|y\|.$$

Hence T^{-1} is bounded, with $\|T^{-1}\| \leq M$.

Question 6: We define a *measure space* to be a triple $(X, \mathcal{R}, \mu)$ where X is a set, $\mathcal{R}$ is a σ-algebra on X and μ is a measure defined on $\mathcal{R}$. Let $Y \in \mathcal{R}$, and define $\mathcal{R}_Y$ by

$$\mathcal{R}_Y = \{S \cap Y : S \in \mathcal{R}\}.$$

Show that $\mathcal{R}_Y$ is a σ-algebra on Y. Define $\mu_Y : \mathcal{R}_Y \to [0, \infty]$ by $\mu_Y(S) = \mu(S \cap Y)$ for $S \in \mathcal{R}_Y$. Show that μ_Y is a measure on $\mathcal{R}_Y$.
Answer: Clearly $\emptyset \in \mathcal{R}_Y$, and as $Y = X \cap Y$, we see that $Y \in \mathcal{R}_Y$. Let $S \cap Y, T \cap Y \in \mathcal{R}_Y$ so that $(S \cap Y) \setminus (T \cap Y) = (S \setminus T) \cap Y \in \mathcal{R}_Y$ as $S \setminus T \in \mathcal{R}$. If (T_n) is a sequence in $\mathcal{R}_Y$, say $T_n = S_n \cap Y$ for some sequence (S_n) in $\mathcal{R}$. Then $S = \bigcup_n S_n \in \mathcal{R}$, so that $\bigcup_n T_n = S \cap Y \in \mathcal{R}_Y$. Hence $\mathcal{R}_Y$ is a σ-algebra on Y (notice that we didn't use that $Y \in \mathcal{R}$).

As $Y \in \mathcal{R}$, for $S \in \mathcal{R}$, we have that $S \cap Y \in \mathcal{R}$, so that $\mu(S \cap Y)$ is defined. Clearly $\mu_Y(\emptyset) = 0$, and if $(S_n \cap Y)$ is a sequence of pairwise-disjoint sets in Y, then

$$\mu_Y\Big(\bigcup_n (S_n \cap Y)\Big) = \mu\Big(\bigcup_n (S_n \cap Y)\Big) = \sum_n \mu(S_n \cap Y) = \sum_n \mu_Y(S_n \cap Y),$$

so that μ_Y is a measure.

Question 7: Let $(X, \mathcal{R}, \mu)$ be a measure space. Define $\overline{\mathcal{R}}$ to be the collection of sets $E \cup N$ where $E \in \mathcal{R}$, and $N \subseteq X$ is a null set. Show that:

1. If (N_n) is a sequence of null sets, then $\bigcup_n N_n$ is null.
2. If $E \cup N \in \overline{\mathcal{R}}$, and M is null, then $(E \cup N) \setminus M \in \overline{\mathcal{R}}$.

Show that $\overline{\mathcal{R}}$ is a σ-algebra.

Answer: For (1), as each N_n is null, there exists $F_n \in \mathcal{R}$ with $N_n \subseteq F_n$ and $\mu(F_n) = 0$. Let $F = \bigcup_n F_n$ so that $\bigcup_n N_n \subseteq F$, and $\mu(F) \leq \sum_n \mu(F_n) = 0$, as μ is a measure.

For (2), notice that

$$(E \cup N) \setminus M = (E \setminus M) \cup (N \setminus M).$$

Clearly $N \setminus M$ is null. As M is null, $M \subseteq F$ for some $F \in \mathcal{R}$ with $\mu(F) = 0$. Then

$$E \setminus M = (E \setminus F) \cup (F \setminus M),$$

so as $F \setminus M \subseteq F$, we have that $F \setminus M$ is null. By (1), we see that $(F \setminus M) \cup (N \setminus M)$ is null. As $E \setminus F \in \mathcal{R}$, we conclude that $E \setminus M \in \overline{\mathcal{R}}$, as required.

Clearly $\emptyset, X \in \overline{\mathcal{R}}$. By (1), if $(E_n \cup N_n)$ is a sequence in $\overline{\mathcal{R}}$, then $\bigcup_n (E_n \cup N_n) = \bigcup_n E_n \cup \bigcup_n N_n \in \overline{\mathcal{R}}$. Let $E_1 \cup N_1, E_2 \cup N_2 \in \overline{\mathcal{R}}$, so that

$$(E_1 \cup N_1) \setminus (E_2 \cup N_2) = \big((E_1 \cup N_1) \setminus E_2\big) \setminus N_2.$$

By (2), if $(E_1 \cup N_1) \setminus E_2 \in \overline{\mathcal{R}}$, then $(E_1 \cup N_1) \setminus (E_2 \cup N_2) \in \overline{\mathcal{R}}$. Notice that

$$(E_1 \cup N_1) \setminus E_2 = (E_1 \setminus E_2) \cup (N_1 \setminus E_2).$$

Here $E_1 \setminus E_2 \in \mathcal{R}$ and $N_1 \setminus E_2 \subseteq N_1$ is null, so we are done.

Question continued: Define $\overline{\mu} : \overline{\mathcal{R}} \to [0, \infty]$ by $\overline{\mu}(E \cap N) = \mu(E)$ for $E \in \mathcal{R}$ and any null set N. Show that $\overline{\mu}$ is a measure on $\overline{\mathcal{R}}$.

Answer: First we should check that $\overline{\mu}$ is well-defined. That is, suppose that $E \cup N = E' \cup N'$ for some $E, E' \in \mathcal{R}$ and null sets N and N'. Then we can find $F, F' \in \mathcal{R}$ with $N \subseteq F$, $N' \subseteq F'$ and $\mu(F) = \mu(F') = 0$. Then $\mu(E) \leq \mu(E \cup F) \leq \mu(E) + \mu(F) = \mu(E)$, so that $\mu(E \cup F) = \mu(E)$. Similarly $\mu(E' \cup F') = \mu(E')$. Finally, as $E \subseteq E \cup N = E' \cup N' \subseteq E' \cup F'$, we see that $\mu(E) \leq \mu(E' \cup F') = \mu(E')$. By symmetry, also $\mu(E') \leq \mu(E)$, so we conclude that $\mu(E) = \mu(E')$. Hence $\overline{\mu}$ is well-defined.

Clearly $\overline{\mu}(\emptyset) = 0$. Let (A_n) be a sequence of pairwise disjoint sets in $\overline{\mathcal{R}}$, say $A_n = E_n \cup N_n$, for each n, where $E_n \in \mathcal{R}$ and N_n is null. Then (E_n) is pairwise disjoint. Observe that

$$\bigcup_n A_n = \bigcup_n E_n \cup \bigcup_n N_n,$$

where as above, $N = \bigcup_n N_n$ is null. Thus

$$\overline{\mu}\Big(\bigcup A_n\Big) = \mu\Big(\bigcup E_n\Big) = \sum_n \mu(E_n) = \sum_n \overline{\mu}(E_n \cup N_n) = \sum_n \overline{\mu}(A_n).$$

So $\overline{\mu}$ is a measure.

Bonus Question 8: For $x, y \in E$ and $t \in \mathbb{K}$, we have, for $f \in E^*$,

$$J(x+ty)(f) = f(x+ty) = f(x) + tf(y) = J(x)(f) + tJ(y)(f).$$

Thus $J(x+ty) = J(x) + tJ(y)$, so that J is linear.

For $x \in E$,

$$\|J(x)\| = \sup\{|J(x)(f)| : f \in E^*, \|f\| \leq 1\} = \sup\{|f(x)| : f \in E^*, \|f\| \leq 1\} = \|x\|,$$

by using Corollary ??? from lectures.

Bonus Question 9: The isometric isomorphism from ℓ^q to $(\ell^p)^*$ is $u \mapsto \phi_u$ where, for $u = (u_n) \in \ell^q$, we have that

$$\phi_u : \ell^p \to \mathbb{K}, \quad \phi_u((x_n)) = \sum_{n=1}^{\infty} x_n u_n.$$

Let this be $\phi : \ell^q \to (\ell^p)^*$. Similarly, let $\psi : \ell^p \to (\ell^q)^*$.

We need to show that J is surjective. Let $F \in (\ell^p)^{**}$. Define $g \in (\ell^q)^*$ by

$$g(\phi^{-1}(f)) = F(f) \qquad (f \in (\ell^p)^*).$$

An equivalent (and less scary) way to define this is as

$$g(u) = F(\phi_u) \qquad (u \in \ell^q).$$

Clearly g is linear, as both F and ϕ are. Also, g is bounded, as $\|g\| \leq \|F\|\|\phi\| = \|F\|$. So $g \in (\ell^q)^*$ as required.

Let $x = (x_n) \in \ell^p$ with $\psi_x = g$. Let $f \in (\ell^p)^*$, and let $u = (u_n) \in \ell^q$ with $\phi_u = f$. Then

$$J(x)(f) = f(x) = \phi_u(x) = \sum_{n=1}^{\infty} x_n u_n = \psi_x(u) = g(u) = F(\phi_u) = F(f).$$

As f was arbitrary, we conclude that $J(x) = F$. Thus J is surjective.

Linear Analysis I: Worked Solutions 4

Question 1: Let E be a Banach space, and let $(x_n)_{n=1}^\infty$ be a sequence of vectors in E such that $\sum_{n=1}^\infty \|x_n\| < \infty$. Show that $\sum_{n=1}^\infty x_n$ converges.
Answer: For $N < M$, we see that by the triangle inequality

$$\Big\| \sum_{n=N}^{M} x_n \Big\| \leq \sum_{n=N}^{M} \|x_n\| \leq \sum_{n=N}^{\infty} \|x_n\|,$$

which is small if N is large, as $\sum_n \|x_n\| < \infty$. So the sequence of partial sums

$$\Big(\sum_{n=1}^{N} x_n \Big)_{N\geq 1}$$

is a Cauchy sequence and hence converges, as E is a Banach space.
Question continued: Let (z_n) be a Cauchy sequence in E. Show that we can find $1 = n(1) < n(2) < \cdots$ such that, if

$$x_1 = z_1, \quad x_k = z_{n(k)} - z_{n(k-1)} \qquad (k \geq 2),$$

then $\sum_n \|x_n\| < \infty$. What is $\sum_{n=1}^N x_n$? Conclude that if $z = \sum_n x_n$ that z is the limit of the Cauchy sequence (z_n).
Answer: As (z_n) is a Cauchy sequence, for each m we can find N_m such that

$$\|z_k - z_l\| < 2^{-m} \qquad (k, l \geq N_m).$$

Set $n(1) = 1$ as required, and then choose $n(k)$ arbitrarily, with the condition that $n(k) \geq N_k$ for all k, and $n(1) < n(2) < \cdots$. Then, as $x_k = z_{n(k)} - z_{n(k-1)}$ and $n(k) > n(k-1) \geq N_{k-1}$, we see that $\|x_k\| < 2^{-(k-1)}$. Thus

$$\sum_k \|x_k\| = \|x_1\| + \sum_{k\geq 2} \|x_k\| \leq \|z_1\| + \sum_{k\geq 2} 2^{1-k} = 1 + \|z_1\| < \infty.$$

Notice also that

$$\sum_{n=1}^{N} x_n = z_1 + (z_{n(2)} - z_1) + (z_{n(3)} - z_{n(2)}) + \cdots + (z_{n(N)} - z_{n(N-1)}) = z_{n(N)}.$$

So if $z = \sum_n x_n$ then $\lim_k z_{n(k)} = z$, so $(z_{n(k)})_k$ converges. This implies that (z_n) converges, as required.

Question 2: Let X be a compact (Hausdorff) space. Let $\phi : X \to X$ be a continuous map. Show that we can define a linear map $T : C_\mathbb{R}(X) \to C_\mathbb{R}(X)$ by

$$T(f) = g \quad \text{where} \quad g(x) = f(\phi(x)).$$

Show that T is bounded, and find $\|T\|$.
Proof: As ϕ is continuous, for $f : X \to \mathbb{R}$ continuous, $x \mapsto f(\phi(x))$ is continuous. So $T(f) \in C_\mathbb{R}(X)$.

We write $T(f)(x) = f(\phi(x))$ for $x \in X$, so that for $f_1, f_2 \in C_\mathbb{R}(X)$ and $\lambda \in \mathbb{R}$, $T(f_1 + \lambda f_2)(x) = f_1(\phi(x)) + \lambda f_2(\phi(x)) = T(f_1)(x) + \lambda T(f_2)(x)$. So T is linear.

Then notice that

$$\|T(f)\|_\infty = \sup_{x\in X} |T(f)(x)| = \sup_x |f(\phi(x))| \leq \sup_x |f(x)| = \|f\|_\infty,$$

so T is bounded with $\|T\| \leq 1$. As $T(1) = 1$, where 1 is the constant function, we see that $\|T\| = 1$.

Bonus Question 3: With notation as in Question 2, now let $X = [0, 1]$ and let ϕ be defined by

$$\phi(t) = \frac{1}{2} + \frac{t - \frac{1}{2}}{2} \qquad (0 \leq t \leq 1).$$

So $\phi(1/2) = 1/2$, $\phi(0) = 1/4$ and $\phi(1) = 3/4$. Define T as in Question 2. Let $T^2 = TT, T^3 = TTT$ and so forth.

Show that for each $f \in C_{\mathbb{R}}([0, 1])$,

$$\lim_{n\to\infty} T^n(f) = g$$

where $g(t) = f(1/2)$ for all $t \in [0, 1]$. That is, g is a constant function.

Proof: Motivated by the contractive mapping theorem, we look at the iterates of ϕ. Clearly ϕ maps $[0, 1]$ onto $[1/4, 3/4]$. Then ϕ maps $[1/4, 3/4]$ onto $[1/2 - 1/8, 1/2 + 1/8] = [3/8, 5/8]$, so that ϕ^2 maps $[0, 1]$ onto $[3/8, 5/8]$. We can show (by induction) that ϕ^n maps $[0, 1]$ onto $[1/2 - 2^{-1-n}, 1/2 + 2^{-1-n}]$. For $f \in C_{\mathbb{R}}([0, 1])$, as f is continuous at $1/2$, for each $\epsilon > 0$ there exists N so that, for $n \geq N$, if $|t - 1/2| < 2^{-1-n}$ then $|f(t) - f(1/2)| < \epsilon$. Thus $|f(\phi^n(t)) - f(1/2)| < \epsilon$ for any $t \in [0, 1]$, that is, $\|T^n(f) - g\|_\infty < \epsilon$. As this was true for all $n \geq N$, we see that $T^n(f) \to g$, as required.

Question continued: Is it true that (T^n) converges in the Banach space $\mathcal{B}(C_{\mathbb{R}}([0, 1]))$?

Proof: Suppose that $T^n \to S$ in norm. Then, for each $f \in C_{\mathbb{R}}([0, 1])$, we have that $T^n(f) \to S(f)$, so $S(f)(t) = f(1/2)$ for all $t \in [0, 1]$. That is, S maps f to the constant function $t \mapsto f(1/2)$.

Hopefully, our intuition from the previous section is that the more f oscillates, the slower the convergence of $T^n(f)$ is. Let $N > 0$ and let $f(t) = \sin(4\pi N t)$ for $t \in [0, 1]$. Then

$$f(1/2) = \sin(2\pi N) = 0, \quad f(1/2 + 1/8N) = \sin(2\pi N + \pi/2) = \sin(\pi/2) = 1.$$

Hence $T^n(f) \to 0$, so $S(f) = 0$. Thus, as ϕ^n maps $[0, 1]$ onto $[1/2 - 2^{-1-n}, 1/2 + 2^{-1-n}]$, if $1/8N \leq 2^{-1-n}$, then $\|T^n(f) - 0\|_\infty = 1$. But $\|f\|_\infty = 1$, so choose N with $1/8N \leq 2^{-1-n}$ to see that

$$\|T^n - S\| \geq \|T^n(f) - S(f)\|_\infty = 1.$$

So (T^n) does not converge to S.

Comment: Saying that $T^n(f)$ converges for each f is saying that (T^n) converges in the *strong operator topology*. Clearly norm convergence implies strong operator convergence, and we have just seen that the converse doesn't hold.

Question 4: Let $(X, \mathcal{R}, \mu)$ be a measure space. Let $f : X \to \mathbb{R}$ be a simple function (see the definition from the lectures). Show carefully that f is measurable, and that f takes finitely many values.

Proof: Let $f = \sum_{k=1}^n t_k \chi_{A_k}$ where (A_k) is a pairwise disjoint family in $\mathcal{R}$ and $(t_k) \subseteq \mathbb{R}$. Let $A_0 = X \setminus (A_1 \cup \cdots \cup A_n) \in \mathcal{R}$. Let $U \subseteq \mathbb{R}$ be open, and define $E \subseteq \{0, 1, \cdots, n\}$ by $0 \in E$ if and only if $0 \in U$, and for $1 \leq k \leq n$, $k \in E$ if and only if $t_k \in U$. You should hopefully see that

$$f^{-1}(U) = \bigcup_{k\in E} A_k \in \mathcal{R}.$$

So f is measurable.

Clearly f only takes the values $\{t_1, t_2, \cdots, t_k\}$, and possibly also 0 if $A_0 \neq \emptyset$.

Question continued: Conversely, show that if $f : X \to \mathbb{R}$ is measurable and takes finitely many values, then f is a simple function.

Proof: Suppose that f takes only the values $\{t_1, \cdots, t_n\}$. Let $A_k = f^{-1}(\{t_k\})$, for $1 \leq k \leq n$. As $\{t_k\}$ is closed in $\mathbb{R}$, the set $\mathbb{R} \setminus \{t_k\}$ is open, and so

$$X \setminus A_k = f^{-1}(\mathbb{R} \setminus \{t_k\}) \in \mathcal{R},$$

so also $A_k \in \mathcal{R}$. (Remember that taking inverse images commutes with unions, intersections and set differences). By definition, (A_k) is a pairwise disjoint family, and so clearly $f = \sum_{k=1}^n t_k \chi_{A_k}$ is a simple function.

Question continued: In particular, show that if $(A_k)_{k=1}^n$ is any collection of subsets of $\mathcal{R}$, and $(t_k)_{k=1}^n \subseteq \mathbb{R}$, then

$$f = \sum_{k=1}^n t_k \chi_{A_k}$$

is simple.

Proof: Just observe that f can only possibly take the values

$$\{0, t_1, \cdots, t_n, t_1 + t_2, \cdots, t_1 + t_n, t_2 + t_3, \cdots, t_2 + t_n, \cdots, t_1 + \cdots + t_n\},$$

which is a finite set.

Question 5: Let X be a set, let $\mathcal{R} = 2^X$, and let μ be the *counting measure* on $\mathcal{R}$, so $\mu(A)$ is the size of A, if A is finite, and is ∞ otherwise. Which functions $f : X \to \mathbb{R}$ are measurable?

Answer: As every subset of X is in $\mathcal{R}$, we see that *any* function $f : X \to \mathbb{R}$ is measurable.

Question Continued: Let $f : X \to [0, \infty)$ be a simple function. Show that f is integrable if and only if f is zero except at finitely many points of X. Conversely, show that if $f : X \to [0, \infty)$ is any function which is zero except at finitely many points, then f is an integrable, simple function.

Answer: Write a simple function $f : X \to [0, \infty)$ as

$$f = \sum_{k=1}^n t_k \chi_{A_k},$$

where we may assume the (A_k) are pairwise disjoint. Then f is integrable if and only if $\mu(A_k) = \infty$ only when $t_k = 0$. As μ is counting measure, we have that $\mu(A_k) = \infty$ if and only if A_k is infinite. Hence f is non-zero only on a finite set.

Conversely, if $f : X \to [0, \infty)$ is non-zero only on a finite set, say A, then we can write

$$f = \sum_{x \in A} f(x) \chi_{\{x\}},$$

a simple function.

Question 6: Let $(X, \mathcal{R}, \mu)$ be a measure space. A function $f : X \to \mathbb{R}$ is *measurable* if $f^{-1}(U) \in \mathcal{R}$ for any open set $U \subseteq \mathbb{R}$. Let $f : X \to \mathbb{R}$ be a function such that $f^{-1}((x, y)) \in \mathcal{R}$ for any $x, y \in \mathbb{R}$ with $x < y$. By thinking about the proof of Corollary 2.7, show that f is measurable.

Answer: Let $D = \{(a,b) : a,b \in \mathbb{Q}, a < b\}$, a countable set of open sets in $\mathbb{R}$. Let $U \subseteq \mathbb{R}$ be open, so for $x \in U$, there exists $(a,b) \in D$ with $x \in (a,b)$ and $(a,b) \subseteq U$. Let $D_U = \{(a,b) \in D : (a,b) \subseteq U\}$, so that $U = \bigcup D_U$. Hence

$$f^{-1}(U) = \bigcup f^{-1}(D_U) \in \mathcal{R},$$

as D_U is countable and $f^{-1}(a,b) \in \mathcal{R}$ for each $(a,b) \in D$. Hence f is measurable.

Question 7: We work with notation as in Question 5. Which measurable functions $f : X \to [0,\infty)$ are integrable? What about functions $f : X \to \mathbb{R}$? You might find it easier to assume that $X = \mathbb{N}$ here.
Answer: Suppose that $f : X \to [0,\infty)$ is integrable. Let $A \subseteq X$ be a finite set, and let $f_A = f\chi_A$. Then f_A is non-zero only on A, so f_A is a simple function, and is integrable. By definition,

$$\sum_{x\in A} f(x) = \sum_{x\in A} f(x)\mu(\{x\}) = \int_X f_A \ d\mu \leq \int_X f \ d\mu < \infty.$$

Hence we see that

$$\sup_{A\subseteq X \text{ finite}} \sum_{x\in A} f(x) < \infty.$$

(This was perhaps a little unfair of me. For positive functions on an infinite, possibly uncountable, set, we *define* $\sum_{x\in X} f(x)$ to be the supremum. I doubt you have seen this before). Conversely, if this supremum is finite, then it is easy to check, by using the previous bit of the question, that if $g : X \to [0,\infty)$ is simple and integrable, with $g \leq f$, then $\int_X g \ d\mu$ is less than the supremum, and hence f is integrable.

By definition, $f : X \to \mathbb{R}$ is integrable if and only if f_+ and f_- are, which is if and only if $|f|$ is integrable. That is, if

$$\sup_{A\subseteq X \text{ finite}} \sum_{x\in A} |f(x)| < \infty.$$

Question Continued: Show that if $X = \mathbb{N}$, then we can identify ℓ^1 with the space of integrable functions $f : X \to \mathbb{R}$.
Answer: $f : \mathbb{N} \to \mathbb{R}$ is integrable if and only if

$$\sup_{A\subseteq \mathbb{N} \text{ finite}} \sum_{n\in A} |f(n)| < \infty.$$

We *claim* that this is equivalent to $\sum_{n=1}^\infty |f(n)| < \infty$, that is, $f \in \ell^1$. Let us check this. Clearly, we have that

$$\sum_{n=1}^\infty |f(n)| = \sup_N \sum_{n=1}^N |f(n)| \leq \sup_{A\subseteq \mathbb{N} \text{ finite}} \sum_{n\in A} |f(n)|.$$

Conversely, let $A \subseteq \mathbb{N}$ be finite, and let $N \leq \max(A)$, so that

$$\sum_{n\in A} |f(n)| \leq \sum_{n=1}^N |f(n)| \leq \sum_{n=1}^\infty |f(n)|,$$

and so

$$\sup_{A\subseteq \mathbb{N} \text{ finite}} \sum_{n\in A} |f(n)| \leq \sum_{n=1}^\infty |f(n)|.$$

Bonus Question: Let $(X, \mathcal{R}, \mu)$ be a measure space. Let $f, g : X \to \mathbb{R}$ be measurable. Show that $f + g$ is measurable.
Proof: We follow the hint; let $a \in \mathbb{R}$, and we try to prove that

$$(f+g)^{-1}((a,\infty)) = \{x \in X : a < f(x)+g(x)\} = \bigcup_{q\in\mathbb{Q}} \{x \in X : q < f(x) \text{ and } a-q < g(x)\}.$$

Firstly, let $x \in X$ with $a < f(x) + g(x)$. We can find $\epsilon > 0$ with $a + \epsilon < f(x) + g(x)$. Then pick $q \in \mathbb{Q}$ with $q < f(x) < q + \epsilon$ (which we can do as $\mathbb{Q}$ is dense in $\mathbb{R}$). Then $g(x) > a + \epsilon - f(x) > a + \epsilon - q - \epsilon = a - q$, as required to show that x is in the left-hand side. Conversely, if $x \in X$ and $q \in \mathbb{Q}$ with $q < f(x)$ and $a - q < g(x)$, then $f(x) + g(x) > q + a - q = a$, as required to show that x is in the right-hand side. So we have proved the equality.

Thus

$$(f + g)^{-1}((a,\infty)) = \bigcup_{q\in\mathbb{Q}} f^{-1}((q,\infty)) \cap g^{-1}((a - q,\infty)).$$

For each $q \in \mathbb{Q}$, as f and g are measurable, $f^{-1}((q,\infty)) \in \mathcal{R}$ and $g^{-1}((a-q,\infty)) \in \mathcal{R}$. So $f^{-1}((q,\infty)) \cap g^{-1}((a-q,\infty) \in \mathcal{R}$. As $\mathbb{Q}$ is countable, we conclude $(f+g)^{-1}((a,\infty)) \in \mathcal{R}$.

Exactly the same sort of argument will show that $(f + g)^{-1}((-\infty, a) \in \mathcal{R}$ for each $a \in \mathbb{R}$. So also

$$(f + g)^{-1}((a, b)) = (f + g)^{-1}((a,\infty)) \cap (f + g)^{-1}((-\infty, b) \in \mathcal{R},$$

for $a < b$. Finally, let $U \subseteq \mathbb{R}$ be open, so as in the proof of Corollary 2.7 we can write U as the *countable* union of open intervals. It follows that $f^{-1}(U) \in \mathcal{R}$, as required.
Question continued: Show that $\{x \in X : f(x) \geq g(x)\} \in \mathcal{R}$.
Proof: If f measurable and $C \subseteq \mathbb{R}$ is closed, then

$$f^{-1}(C) = f^{-1}(\mathbb{R} \setminus (\mathbb{R} \setminus C)) = X \setminus f^{-1}(\mathbb{R} \setminus C) \in \mathcal{R},$$

as $\mathbb{R} \setminus C$ is open, and so $f^{-1}(\mathbb{R} \setminus C) \in \mathcal{R}$.

We follow the hint:

$$\{x \in X : f(x) \geq g(x)\} = \bigcap_{q\in\mathbb{Q},q>0} \bigcup_{r\in\mathbb{Q}} \{x \in X : f(x) > r > g(x) - q\}.$$

To prove this, first let $x \in X$ with $f(x) \geq g(x)$. Then, for every $q \in \mathbb{Q}$ with $q > 0$, we have that $f(x) > g(x) - q$, and so there exists $r \in \mathbb{Q}$ with $f(x) > r > g(x) - q$. So we have "$\subseteq$". Conversely, suppose that for all $q \in \mathbb{Q}$ with $q > 0$, for some $r \in \mathbb{Q}$, we have that $f(x) > r > g(x) - q$. In particular, $f(x) > g(x) - q$ for all $q > 0$ with $q \in \mathbb{Q}$, so that $f(x) \geq g(x)$. Hence we have equality, as claimed.

Now, for $q, r \in \mathbb{Q}$ with $q > 0$, we have that

$$\begin{aligned}\{x \in X : f(x) > r > g(x) - q\} &= \{x \in X : f(x) > r\} \cap \{x \in X : r + q > g(x)\} \\ &= f^{-1}((r,\infty)) \cap g^{-1}((-\infty, r + q)) \in \mathcal{R}.\end{aligned}$$

Hence, for $q \in \mathbb{Q}$ with $q > 0$, we have that

$$\bigcup_{r\in Q} \{x \in X : f(x) > r > g(x) - q\} \in \mathcal{R},$$

as this is a countable union. Similarly, by taking a countable intersection, we see that

$$\bigcap_{q\in\mathbb{Q},q>0}\bigcup_{r\in\mathbb{Q}}\{x\in X : f(x)>r>g(x)-q\}\in\mathcal{R}.$$

So $\{x\in X : f(x)\geq g(x)\}\in\mathcal{R}$, as required.

Question continued: Show that fg is measurable.

Cheeky proof: Let Y be a topological space. We say that a map $f : X\to Y$ is *measurable* if $f^{-1}(U)\in\mathcal{R}$ for every open set $U\subseteq Y$. This generalises our definition for maps to $\mathbb{R}$.

Let $\alpha : X\to\mathbb{R}^2$ be the map $\alpha(x)=(f(x),g(x))$, and let $c:\mathbb{R}^2\to\mathbb{R}$ be some continuous map. In particular, we can take $c(t,s)=t+s$ or $c(t,s)=ts$, so that $c\circ\alpha=f+g$ or fg, respectively.

We first check that α is measurable. Firstly, let $a<b$ and $c<d$, so that

$$\alpha^{-1}((a,b)\times(c,d))=\{x : a<f(x)<b, c<g(x)<d\}=f^{-1}((a,b))\cap g^{-1}((c,d)),$$

which is in $\mathcal{R}$, as f and g are measurable. Now we use our usual trick. Let $U\subseteq\mathbb{R}^2$ be open, and let $x\in U$. Then we can find rationals a,b,c,d with $x\in(a,b)\times(c,d)\subseteq U$. Hence

$$U=\bigcup_{a,b,c,d\in\mathbb{Q},(a,b)\times(c,d)\subseteq U}(a,b)\times(c,d),$$

which is a countable union. Hence

$$\alpha^{-1}(U)=\bigcup_{a,b,c,d\in\mathbb{Q},(a,b)\times(c,d)\subseteq U}\alpha^{-1}((a,b)\times(c,d)),$$

which is in $\mathcal{R}$.

Finally, consider $U\subseteq\mathbb{R}$ open. As c is continuous, $c^{-1}(U)\subseteq\mathbb{R}^2$ is open, and so $(c\alpha)^{-1}=\alpha^{-1}c^{-1}(U)\in\mathcal{R}$. Hence $c\alpha$ is measurable, as required.

Linear Analysis I: Worked Solutions 5

Question 1: Let (a_n) be a convergent sequence of positive reals. Prove that

$$\lim_n a_n = \limsup_n a_n = \liminf_n a_n.$$

Let (a_n) be any sequence of positive reals. Show that

$$\liminf_n a_n \leq \limsup_n a_n,$$

where these may be $\pm\infty$. Show that if

$$\liminf_n a_n = \limsup_n a_n,$$

then (a_n) converges.

Answer: Let (a_n) be convergent, with limit a. For $\epsilon > 0$, there exists N_ϵ such that $|a_n - a| < \epsilon$ for $n \geq N_\epsilon$. Hence, for $m \geq N_\epsilon$,

$$a + \epsilon \geq \sup_{n\geq m} a_n \geq a - \epsilon, \quad a - \epsilon \leq \inf_{n\geq m} a_n \leq a + \epsilon,$$

which is enough to ensure that

$$\lim_n a_n = \limsup_n a_n = \liminf_n a_n.$$

Now let (a_n) be an arbitrary sequence in $\mathbb{R}$. Then, for all n,

$$\inf_{k\geq n} a_k \leq \sup_{k\geq n} a_k,$$

and so, by taking the limit, $\liminf_n a_n \leq \limsup_n a_n$. Now suppose that $\liminf_n a_n = \limsup_n a_n$, which means that for all $\epsilon > 0$, there exists N_ϵ such that

$$\big| \inf_{n\geq N_\epsilon} a_n - \sup_{n\geq N_\epsilon} a_n \big| = \sup_{n\geq N_\epsilon} a_n - \inf_{n\geq N_\epsilon} a_n < \epsilon.$$

This implies that $|a_n - a_m| < \epsilon$ for any $n, m \geq N_\epsilon$. Hence (a_n) is a Cauchy sequence, and hence converges.

Question 2: Use the monotone convergence theorem to evaluate $\int_{\mathbb{R}} f(x)\ d\mu(x)$ for the following.

1. $f(x) = e^{-|x|}$.

   Answer: f is continuous and hence measurable. Let $f_n(x) = e^{-|x|}\chi_{[-n,n]}$, so that $f_n \uparrow f$, and hence by MCT

$$\begin{aligned}\int_{\mathbb{R}} f\ d\mu &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n\ d\mu = \lim_{n\to\infty} \int_{-n}^{n} e^{-|x|}\ dx \\ &= \lim_{n\to\infty} \int_{-n}^{0} e^{x}\ dx + \int_0^n e^{-x}\ dx = \lim_{n\to\infty} 2(1 - e^{-n}) = 2.\end{aligned}$$

2. $f(x) = x^{-1/2}\chi_{(0,1]}$.

   Answer: f is continuous on $(0,1]$ and zero elsewhere, so as $(0,1]$ is measurable, f is measurable (check this if you don't believe it!) Let $f_n(x) = x^{-1/2}\chi_{[1/n,1]}$, so that $f_n \uparrow f$, and hence by MCT

$$\begin{aligned}\int_{\mathbb{R}} f\ d\mu &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n\ d\mu = \lim_{n\to\infty} \int_{1/n}^{1} x^{-1/2}\ dx \\ &= \lim_{n\to\infty} \Big[2x^{1/2}\Big]_{x=1/n}^{1} = \lim_{n\to\infty} 2 - 2/\sqrt{n} = 2.\end{aligned}$$

Similarly, establish that the following have finite integral.

1. $f(x) = e^{-x^2}$.

   Answer: f is continuous, so measurable. Let $f_n(x) = e^{-x^2}\chi_{[-n,n]}$. Then $f_n \uparrow f$, and so by MCT

$$\begin{aligned}\int_{\mathbb{R}} f \, d\mu &= \lim_{n\to\infty} \int_{\mathbb{R}} f_n \, d\mu = \lim_{n\to\infty} \int_{-n}^{n} e^{-x^2} \, dx = \lim_{n\to\infty} 2 \int_{0}^{n} e^{-x^2} \, dx \\ &\leq \lim_{n\to\infty} 2 \int_0^1 1 \, dx + 2 \int_1^n e^{-x} \, dx = 2 + \lim_{n\to\infty} 2(e^{-1} - e^{-n}) = 2 + 2e^{-1}.\end{aligned}$$

   This uses that $-x^2 \leq -x$ for $x \geq 1$, and that $e^{-x^2} \leq 1$ for $x \in [0,1]$.

2. $f(x) = x^{-2}\sin(x)\chi_{[\pi,\infty)}$.

   Answer: f is the restriction of a continuous function to the measurable set $[\pi,\infty)$, and so f is measurable. Notice that $f(x) \geq 0$ if and only if $x < \pi$ or $\sin(x) \geq 0$, that is, if and only if $x < \pi$ or $2k\pi \leq x \leq (2k+1)\pi$ for some $k \in \mathbb{N}$. Hence let

$$A = (-\infty, \pi) \cup \bigcup_{k\in\mathbb{N}} [2k\pi, (2k+1)\pi],$$

   so that

$$f_+ = f\chi_A, \qquad f_- = -f\chi_{\mathbb{R}\setminus A}.$$

   Then, by monotone convergence,

$$\begin{aligned}\int_{\mathbb{R}} f_+ \, d\mu = \int_A f \, d\mu &= \lim_{n\to\infty} \sum_{k=1}^{n} \int_{2k\pi}^{(2k+1)\pi} x^{-2}\sin(x) \, dx \\ &\leq \lim_{n\to\infty} \sum_{k=1}^{n} \int_{2k\pi}^{(2k+1)\pi} x^{-2} \, dx = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{2k\pi} - \frac{1}{(2k+1)\pi} \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\pi}{2k(2k+1)\pi^2} \leq \sum_{k=1}^{\infty} k^{-2} < \infty.\end{aligned}$$

   A similar argument applies to f_-.

Finally, show that the following are not Lebesgue integrable (that is, they have infinite integrals).

1. $f(x) = x^{-1}\chi_{[1,\infty)}$.

   Answer: Let $f_n = x^{-1}\chi_{[1,n]}$, so $f_n \uparrow f$, and hence

$$\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_1^n \frac{1}{x} \, dx = \lim_{n\to\infty} \log(n) = \infty.$$

2. $f(x) = \log(x)\chi_{[1,\infty)}$.

   Answer: Let $f_n = \log(x)\chi_{[1,n]}$, so $f_n \uparrow f$, and hence

$$\int_{\mathbb{R}} f \, d\mu = \lim_{n\to\infty} \int_1^n \log(x) \, dx = \lim_{n\to\infty} n\log(n) - n + 1 = \infty.$$

Question 3: Recall that $f(x) = \sin(x)/x$ is a continuous function $\mathbb{R} \to \mathbb{R}$. This is not Lebesgue integrable, as f_+ and/or f_- do not have finite integral. Carefully prove this.
Answer: Notice that

$$f(x) \geq 0 \Leftrightarrow \begin{cases} x \geq 0, 2k\pi \leq x \leq (2k+1)\pi \text{ for some } k \in \mathbb{N}, \\ x < 0, (2k+1)\pi \leq x \leq (2k+2)\pi \text{ for some } k \in \mathbb{Z}. \end{cases}$$

So by Monotone convergence,

$$\int_{\mathbb{R}} f_+ \, d\mu = \lim_{n\to\infty} \sum_{k=0}^{n} \int_{2k\pi}^{(2k+1)\pi} \frac{\sin(x)}{x} \, dx + \int_{(-2k-1)\pi}^{-2k\pi} \frac{\sin(x)}{x} \, dx.$$

Notice that

$$\Big(2k + \frac{1}{4}\Big)\pi \leq x \leq \Big(2k + \frac{3}{4}\Big)\pi \quad \implies \quad \sin(x) \geq \frac{1}{\sqrt{2}},$$

and so

$$\begin{aligned}\int_{\mathbb{R}} f_+ \, d\mu &\geq \sum_{k=0}^{\infty} \int_{(2k+1/4)\pi}^{(2k+3/4)\pi} \frac{1}{x\sqrt{2}} \, dx + \int_{(-2k-3/4)\pi}^{(-2k-1/4)\pi} \frac{1}{|x|\sqrt{2}} \, dx \\ &\geq \frac{1}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{\pi/2}{(2k+3/4)\pi} + \frac{\pi/2}{(2k+1/4)\pi} \geq \sqrt{2} \sum_{k=1}^{\infty} \frac{1}{k} = \infty.\end{aligned}$$

It might be easier to see this if you draw a sketch!!

Question 4: For each n, let $f_n(x) = n^{3/2}x(1 + n^2x^2)^{-1}$ for $x \in [0,1]$. By using the Dominated Convergence Theorem, find

$$\lim_n \int_0^1 f_n(x) \, dx.$$

Answer: For $0 < x \leq 1$, consider the function

$$\theta_x : [1,\infty) \to [0,\infty), \quad t \mapsto \frac{1}{t^{-3/2} + t^{1/2}x^2},$$

which has a turning point at $\sqrt{3}/x$. We check that

$$\theta_x(1) = \frac{1}{1+x^2}, \quad \theta_x(\sqrt{3}/x) = \frac{1}{3^{-3/4}x^{3/2} + 3^{1/4}x^{3/2}} = \frac{1}{x^{3/2}(3^{-3/4} + 3^{1/4})},$$

and clearly $\theta_x(t) \to 0$ as $t \to \infty$. So the maximum of θ_x is $(x^{3/2}(3^{-3/4} + 3^{1/4}))^{-1}$.

So if we define $g : [0,1] \to [0,\infty)$ by $g(0) = 0$, and

$$g(x) = \sup_n f_n(x) = \sup_n \frac{n^{3/2}x}{1+n^2x^2} = \sup_n x\theta_x(n),$$

then we get the crude estimate that $g(x) \leq x^{-1/2}$ for $x > 0$. Hence

$$\int_{[0,1]} g \, d\mu = \Big[2x^{1/2}\Big]_0^1 = 2.$$

So g is integrable, and $|f_n| = f_n \leq g$ for all n. So by Dominated Convergence,

$$\lim_n \int_{[0,1]} f_n \, d\mu = \int_{[0,1]} \lim_n f_n \, d\mu = 0.$$

Question 5: Use the Dominated Convergence Theorem to show that $f : [0,4] \to \mathbb{R}$, defined by

$$f(x) = \begin{cases} 0 & : x = 0, \\ x^{-1/2} \sin(1/x) & : 0 < x \leq 4, \end{cases}$$

is integrable.

Answer: Set $f_n(x) = x^{-1/2} \sin(1/x) \chi_{(1/n,4]}$, so $f_n \to f$ pointwise, but f_n does not *increase* to f, so we cannot apply the Monotone Convergence Theorem. Instead, we notice that $|f_n(x)| \leq x^{-1/2}$ for $0 < x \leq 4$. So define

$$g(x) = \begin{cases} x^{-1/2} & : 0 < x \leq 4, \\ 0 & : x \leq 0, x > 4. \end{cases}$$

We can use Monotone Convergence to show that

$$\int_{\mathbb{R}} g \; d\mu = \lim_{n\to\infty} \int_{1/n}^{4} x^{-1/2} \; dx = \lim_{n\to\infty} 2(2 - 1/\sqrt{n}) = 4.$$

Hence g is integrable, and as $|f_n| \leq g$, each f_n is integrable. Apply the Dominated Convergence Theorem, we see that f is also integrable, as required.

Question 6: Define $f_n : [0,1] \to [0,\infty)$ by

$$f_n(x) = \begin{cases} n & : 0 \leq x \leq 1/n, \\ 0 & : x > 1/n. \end{cases}$$

Show that $f_n(x) \to 0$ almost everywhere, but that

$$\int_0^1 f_n \; d\mu = 1,$$

for all n. Why can we not apply either the Monotone or the Dominated Convergence Theorems in this case?

Answer: For $x > 0$, if n is large enough, then $x > 1/n$, implying that $f_n(x) = 0$. Hence $f_n \to 0$ except on $\{0\}$. But a singleton is a null set, so $f_n \to 0$ almost everywhere. However, as f_n is a simple function,

$$\int_0^1 f_n \; d\mu = n\mu([0, 1/n]) = 1,$$

for all n.

Clearly f_n is *not* an increasing sequence, so Monotone Convergence does not apply. Let

$$g(x) = \sup_n f_n(x) = \sup\{n \in \mathbb{N} : x \leq 1/n\}.$$

Hence $g(x) = n$ for $(n+1)^{-1} < x \leq 1/n$, and so, for each N,

$$\int_0^1 g \; d\mu \geq \sum_{n=1}^{N} n\Big(\frac{1}{n} - \frac{1}{n+1}\Big) = \sum_{n=1}^{N} \frac{n}{n(n+1)} = \sum_{n=2}^{N+1} n^{-1}.$$

This sum diverges (as $N \to \infty$), and so g has infinite integral. Hence we cannot bound the sequence (f_n) by an integrable function, and so we cannot apply the Dominated Convergence Theorem.

Question 7: Let $(X, \mathcal{R}, \mu)$ be a measure space, and let $Y \in \mathcal{R}$. On a previous example sheet, we saw how to define the sub-measure space $(Y, \mathcal{R}_Y, \mu_Y)$. Let $f : X \to \mathbb{R}$ be measurable, and let f_Y be the restriction of f to Y. Show that f_Y is measurable with respect to $\mathcal{R}_Y$. Show that $f\chi_Y$ is measurable. Show that

$$\int_Y f_Y \, d\mu_Y = \int_X f\chi_Y \, d\mu.$$

Hence integrating with respect to a sub-measure space, or just multiplying by the characteristic function of a measurable subset, gives the same answer.

Answer: Recall that $\mathcal{R}_Y = \{A \cap Y : A \in \mathcal{R}\}$, and μ_Y is simply the restriction of μ to Y. Firstly we check that f_Y is $\mathcal{R}_Y$-measurable. Let $U \subseteq \mathbb{R}$ be open, so that $f^{-1}(U) \in \mathcal{R}$, as f is $\mathcal{R}$-measurable. Hence

$$f_Y^{-1}(U) = \{y \in Y : f(y) \in Y\} = f^{-1}(U) \cap Y \in \mathcal{R}_Y,$$

and so we conclude that f_Y is $\mathcal{R}_Y$-measurable.

Next we show that $f\chi_Y$ is $\mathcal{R}$-measurable. Again, let $U \subseteq \mathbb{R}$ be open with $0 \in U$, so that

$$\begin{aligned}(f\chi_Y)^{-1}(U) &= \{x \in X : f(x)\chi_Y(x) \in U\} = \{x \in Y : f(x) \in U\} \cup \{x \in X \setminus Y\} \\ &= \big(f^{-1}(U) \cap Y\big) \cup (X \setminus Y) \in \mathcal{R}.\end{aligned}$$

If $0 \notin U$, then

$$(f\chi_Y)^{-1}(U) = \{x \in Y : f(x) \in U\} = f^{-1}(U) \cap Y \in \mathcal{R}.$$

So $f\chi_Y$ is measurable (this uses that $Y \in \mathcal{R}$).

Let $f = \sum_{k=1}^n t_k \chi_{A_k}$ be a simple function, with the (A_k) disjoint, so we have that

$$\int_Y f_Y \, d\mu_Y = \sum_{k=1}^n t_k \mu_Y(A_k \cap Y) = \sum_{k=1}^n t_k \mu(A_k \cap Y) = \int_X f\chi_Y \, d\mu.$$

If $f : X \to [0, \infty)$ is measurable, then let

$$f_n = 2^{-1} \lfloor 2^n f \rfloor \qquad (n \in \mathbb{N}),$$

so that each f_n is simple, and $f_n \uparrow f$. Obviously also $f_n\chi_Y \uparrow f\chi_Y$, so by the MCT,

$$\int_Y f_Y \, d\mu_Y = \lim_n \int_Y (f_n)_Y \, d\mu_Y = \lim_n \int_X f_n\chi_Y \, d\mu = \int_X f\chi_Y \, d\mu.$$

Finally, to handle a general measurable $f : X \to \mathbb{R}$, we simply consider positive and negative parts.

Linear Analysis I: Worked Solutions 6

Question 1: Let $(X, \mathcal{R}, \mu)$ be a measure space. Let $f : X \to [0, \infty)$ be measurable. For each $A \in \mathcal{R}$, define

$$\mu_f(A) = \int_X f\chi_A \, d\mu \qquad (A \in \mathcal{R}).$$

Show that μ_f is a measure.
Answer: Clearly $\mu_f(\emptyset) = 0$. Let (A_n) be a sequence of pairwise disjoint sets in $\mathcal{R}$, and let $A = \bigcup_n A_n$. Let $g = \chi_A$ and $f_n = \chi_{A_1 \cup \cdots \cup A_n}$ for each n. Then

$$A_1 \cup \cdots \cup A_n \subseteq A_1 \cup \cdots \cup A_{n+1} \implies f_n \leq f_{n+1}, \qquad (n \in \mathbb{N}),$$

so (f_n) is an increasing sequence. If $x \in A$, then for some n, we have $x \in A_n$, and so $f_n(x) \to 1$. Thus $f_n(x) \to g(x)$. If $x \notin A$ then $x \notin A_n$ for each n, so that $0 = g(x) = f_n(x)$ for all n. Thus $g = \chi_A = \lim_n f_n$.

As f is positive, we see that ff_n increases to $f\chi_A$. By the Monotone Convergence Theorem,

$$\mu_f(A) = \int_X f\chi_A \, d\mu = \lim_{n\to\infty} \int_X ff_n \, d\mu = \lim_{n\to\infty} \int_X f\chi_{A_1 \cup \cdots \cup A_n} \, d\mu.$$

As integration is linear,

$$\mu_f(A) = \lim_{n\to\infty} \sum_{k=1}^n \int_X f\chi_{A_k} \, d\mu = \lim_{n\to\infty} \sum_{k=1}^n \mu_f(A_k) = \sum_{n=1}^\infty \mu_f(A_n).$$

So μ_f is countably additive, and hence a measure.
Question continued: Furthermore, show that if g is a simple function, then

$$\int_X g \, d\mu_f = \int_X gf \, d\mu.$$

Conclude (using Monotone convergence) that this holds for any integrable function $g : X \to \mathbb{R}$.
Answer: We can write a simple function as

$$g = \sum_{k=1}^n a_k \chi_{A_k},$$

for some pairwise disjoint (A_k), and scalars (a_k). Then, by definition, and using linearity,

$$\int_X g \, d\mu_f = \sum_{k=1}^n a_k \mu_f(A_k) = \sum_{k=1}^n a_k \int_X f\chi_{A_k} \, d\mu = \int_X \sum_{k=1}^n a_k f\chi_{A_k} \, d\mu = \int_X fg \, d\mu.$$

Now let $g : X \to [0, \infty)$ be measurable, and as usual, set

$$g_n = \min(n, 2^{-n}\lfloor 2^n g \rfloor),$$

so each g_n is simple, and $g_n \uparrow g$. Similarly, $g_n f \uparrow fg$. Thus, by Monotone Convergence,

$$\int_X g \, d\mu_f = \lim_{n\to\infty} \int_X g_n \, d\mu_f = \lim_{n\to\infty} \int_X g_n f \, d\mu = \int_X gf \, d\mu,$$

as required. The claim then follows by taking positive and negative parts.

Question 2: Let $(X, \mathcal{R}, \mu)$ be a measure space. A function $f : X \to \mathbb{R}$ is *essentially bounded* if there exists $K > 0$ such that $|f| \leq K$ almost everywhere. The inf of such K is called the *essential supremum* of f, and is denoted by

$$\text{ess-sup}_{x\in X} |f(x)| \quad \text{or simply} \quad \text{ess-sup}_X |f|.$$

Let f be essentially bounded, and suppose that $g : X \to \mathbb{R}$ is measurable and integrable. Show that fg is integrable, and that

$$\int_X |fg| \; d\mu \leq \Big(\text{ess-sup}_X |f|\Big) \int_X |g| \; d\mu.$$

Answer: Let $\epsilon > 0$, and set $K = \text{ess-sup}_X |f|$. Then $|f| \leq K + \epsilon$ almost everywhere, so $A = \{x \in X : |f(x)| \geq K + \epsilon\}$ is a null set. Thus $f = f\chi_{X\setminus A}$ almost everywhere, and $|f\chi_{X\setminus A}| \leq K + \epsilon$.

Then $|f\chi_{X\setminus A}g| \leq (K + \epsilon)|g|$, and so

$$\int_X |f\chi_{X\setminus A}g| \; d\mu \leq (K + \epsilon) \int_X |g| \; d\mu < \infty.$$

As $|f\chi_{X\setminus A}g| = |fg|$ almost everywhere, we also have that

$$\int_X |fg| \; d\mu = \int_X |f\chi_{X\setminus A}g| \; d\mu \leq (K + \epsilon) \int_X |g| \; d\mu.$$

As $\epsilon > 0$ was arbitrary, we are done.

Question 3: We define Lebesgue measure on $\mathbb{R}^3$ by identifying $\mathbb{R}^3$ with $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The *volume* of a measurable set $A \subseteq \mathbb{R}^3$ is then simply the integral of the characteristic function of A. Carefully apply Fubini's Theorem to find the volumes of the sets:

1. $\{(x, y, z) : 0 \leq z \leq 2 - x^2 - y^2\}$.

2. $\{(x, y, z) : x + y + z \leq 1, x \geq 0, y \geq 0, z \geq 0\}$.

Notice that these sets are bounded, so we can work in a finite measure space if we wish.

Answer: *It is quite possible that I have made mistakes here, so check these integrals!* For (1), we have, being careful,

$$\text{Volume} = \int_{\mathbb{R}^3} \chi_{\{(x,y,z):0\leq z\leq 2-x^2-y^2\}} \; d\mu_3,$$

where here I write μ_3 for Lebesgue measure on $\mathbb{R}^3$. The set we are integrating over is bounded, and hence has finite measure. So we can apply Fubini. Hence

$$\text{Volume} = \int_{\mathbb{R}^2} \chi_{\{(x,y):x^2+y^2\leq 2\}} \Big(\int_0^{2-x^2-y^2} 1 \; dz \Big) \; d\mu_2.$$

Then, for x fixed with $x^2 \leq 2$, we have that $x^2 + y^2 \leq 2$ if and only if $(x^2 - 2)^{1/2} \leq y \leq$

$(2-x^2)^{1/2}$. Hence

$$\begin{aligned}\text{Volume} &= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{(x^2-2)^{1/2}}^{(2-x^2)^{1/2}} \int_0^{2-x^2-y^2} 1 \; dz \; dy \; dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \int_{(x^2-2)^{1/2}}^{(2-x^2)^{1/2}} 2 - x^2 - y^2 \; dy \; dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} 2(2-x^2)^{1/2}(2-x^2) - \left[\frac{y^3}{3}\right]_{y=(x^2-2)^{1/2}}^{(2-x^2)^{1/2}} dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} 2(2-x^2)^{3/2} - \frac{2}{3}(2-x^2)^{3/2} \; dx \\ &= \frac{4}{3}\int_{-\sqrt{2}}^{\sqrt{2}} (2-x^2)^{3/2} \; dx.\end{aligned}$$

Let $x = \sqrt{2}\sin(t)$, so that $dx/dt = \sqrt{2}\cos(t)$, and hence

$$\begin{aligned}\text{Area} &= \frac{4}{3}\int_{-\pi/2}^{\pi/2} 2^{3/2}\sqrt{2}\cos^4(t) \; dt = \frac{16}{3}\int_{-\pi/2}^{\pi/2} \cos^4(t) \; dt \\ &= \frac{2}{3}\int_{-\pi/2}^{\pi/2} \cos(4t) + 4\cos(2t) + 3 \; dt = \frac{2}{3}\left[\frac{\sin(4t)}{4} + 2\sin(2t) + 3t\right]_{t=-\pi/2}^{\pi/2} = 2\pi\end{aligned}$$

If you'd seen this question in a Calculus Course, you would probably change into plane polar coordinates. There is a way to handle change of variables for Lebesgue (or more general) integrable functions. I haven't covered this in the course, in the interests of time, but in an easy form, it is rather similar to change of variables for Riemann integration. In a more complicated form, it is not very useful for practical calculations.

For (2), with much less justification this time, we have

$$\begin{aligned}\text{Volume} &= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 1 \; dz \; dy \; dx = \int_0^1 \int_0^{1-x} 1 - x - y \; dy \; dx \\ &= \int_0^1 (1-x)^2 - \left[\frac{y^2}{2}\right]_{y=0}^{1-x} dx = \frac{1}{2}\int_0^1 (1-x)^2 \; dx = \frac{1}{2}\left[x - x^2 + \frac{x^3}{3}\right]_{x=0}^{1} = \frac{1}{6}.\end{aligned}$$

Question 4: Let $(X, \mathcal{R}, \mu)$ and $(Y, \mathcal{S}, \lambda)$ be finite measure spaces. Let $f : X \to \mathbb{R}$ be $\mathcal{R}$-measurable, and let $g : Y \to \mathbb{R}$ be $\mathcal{S}$-measurable. Let $h : X \times Y \to \mathbb{R}$ be defined by $h(x, y) = f(x)g(y)$. Show that h is $(\mathcal{R} \otimes \mathcal{S})$-measurable. Suppose that f and g are integrable with respect to μ and λ, respectively. Use Fubini to show that

$$\int_{X\times Y} h \; d(\mu \otimes \lambda) = \int_X f \; d\mu \int_Y g \; d\lambda.$$

Answer: Let $U \subseteq \mathbb{R}$ be open. Consider the continuous map $\alpha : \mathbb{R}^2 \to \mathbb{R}$ defined by $\alpha(x, y) = xy$. Consider also the map $\beta : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $\beta(x, y) = (f(x), g(y))$. Then $h = \alpha\beta$, and so $h^{-1}(U) = \beta^{-1}\alpha^{-1}(U)$. As α is continuous, $\alpha^{-1}(U)$ is open. Suppose that β is $\mathcal{R} \otimes \mathcal{S}$-measurable, in the sense that if $V \subseteq \mathbb{R} \times \mathbb{R}$ is open, then $\beta^{-1}(V) \in \mathcal{R} \otimes \mathcal{S}$. Then we have that $h^{-1}(U) \in \mathcal{R} \otimes \mathcal{S}$, showing that h is $\mathcal{R} \otimes \mathcal{S}$-measurable.

So we want β to be measurable. Let $U, V \subseteq \mathbb{R}$ be open, so that $f^{-1}(U) \in \mathcal{R}$ and $g^{-1}(V) \in \mathcal{S}$, as f and g are measurable. So, by the definition of $\mathcal{R} \otimes \mathcal{S}$, we have that

$\beta^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V) \in \mathcal{R} \otimes \mathcal{S}$. We now "exploit the rationals". Let $U \subseteq \mathbb{R}^2$ be open, let $\mathcal{D}$ be the collection of all open intervals (a, b) with $a, b \in \mathbb{Q}$. Then

$$U = \bigcup_{\{A,B \in \mathcal{D} : A \times B \subseteq U\}} A \times B,$$

a countable union, so

$$\beta^{-1}(U) = \bigcup_{\{A,B \in \mathcal{D} : A \times B \subseteq U\}} f^{-1}(A) \times g^{-1}(B)$$

is in $\mathcal{R} \otimes \mathcal{S}$. Hence β is measurable.

As f and g are measurable, by Fubini (for positive functions) we see that

$$\int_{X \times Y} |h| \; d(\mu \otimes \lambda) = \int_X |h|_1 \; d\mu,$$

where

$$|h|_1(x) = \begin{cases} \int_Y |h|(x, y) \; d\lambda(y) & : \text{this is finite,} \\ 0 & : \text{otherwise.} \end{cases}$$

However, in this case

$$\int_Y |h|(x, y) \; d\lambda(y) = \int_Y |f|(x)|g|(y) \; d\lambda(y) = |f(x)| \int_Y |g| \; d\lambda.$$

Thus

$$\begin{aligned} \int_{X \times Y} |h| \; d(\mu \otimes \lambda) &= \int_X |f(x)| \int_Y |g| \; d\lambda \; d\mu(x) \\ &= \int_X |f(x)| \; d\mu(x) \int_Y |g(y)| \; d\lambda(y) < \infty. \end{aligned}$$

Hence h is integrable, and so by Fubini,

$$\int_{X \times Y} h \; d(\mu \otimes \lambda) = \int_X f \; d\mu \int_Y g \; d\lambda,$$

by just repeating the argument.

Question 5: Define $f : [0, 1]^2 \to \mathbb{R}$ by

$$f(x, y) = \begin{cases} \frac{x^2 - y^2}{(x^2 + y^2)^2} & : (x, y) \neq (0, 0), \\ 0 & : \text{otherwise.} \end{cases}$$

Show by calculation that

$$\int_0^1 \int_0^1 f(x, y) \; dx \; dy \neq \int_0^1 \int_0^1 f(x, y) \; dy \; dx.$$

Why can we not apply Fubini's Theorem in this case?

Answer: We see that for $y > 0$,

$$\int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \; dx = \Big[\frac{-x}{x^2 + y^2} \Big]_{x=0}^1 = \frac{-1}{1 + y^2}.$$

Hence we have that

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, dx \, dy = \int_0^1 \frac{-1}{1+y^2} \, dy = \Big[-\tan^{-1}(y) \Big]_{y=0}^1 = -\pi/4.$$

By symmetry, we have that for $x > 0$,

$$\int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, dy = \Big[\frac{y}{x^2+y^2} \Big]_{y=0}^1 = \frac{1}{1+x^2}.$$

and consequently,

$$\int_0^1 \int_0^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, dy \, dx = \pi/4.$$

We cannot apply Fubini's Theorem as $|f|$ has infinite integral over $[0,1]^2$, that is, f is NOT integrable. This follows as

$$\begin{aligned} \int_0^1 \frac{|x^2 - y^2|}{(x^2+y^2)^2} \, dx &= \int_0^y \frac{y^2 - x^2}{(x^2+y^2)^2} \, dx + \int_y^1 \frac{x^2 - y^2}{(x^2+y^2)^2} \, dx \\ &= \Big[\frac{x}{x^2+y^2} \Big]_{x=0}^y + \Big[\frac{-x}{x^2+y^2} \Big]_{x=y}^1 \\ &= \frac{y}{y^2+y^2} + \frac{-1}{1+y^2} - \frac{-y}{y^2+y^2} = \frac{1}{y} - \frac{1}{1+y^2}. \end{aligned}$$

Hence

$$\int_0^1 \int_0^1 \frac{|x^2 - y^2|}{(x^2+y^2)^2} \, dx \, dy = \int_0^1 \frac{1}{y} - \frac{1}{1+y^2} \, dy = \infty - \int_0^1 \frac{1}{1+y^2} \, dy = \infty.$$

Formally, we should use Monotone Convergence in this last calculation.

Question 6: Let $(X, \mathcal{R}, \mu)$ and $(Y, \mathcal{S}, \lambda)$ be finite measure spaces, and let $E \in \mathcal{R} \otimes \mathcal{S}$. For each $x \in X$, let $E_x = \{y \in Y : (x,y) \in E\}$, a *cross-section* of E. Show that the following are equivalent:

1. $(\mu \otimes \lambda)(E) = 0$;
2. $\lambda(E_x) = 0$ for almost all x with respect to μ (that is, $\mu(\{x \in X : \lambda(E_x) \neq 0\}) = 0$).

Answer: Let $f = \chi_E$ a measurable function on $X \times Y$. By the results in lectures, we know that each E_x is in $\mathcal{S}$, so we can let $f_x = \chi_{E_x}$ a measurable function on Y. Notice that $f_x(y) = \chi_E(x,y)$ for $x \in X$ and $y \in F$. As f is positive and bounded, we can apply (the easiest form of) Fubini to see that

$$\begin{aligned} (\mu \otimes \lambda)(E) &= \int_{X \times Y} f \, d(\mu \otimes \lambda) = \int_X \int_Y f(x,y) \, d\lambda(y) \, d\mu(x) \\ &= \int_X \int_Y f_x \, d\lambda \, d\mu(x) = \int_X \lambda(E_x) \, d\mu(x). \end{aligned}$$

So $(\mu \otimes \lambda)(E) = 0$ if and only if $x \mapsto \lambda(E_x)$ has zero integral over X, which is if and only if $\lambda(E_x) = 0$ for almost every x with respect to μ.

Question 7: Let X be a set, and let $\mathcal{R}$ be a σ-algebra on X. For $x \in X$, define a map $\delta_x : \mathcal{R} \to [0, \infty)$ by

$$\delta_x(A) = \begin{cases} 1 & : x \in A, \\ 0 & : x \notin A. \end{cases}$$

Show that δ_x is a measure.
Answer: Clearly $\delta_x(\emptyset) = 0$. For (A_n) a sequence of pairwise disjoint sets in $\mathcal{R}$, let $A = \bigcup_n A_n$. If $x \notin A$, then $x \notin A_n$ for all n, and so

$$0 = \delta_x(A) = \sum_n \delta_x(A_n).$$

If $x \in A$, then by pairwise disjointness, there exists a unique n_0 with $x \in A_{n_0}$. Then

$$1 = \delta_x(A) = \sum_n \delta_x(A_n) = \delta_x(A_{n_0}) = 1,$$

as required to show that δ_x is countably additive. So δ_x is a measure.
Question continued: Determine the completion of δ_x (that is, what are the null sets for δ_x?)
Answer: This is slightly a trick question! If $\{x\} \in \mathcal{R}$, then also $X \setminus \{x\} \in \mathcal{R}$, and $\delta_x(X \setminus \{x\}) = 0$. It follows easily now that *every* set not containing x is null, as such sets are contained in $X \setminus \{x\}$. In the completed σ-algebra, *every* set is measurable, and δ_x is defined in the same way as before.

However, maybe $\mathcal{R}$ is the trivial σ-algebra, $\mathcal{R} = \{X, \emptyset\}$. Then $\delta_x(X) = 1$, so the only set in $\mathcal{R}$ which has zero measure is $\emptyset$. So completing does nothing in this case.
Question continued: For a measurable function $f : X \to [0, \infty)$, what is $\int_X f \, d\delta_x$? Which functions $f : X \to \mathbb{R}$ are integrable for δ_x?
Answer: Intuition suggests that $\int f \, d\delta_x = f(x)$. Let us prove this! For $A \in \mathcal{R}$, we have $\int \chi_A \, d\delta_x = \delta_x(\chi_A) = \chi_A(x)$. By taking linear combinations, it is easy to see that $\int g \, d\delta_x = g(x)$ for any simple function $g : X \to [0, \infty)$. We could now use Monotone Convergence, in the usual way.

However, let's be different, and use the *definition* of the integral. So, if $g \leq f$ and g is simple, then $\int g \, d\delta_x = g(x) \leq f(x)$, so by definition,

$$\int f \, d\delta_x \leq f(x).$$

Conversely, for $\epsilon > 0$, notice that

$$A = \{y \in X : f(x) - \epsilon \leq f(y) \leq f(x)\} = f^{-1}([f(x) - \epsilon, f(x)])$$

is in $\mathcal{R}$, as f is measurable. Then $x \in A$, and for any $y \in A$, $f(y) \geq f(x) - \epsilon$. So

$$f\chi_A \geq (f(x) - \epsilon)\chi_A \implies \int f \, d\delta_x \geq (f(x) - \epsilon)\delta_x(\chi_A) = f(x) - \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that $\int f \, d\delta_x = f(x)$, as required.

Maybe (or maybe not!) you worry that we haven't used any facts about $\mathcal{R}$ here! Well, if $\mathcal{R} = \{X, \emptyset\}$, then there are very few measurable functions $f : X \to [0, \infty)$. Indeed, a moment's thought shows that f must actually be constant (prove this!)

So any positive measurable function has a finite integral. By taking positive and negative parts, we see that any measure function $f : X \to \mathbb{R}$ is integrable, with $\int f \, d\delta_x = f(x)$.

Question 8: Let $A \subseteq \mathbb{R}$ be a Lebesgue measurable set with finite Lebesgue measure. Show that for $\epsilon > 0$, we can find an open set U with $A \subseteq U$ and $\mu(U) < \mu(A) + \epsilon$.
Answer: By the definition of Lebesgue outer measure, for $\epsilon > 0$, we can find U, a countable union of open intervals, with $A \subseteq U$ and $\mu(U) < \mu(A) + \epsilon$. (This follows, as by definition, $\mu(A)$ is the infimum of $\mu(U)$ for such U).

Question continued: Show that for $\epsilon > 0$, we can find a compact (that is, closed and bounded) set K with $K \subseteq A$ and $\mu(K) > \mu(A) - \epsilon$.

Answer: Suppose first that A is bounded, that is, $A \subseteq [-n, n]$ for some $n > 0$. Then let $B = [-n, n] \setminus A$ which is also Lebesgue measurable, so by the first bit of the question, we can find some open set U with $B \subseteq U$ and $\mu(U) < \mu(B) + \epsilon$. *At this point, drawing a diagram may help!* Then let $K = [-n, n] \setminus U = [-n, n] \cap (\mathbb{R} \setminus U)$ a closed and bounded set. For $k \in K$, we have that $k \in [-n, n]$ but $k \notin U$, so certainly $k \notin B$. Thus $k \in A$ (use the definition of B). So $K \subseteq A$. A bit of thought shows that $A \setminus K = U \cap A \subseteq U \setminus B$. Thus

$$\mu(A \setminus K) = \mu(A) - \mu(K) \leq \mu(U \setminus B) = \mu(U) - \mu(B) < \epsilon,$$

so that $\mu(A) - \epsilon < \mu(K)$.

Let $A_n = A \cap [-n, n]$, so that $A_1 \subseteq A_2 \subseteq \cdots$ and $A = \bigcup_n A_n$. Then $\mu(A) = \lim_n \mu(A_n)$, and so as $\mu(A) < \infty$, there exists n with $\mu(A_n) > \mu(A) - \epsilon/2$. As $A_n \subseteq [-n, n]$, then above shows that there exists a closed and bounded K with $K \subseteq A_n \subseteq A$ with $\mu(K) > \mu(A_n) - \epsilon/2$. Thus $\mu(K) > \mu(A) - \epsilon$, as we want.

Question continued: Conclude that

$$\sup\{\mu(K) : K \subseteq A \text{ is compact }\} = \mu(A) = \inf\{\mu(U) : A \subseteq U \text{ is open }\}.$$

This shows that μ is a *regular* measure. We will learn more about regular measures later in the course.

Answer: This is immediate, as $\epsilon > 0$ was arbitrary.

Linear Analysis I: Worked Solutions 7

Question 1: Consider the set $\mathbb{N}$ together with the trivial σ-algebra consisting of all subsets of $\mathbb{N}$. Let (ω_n) be a sequence of positive reals, with $(\omega_n) \in \ell^1$. Show that we may define a measure μ by

$$\mu(A) = \sum_{n \in A} \omega_n \qquad (A \subseteq \mathbb{N}).$$

What are the null sets for this measure?

Proof: Firstly we remark that as $\omega_n \geq 0$ for all n, the order which we take the sum does not matter. Clearly $\mu(\emptyset) = 0$; if (A_n) is a pairwise disjoint collection of subsets of $\mathbb{N}$, and $A = \bigcup_n A_n$, then it is pretty clear that

$$\sum_n \mu(A_n) = \sum_n \sum_{k \in A_n} \omega_k = \sum_{k \in A} \omega_k.$$

Exercise: Give an ϵ-δ proof of this!

We claim that $A \subseteq \mathbb{N}$ is null if and only if $\omega_n = 0$ for each $n \in A$. The "if" case is easy; conversely, if $\mu(A) = 0$ then $\sum_{n \in A} \omega_n = 0$, so as each ω_n is positive, we must have that $\omega_n = 0$ for each $n \in A$, as claimed.

Question 2: This follows on from Question 1. Determine when a function $f : \mathbb{N} \to \mathbb{C}$ is in $L^p(\mu)$. Describe, briefly, the space $\mathcal{L}^p(\mu)$.

Proof: Let's do this carefully (having told you not to bother being too careful, maybe it's good to see the details). Let $f : \mathbb{N} \to \mathbb{R}$ be simple, say

$$f = \sum_{k=1}^n a_k \chi_{A_k}$$

for some pairwise disjoint (A_k). Then

$$|f|^p = \sum_{k=1}^n |a_k|^p \chi_{A_k} \implies \int |f|^p \ d\mu = \sum_{k=1}^n |a_k|^p \sum_{j \in A_k} \omega_j = \sum_{j \in \mathbb{N}} |f(j)|^p \omega_j.$$

Now let $f : \mathbb{N} \to \mathbb{R}$ be arbitrary, and let $g \leq f$ be simple. Then

$$\int |g|^p \ d\mu = \sum_{j \in \mathbb{N}} |g(j)|^p \omega_j \leq \sum_{j \in \mathbb{N}} |f(j)|^p \omega_j.$$

By the definition of the integral, taking the supremum over such g, we conclude that

$$\int |f|^p \ d\mu \leq \sum_{j \in \mathbb{N}} |f(j)|^p \omega_j.$$

Conversely, let $n \in \mathbb{N}$, and define $g : \mathbb{N} \to \mathbb{R}$ by $g(j) = f(j)$ if $j \leq n$, and $g(j) = 0$ otherwise. Then g is simple, so we see that

$$\int |f|^p \ d\mu \geq \int |g|^p \ d\mu = \sum_{j=1}^n |f(j)|^p \omega_j.$$

Letting $n \to \infty$, we have that

$$\int |f|^p \ d\mu \geq \sum_{j \in \mathbb{N}} |f(j)|^p \omega_j.$$

So we have equality. As usual, we could have used a Monotone Convergence argument instead.

So $f : \mathbb{N} \to \mathbb{C}$ is in $L^p(\mu)$ if and only if

$$\sum_{j=1}^{\infty} |f(j)|^p \omega_j < \infty.$$

Then $\mathcal{L}^p(\mu)$ is $L^p(\mu)$, modulo functions which are equal almost everywhere. Using Question 1, we see that $f \sim g$ if and only if $f(j) \neq g(j)$ implies that $\omega_j = 0$.

So, if we let $A = \{j \in \mathbb{N} : \omega_j \neq 0\}$, we could consider $\mathcal{L}^p(\mu)$ to be the space of functions $f : A \to \mathbb{C}$ with

$$\sum_{j \in A} |f(j)|^p \omega_j < \infty.$$

Question 3: Let $(X, \mathcal{R}, \mu)$ be a *finite* measure space. Show that if $1 \leq p < r < \infty$, then $L^r(\mu) \subseteq L^p(\mu)$. *Hint:* Given a function $f \in L^r(\mu)$, write

$$f = f\chi_{\{x:|f(x)|\leq 1\}} + f\chi_{\{x:|f(x)|>1\}},$$

then think about whether these two functions are in $L^p(\mu)$.

Proof: We follow the hint. Let $f \in \mathcal{L}^r(\mu)$, and fix a representative of f. If $|f(x)| > 1$, then $|f(x)|^p \leq |f(x)|^r$ as $p < r$. Thus

$$\begin{aligned}\int |f|^p \, d\mu &= \int_{\{x\in X:|f(x)|\leq 1\}} |f|^p \, d\mu + \int_{\{x\in X:|f(x)|>1\}} |f|^p \, d\mu \\ &\leq \int_{\{x\in X:|f(x)|\leq 1\}} 1 \, d\mu + \int_{\{x\in X:|f(x)|>1\}} |f|^r \, d\mu \\ &\leq \mu(\{x \in X : |f(x)| \leq 1\}) + \|f\|_r^r \leq \mu(X) + \|f\|_r^r < \infty,\end{aligned}$$

as μ is finite. Hence $f \in L^p(\mu)$ and so defines a member of $\mathcal{L}^p(\mu)$ (and notice that if $f \sim g$ in $\mathcal{L}^r(\mu)$, the same is true in $\mathcal{L}^p(\mu)$).

Question continued: Try to use the Holder inequality!

Proof: How might we use Holder? Well, let $s \in (1, \infty)$, so by Holder

$$\int |f|^p \, d\mu = \int |f|^p 1 \, d\mu \leq \Big(\int |f|^{ps} \, d\mu \Big)^{1/s} \Big(\int 1^t \, d\mu \Big)^{1/t},$$

where $1/s + 1/t = 1$, as usual. We only know that $\int |f|^r \, d\mu < \infty$, so it seems natural to let $ps = r$, that is, $s = r/p$. As $p < r$, we see that $r/p > 1$, so s is in the interval $(1, \infty)$. Then $1/t = 1 - 1/s = 1 - p/r$. Thus

$$\int |f|^p \, d\mu \leq \Big(\int |f|^r \, d\mu \Big)^{p/r} \mu(X)^{1-p/r} < \infty,$$

as $\int |f|^r < \infty$ and $\mu(X) < \infty$.

Question 4: By considering $\mathbb{R}$ with Lebesgue measure, or otherwise, show that the conclusions of Question 5 no longer hold if we are not working with a finite measure space.

Proof: We first do the $p = 1$ case. So we try to find $f \in L^r(\mu)$ with $\int |f| \, d\mu = \infty$, so that $f \notin L^1(\mu)$. Try

$$f(x) = x^{-1}\chi_{(1,\infty)}.$$

Then (formally, we use Monotone convergence here)

$$\int |f|^r \, d\mu = \lim_n \int_1^n x^{-r} \, d\mu = \lim_n \left[\frac{x^{1-r}}{1-r}\right]_1^n = \lim_n \frac{1-n^{1-r}}{r-1} = \frac{1}{r-1}.$$

So $f \in L^r(\mu)$. However,

$$\int |f| \, d\mu = \lim_n \int_1^n x^{-1} \, d\mu = \lim_n \Big[\log(x)\Big]_1^n = \lim_n \log(n) = \infty,$$

so $f \not\in L^1(\mu)$.

To do the general case, just use

$$f(x) = x^{-1/p}\chi_{(1,\infty)},$$

so that $|f|^p = x^{-1}\chi_{(1,\infty)}$ while $|f|^r = x^{-r/p}\chi_{(1,\infty)}$, which has finite integral, as $r/p > 1$.

Question 5: Let $(\mathbb{R}, \mathcal{R}, \mu)$ be Lebesgue measure on the real line. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function. Define a map $\lambda : \mathcal{R} \to \mathbb{R}$ by

$$\lambda(E) = \int_E f \, d\mu \qquad (E \in \mathcal{R}).$$

Show quickly that λ is a signed measure. Let $A \cup B$ be a Hahn-Decomposition for λ. How can we relate the sets A and B to the function f?

Proof: Clearly $\lambda(\emptyset) = 0$. Let (E_n) be a pairwise disjoint sequence in $\mathcal{R}$, and let $E = \bigcup_n E_n$. As the (E_n) are pairwise disjoint, we have that

$$\big(f\chi_{E_1\cup\cdots\cup E_n}\big)_\pm = \sum_{k=1}^n f_\pm \chi_{E_k}, \qquad \big(f\chi_E\big)_\pm = f_\pm\chi_E.$$

Then $\big(f\chi_{E_1\cup\cdots\cup E_n}\big)_\pm \uparrow f_\pm\chi_E$, so by Monotone convergence,

$$\begin{aligned}\sum_k \int_{E_k} f \, d\mu &= \sum_k \Big(\int_{E_k} f_+ \, d\mu - \int_{E_k} f_- \, d\mu\Big) = \lim_n \sum_{k=1}^n \int f_+\chi_{E_k} - f_-\chi_{E_k} \, d\mu \\ &= \lim_n \int f_+\chi_{E_1\cup\cdots\cup E_n} \, d\mu - \lim_n \int f_-\chi_{E_1\cup\cdots\cup E_n} \, d\mu \\ &= \int f_+\chi_E \, d\mu - \int f_-\chi_E \, d\mu = \int f\chi_E \, d\mu,\end{aligned}$$

showing that λ is countably additive.

Let $A = \{x \in X : f(x) \geq 0\}$ and $B = \{x \in X : g(x) < 0\}$. As f is measurable, $A \in \mathcal{R}$ and $B \in \mathcal{R}$. Then, for any $E \in \mathcal{R}$, we see that f is positive on $E \cap A$, and negative on $E \cap B$, so that

$$\lambda(E \cap A) = \int_{E\cap A} f \, d\mu \geq 0, \quad \lambda(E \cap B) \leq 0.$$

So (A, B) is a Hahn-Decomposition.

Question 6: Let $(\mathbb{R}, \mathcal{R}, \mu)$ be Lebesgue measure on the real line. Show, quickly, that we can define a measure ν on $\mathbb{R}$ by

$$\nu(A) = \int_A |x| \, d\mu(x) \qquad (A \in \mathcal{R}).$$

Show that $\nu << \mu$. However, show that for any $\epsilon > 0$, there does not exist $\delta > 0$ such that $\mu(A) \leq \delta$ implies that $\nu(A) \leq \epsilon$.

Proof: Clearly $\nu(\emptyset) = 0$; if (A_n) are pairwise disjoint, then for $A = \bigcup_n A_n$,

$$\sum_n \nu(A_n) = \sum_n \int_{A_n} |x| \ d\mu(x) = \int_A |x| \ d\mu(x) = \nu(A),$$

by Monotone Convergence, as

$$\chi_{A_1 \cup \cdots \cup A_n} |x| \uparrow \chi_A |x|.$$

If $\mu(A) = 0$, then $\nu(A) = \int |x| \chi_A \ d\mu(x) = 0$, as $|x|\chi_A = 0$ almost everywhere for μ. Hence $\nu << \mu$.

However, let $\delta > 0$, and let $t > 0$ be very large, so that

$$\nu((t, t+\delta)) = \int_t^{t+\delta} x \ dx \geq t\delta.$$

Thus, for all $\delta > 0$, there exists $A \in \mathcal{R}$ with $\mu(A) = \delta$, but $\nu(A)$ arbitrarily large.

Question 7: Let $(X, \mathcal{R})$ be a set with a σ-algebra, and let μ, λ be *finite* measures on $\mathcal{R}$. Show that the following are equivalent:

1. $\mu << \lambda$ and $\lambda << \mu$;
2. $A \in \mathcal{R}$ is μ-null if and only if it is λ-null;
3. there exists a measurable function $f : X \to (0, \infty)$ (*note that I am not using* $[0, \infty)$ *or* $[0, \infty]$) such that $\lambda(A) = \int_A f \ d\mu$ for all $A \in \mathcal{R}$.

Proof: Clearly (1) if and only if (2). If (1) holds, then by applying Radon-Nikodym, we can find a measurable $f : X \to [0, \infty)$ such that

$$\lambda(A) = \int_A f \ d\mu \qquad (A \in \mathcal{R}).$$

Let $B = \{x \in X : f(x) = 0\}$, so that

$$\lambda(B) = \int_B f \ d\mu = \int_B 0 \ d\mu = 0.$$

As $\mu << \lambda$, we also have that $\mu(B) = 0$. Define $\tilde{f} : X \to (0, \infty)$ by

$$\tilde{f}(x) = \begin{cases} f(x) & : x \notin B, \\ 1 & : x \in B. \end{cases}$$

Then for $A \in \mathcal{R}$, we have

$$\begin{aligned} \int_A \tilde{f} \ d\mu &= \int_{A \cap B} 1 \ d\mu + \int_{A \setminus B} f \ d\mu = \mu(A \cap B) + \lambda(A \setminus B) \\ &= \lambda(A \setminus B) = \lambda(A) - \lambda(A \cap B) = \lambda(A), \end{aligned}$$

as $\mu(A \cap B) \leq \mu(B) = 0$, and $\lambda(A \cap B) \leq \lambda(B) = 0$. So we have shown (3).

Finally, suppose (3) holds. Let $A \in \mathcal{R}$ be such that $\mu(A) = 0$, so clearly $\lambda(A) = 0$. Conversely, if $\lambda(A) = 0$, then for each $\epsilon > 0$,

$$\begin{aligned} 0 = \lambda(A) &= \int_A f \ d\mu = \int_{A \cap \{x \in X : f(x) \geq \epsilon\}} f \ d\mu + \int_{A \cap \{x \in X : f(x) < \epsilon\}} f \ d\mu \\ &\geq \epsilon \mu\big(A \cap \{x \in X : f(x) \geq \epsilon\}\big). \end{aligned}$$

Hence $A \cap \{x \in X : f(x) \geq \epsilon\}$ is a μ-null set for each $\epsilon > 0$. Thus

$$A = \bigcup_{n=1}^{\infty} A \cap \{x \in X : f(x) \geq 1/n\}$$

is also a μ-null set, which follows as $f > 0$ everywhere. Hence we have shown (2).

Question 8: Let $(\mathbb{R}, \mathcal{R}, \mu)$ be Lebesgue measure on the real line. Let (r_n) be an enumeration of the rationals. For each n, let

$$A_n = (r_n - 2^{-n}, r_n + 2^{-n}), \quad f_n = 2^n \chi_{A_n}.$$

Hence $f_n \geq 0$ and $\int_X f_n \, d\mu = 2$.

Let B be the set of $x \in \mathbb{R}$ such that x is in infinitely many of the sets A_n. Show that

$$B = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

Using Proposition 2.3, show that $\mu(B) = 0$. Hence show that $\sum_n f_n < \infty$ almost everywhere.

Proof: If x is in infinitely many of the sets A_n, then for each n, we have $x \in \bigcup_{k=n}^{\infty}$, and so $x \in B$. Conversely, if $x \in B$, then for all n, $x \in \bigcup_{k=n}^{\infty}$, so we must have that x is in infinitely many A_k, as required.

We see that

$$\mu(B) = \lim_{n\to\infty} \mu\Big(\bigcup_{k=n}^{\infty} A_k \Big) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} 2^{1-k} = \lim_{n\to\infty} 2^{-n} = 0.$$

Finally, let $f(x) = \sum_n f_n(x) = \sum_n 2^n \chi_{A_n}(x)$, which we allow to be infinity. Then $f(x) = \infty$ if and only if x is in infinitely many of the A_n, which is if and only if $x \in B$. As $\mu(B) = 0$, we see that $f < \infty$ almost everywhere.

Question continued: Define a measure $\lambda : \mathcal{R} \to [0, \infty]$ by

$$\lambda(A) = \sum_{n=1}^{\infty} \int_A f_n \, d\mu.$$

For $a < b$, show that $\lambda((a, b)) = \infty$. *Hint:* There must be infinitely many rational numbers in the open set (a, b). Conclude that $\lambda(U) = \infty$ for any open set $U \subseteq \mathbb{R}$.

Show, however, that $\lambda << \mu$. Hence absolutely continuous measures can be pretty nasty!

Proof: First notice that for $A \in \mathcal{R}$,

$$\lambda(A) = \sum_{n=1}^{\infty} \int_A f_n \, d\mu = \sum_{n=1}^{\infty} \int_A 2^n \chi_{A_n} \, d\mu = \sum_{n=1}^{\infty} 2^n \mu(A \cap A_n).$$

For $a < b$, let $X = \{n \in \mathbb{N} : a < r_n < b\}$, so that X is infinite. For $n \in X$, we see that

$$A \cap A_n = A \cap (r_n - 2^{-n}, r_n + 2^{-n}) = \big(\max(a, r_n - 2^{-n}), \min(b, r_n + 2^{-n})\big).$$

If $2^{-n} < (b - a)/2$, then we crudely estimate that

$$\mu(A \cap A_n) \geq 2^{-n}.$$

Hence we conclude that

$$\lambda(A) \geq \sum_{n \in X} 2^n \mu(A \cap A_n) \geq \sum_{n \in X} 1 = \infty.$$

If U is open, then we can find $a < b$ with $(a, b) \subseteq U$, so that

$$\lambda(U) \geq \lambda((a, b)) = \infty.$$

However, if $\mu(A) = 0$, then clearly $\lambda(A) = 0$, so $\lambda << \mu$.

Linear Analysis I: Worked Solutions 8

Question 1: Let $(\mathbb{R}, \mathcal{R}, \mu)$ be Lebesgue measure on the real line. Let X be the subset of $L^1(\mu)$ consisting of those $f \in L^1(\mu)$ such that, for some $K > 0$, we have that $|f| \leq K$ almost everywhere (loosely, we could write $f \in L^1(\mu) \cap L^\infty(\mu)$). Hence X is also a subspace of $\mathcal{L}^1(\mu)$.

Show that $f : \mathbb{R} \to [0, \infty)$ defined by

$$f(x) = \begin{cases} n^{1/2} & : (n+1)^{-1} < x \leq n^{-1} \text{ for some } n \in \mathbb{N}, \\ 0 & : \text{otherwise}, \end{cases}$$

is in $L^1(\mu)$. Hence, or otherwise, show *carefully* show that $X \neq \mathcal{L}^1(\mu)$.

Show, however, that X is dense in $\mathcal{L}^1(\mu)$.

Answer: We see, again technically by Monotone Convergence, that

$$\int_{\mathbb{R}} |f| \, d\mu = \sum_{n=1}^{\infty} n^{1/2} \Big(\frac{1}{n} - \frac{1}{n+1} \Big) = \sum_{n=1}^{\infty} \frac{n^{1/2}}{n(n+1)} \leq \sum_{n=1}^{\infty} n^{-3/2} < \infty.$$

Hence $f \in L^1(\mu)$. However, for any n, we see that $|f| > n^{1/2}$ on, say, the interval $((n+2)^{-1}, (n+1)^{-1})$, which does not have measure zero. Hence there can exist no $K > 0$ with $|f| \leq K$ almost everywhere.

Suppose there exists $g \in L^1(\mu)$ and $K > 0$ with $|g| \leq K$ almost everywhere, and yet $f = g$ almost everywhere (so that f and g define the same vector in $\mathcal{L}^1(\mu)$). Then $|f| \leq K$ almost everywhere, which is a contradiction. So $f \notin X$.

Suppose there exists $h \in \mathcal{L}^1(\mu)$ and $\epsilon > 0$, such that for every $g \in X$, we have $\|h - g\|_1 \geq \epsilon$. In particular, for each $n \in \mathbb{N}$,

$$h_n = h\chi_{\{x \in X : |h| \leq n\}} \in X,$$

because $|h_n| \leq n$, and as $|h_n| \leq |h|$, also $h_n \in \mathcal{L}^1(\mu)$. Thus, for each n,

$$\epsilon \leq \|h - h_n\|_1 = \int_{\{x \in X : |h| > n\}} |h| \, d\mu.$$

However, we clearly have that $|h_n| \uparrow |h|$, and so by Monotone Convergence,

$$\int_X |h| \, d\mu = \lim_{n \to \infty} \int_X |h_n| \, d\mu.$$

Hence

$$\epsilon \leq \lim_{n \to \infty} \int_X |h| \chi_{\{x \in X : |h| > n\}} \, d\mu = \lim_{n \to \infty} \int_X |h| - |h_n| \, d\mu = 0,$$

a contradiction.

Question 2: This continues from Question 1. Show that the mapping

$$T(f) = g \quad \text{where} \quad g(t) = \int_{[0,t]} f \, d\mu \qquad (t \geq 0),$$

is a well-defined map $X \to C_{\mathbb{K}}([0, \infty))$.

As usual, we give $C_{\mathbb{K}}([0, \infty))$ the $\| \cdot \|_\infty$ norm. Show that T is linear and bounded. What is $\|T\|$?

Does the definition of T make sense on $\mathcal{L}^1(\mu)$?

My thanks to Thomas for pointing out that I did something a bit cheeky here. We haven't studied that space $C_{\mathbb{K}}([0, \infty))$ before, as $[0, \infty)$ is not compact. Here are some solutions out of this problem:

- Just work in $C_{\mathbb{K}}([0,N])$ for some N.
- If we interpret $C_{\mathbb{K}}([0,\infty))$ to mean the vector space of *bounded* continuous functions $[0,\infty)\to\mathbb{K}$, then actually $C_{\mathbb{K}}([0,\infty))$ is a Banach space: the proof I have still works in the non-compact case.
- The really sophisticated method might be to work with $[0,\infty]$, defined here as the *one-point compactification*[1] of $[0,\infty)$. The $C_{\mathbb{K}}([0,\infty])$ can be identified with the space of continuous functions $f:[0,\infty)\to\mathbb{K}$ with $\lim_{t\to\infty} f(t)$ existing.

Proof: Obviously (because we are integrating) T is well-defined on $\mathcal{L}^1(\mu)$, and so also on X. Let $f\in X$, so there exists $K>0$ with $|f|\leq K$ a.e. and so for $t\geq 0$ and $h>0$, we have

$$|g(t+h)-g(t)| = \Big|\int_{(t,t+h]} f \ d\mu\Big| \leq \int_{(t,t+h]} |f| \ d\mu \leq Kh.$$

Hence g is continuous. Clearly T is linear. We see that

$$\|g\|_\infty = \sup_{t\geq 0}\Big|\int_{[0,t]} f \ d\mu\Big| \leq \sup_{t\geq 0}\int_{[0,t]} |f| \ d\mu \leq \int_{\mathbb{R}} |f| \ d\mu = \|f\|_1.$$

Thus $\|T\|\leq 1$. If $f=\chi_{[0,1]}$, then

$$g(t) = \int_{[0,t]} \chi_{[0,1]} \ d\mu = \mu([0,1]\cap[0,t]) = \mu([0,\min(t,1)]) = \min(t,1).$$

So $\|g\|_\infty = 1 = \|f\|_1$, and so $\|T\|=1$.

Finally, as X is dense in $\mathcal{L}^1(\mu)$, if $f\in\mathcal{L}^1(\mu)$, then there exists a sequence (f_n) in X with $f_n\to f$. In particular, (f_n) is Cauchy, so for $\epsilon>0$, there exists N such that $\|f_n-f_m\|<\epsilon$ for $n,m\geq N$. Then

$$\|T(f_n)-T(f_m)\| \leq \|f_n-f_m\| < \epsilon \qquad (n,m\geq N).$$

So $(T(f_n))$ is Cauchy in $C_{\mathbb{K}}([0,\infty))$, which is a Banach space, and hence converges to $T(f)$ say. This is well-defined, for if $g_n\to f$ as well, then for each $\epsilon>0$, there exists M with $\|f-f_n\|<\epsilon/2$ and $\|f-g_n\|<\epsilon/2$ for $n\geq M$. Thus $\|f_n-g_n\|<\epsilon$ for $n\geq M$, showing that $\|T(f_n)-T(g_n)\|<\epsilon$ for $n\geq M$. Hence $\lim_n T(f_n)=\lim_n T(g_n)$.

We can similarly show that T is linear, bounded, and that $\|T\|=1$.

However, notice that it's not obvious, just from the definition, that T is defined on $\mathcal{L}^1(\mu)$ (because why would we get a *continuous* function by integrating an $\mathcal{L}^1$ function?)

Question 3: With notation as from Question 1: for $1<p<\infty$, let $X_p\subseteq\mathcal{L}^p(\mu)$ have the same definition as X. Show quickly that X_p is a subspace. By using Question 1, and the fact that $\mathcal{L}^p(\mu)^*=\mathcal{L}^q(\mu)$, show that X_p is dense in $\mathcal{L}^p(\mu)$.

Proof: It is simple to show that X_p is a subspace. If X_p is not dense, that we could find a non-zero $g\in\mathcal{L}^p(\mu)^*=\mathcal{L}^q(\mu)$ which kills all[2] of X_p. We shall show that this is not possible, so that X_p is dense.

So suppose $g\in\mathcal{L}^q(\mu)$ is such that

$$\int_{\mathbb{R}} fg \ d\mu = 0 \qquad (f\in X_p).$$

[1] See Wikipedia

[2] In Chapter 1, we used the Hahn-Banach theorem to show that if X is a Banach space, and Y a subspace, then for $x\in X$, we have that x is in the closure of Y if and only if $\mu(x)=0$ whenever $\mu\in X^*$ has $Y\subseteq\ker\mu$. So it follows that Y is dense in X (that is, the closure of Y is all of X) if and only if, whenever $\mu\in X^*$ with $Y\subseteq\ker\mu$, we actually have that $\mu=0$.

In particular, if $A \subseteq \mathbb{R}$ has finite measure, then $\chi_A \in X_p$, as $\|\chi_A\|_p = \mu(A)^{1/p} < \infty$. Thus

$$\int_A g \, d\mu = \int_{\mathbb{R}} \chi_A g \, d\mu = 0.$$

So let $\epsilon > 0$, let $B = \{x \in \mathbb{R} : g(x) \geq \epsilon\}$, and suppose towards a contradiction that $\mu(B) \neq 0$. Then

$$0 \neq \mu(B) = \lim_{n\to\infty} \mu\big(B \cap [-n, n]\big),$$

so for some $n > 0$, we have that $\mu(B \cap [-n, n]) > 0$. As $B \cap [-n, n]$ has finite measure, it follows that

$$0 = \int_{B\cap[-n,n]} g \, d\mu \geq \epsilon\mu(B \cap [-n, n]) > 0,$$

a contradiction. Thus $\{x \in \mathbb{R} : g(x) \geq \epsilon\}$ is null for all $\epsilon > 0$, and so $\{x \in \mathbb{R} : g(x) > 0\}$ is null.[3] Similarly, $\{x \in \mathbb{R} : g(x) < 0\}$ is null. So $g = 0$ a.e. as required.

Question 4: We show that $C([0, 1])$ is not dense in $\mathcal{L}^\infty([0, 1])$ (over either $\mathbb{R}$ or $\mathbb{C}$). Let $f : [0, 1] \to [-1, 1]$ be defined by

$$f(x) = \begin{cases} 0 & : x = 0, \\ \sin(1/x) & : 0 < x \leq 1. \end{cases}$$

As f is continuous, except at 0, it is measurable. Clearly f is bounded everywhere, so $f \in \mathcal{L}^\infty([0, 1])$. By considering what happens at zero, show that for any $g \in C([0, 1])$, we have that $\|f - g\|_\infty \geq 1$.

Answer: Notice that

$$f\Big(\frac{1}{2\pi n + \pi/2}\Big) = 1, \quad f\Big(\frac{1}{2\pi n - \pi/2}\Big) = -1 \qquad (n \in \mathbb{N}).$$

As g is continuous, for $\epsilon > 0$, there exists $\delta > 0$ such that $|g(0) - g(t)| < \epsilon$ when $|t| < \delta$. Then there exists n with $(2\pi n + \pi/2)^{-1} < \delta$, and $(2\pi n - \pi/2)^{-1} < \delta$, so

$$\Big|f\Big(\frac{1}{2\pi n + \pi/2}\Big) - g\Big(\frac{1}{2\pi n + \pi/2}\Big)\Big| \geq |1 - g(0)| - \epsilon,$$

and

$$\Big|f\Big(\frac{1}{2\pi n - \pi/2}\Big) - g\Big(\frac{1}{2\pi n - \pi/2}\Big)\Big| \geq | - 1 - g(0)| - \epsilon,$$

so for some choice, we certainly get a number greater than $1 - \epsilon$, and so we have that $\sup_{0\leq t\leq 1} |f(t) - g(t)| \geq 1 - \epsilon$.

However, we need to deal with essential supremums. Suppose for the moment that $g(0) \leq 0$. Then let N be such that $2\pi N + \pi/2 > 1/\delta$, let $\gamma > 0$ be small, and let

$$A_\epsilon = \bigcup_{n\geq N} \Big(\frac{1}{2\pi n + \pi/2 + \gamma}, \frac{1}{2\pi n + \pi/2 - \gamma}\Big).$$

Then if γ is sufficiently small, we have that $t \in A_\epsilon$ implies that $f(t) > 1 - \epsilon$. Notice that A_ϵ is not a null set. Then, if $t \in A_\epsilon$, then

$$|f(t) - g(t)| \geq |f(t) - g(0)| - |g(0) - g(t)| \geq 1 - \epsilon - g(0) - \epsilon \geq 1 - 2\epsilon.$$

[3] If you don't see this, think about the proof from lectures of the fact that for $f \in L^\infty(\mu)$, we have that $|f| \leq \|f\|_\infty$ almost everywhere.

Hence we see that

$$\text{ess-sup}_{[0,1]} |f - g| \geq 1 - 2\epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that $\|f - g\|_\infty \geq 1$ in $\mathcal{L}^\infty([0,1])$. A similar argument applies when $g(0) \geq 0$.

Question 5: Let $([0,1], \mathcal{R}, \mu)$ be the restriction of the Lebesgue measure to $[0,1]$. Let $f \in \mathcal{L}^\infty(\mu)$. Show that $f \in \mathcal{L}^p(\mu)$ for $1 \leq p < \infty$, and $\sup\{\|f\|_p : 1 \leq p < \infty\} < \infty$.
Answer: As $|f| \leq \|f\|_\infty$ almost everywhere, for any $p \geq 1$, we have $|f|^p \leq \|f\|_\infty^p$ almost everywhere. Hence

$$\|f\|_p = \Big(\int_{[0,1]} |f|^p \, d\mu \Big)^{1/p} \leq \big(\|f\|_\infty^p \big)^{1/p} = \|f\|_\infty.$$

Question continued: Conversely, suppose that $f : [0,1] \to \mathbb{K}$ is measurable, that $f \in \mathcal{L}^p(\mu)$ for each $1 \leq p < \infty$, and that $\sup\{\|f\|_p : 1 \leq p < \infty\} < \infty$. Show that $f \in \mathcal{L}^\infty(\mu)$.
Answer: Let $K > 0$, and suppose $A = \{x \in [0,1] : |f(x)| \geq K\}$ is not null. Hence $|f| \geq K\chi_A$, and so for $p \geq 1$, also $|f|^p \geq K^p \chi_A$, so

$$K\mu(A)^{1/p} = \big(K^p \mu(A)\big)^{1/p} = \Big(\int_{[0,1]} K^p \chi_A \, d\mu \Big)^{1/p} \leq \Big(\int_{[0,1]} |f|^p \, d\mu \Big)^{1/p} = \|f\|_p.$$

For $0 < t \leq 1$, we have that $\sup_{p \geq 1} t^{1/p} = 1$, so

$$K = K \sup_{p \geq 1} \mu(A)^{1/p} \leq \sup_{p \geq 1} \|f\|_p.$$

We hence conclude that

$$\|f\|_\infty \leq \sup_{p \geq 1} \|f\|_p,$$

showing that $f \in \mathcal{L}^\infty(\mu)$.
Question continued: Finally, show that if $f \in \mathcal{L}^\infty(\mu)$, then

$$\|f\|_\infty = \lim_{p \to \infty} \|f\|_p.$$

Answer: From the above, we saw that if $|f| \geq K$ on a non-null set, then

$$K \leq \lim_{p \to \infty} \|f\|_p.$$

Hence we see that

$$\|f\|_\infty \leq \liminf_{p \to \infty} \|f\|_p.$$

Conversely, by the first part, we see that

$$\limsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty.$$

In conclusion,

$$\limsup_{p \to \infty} \|f\|_p \leq \|f\|_\infty \leq \liminf_{p \to \infty} \|f\|_p \leq \limsup_{p \to \infty} \|f\|_p,$$

so we have equality throughout, and by a previous sheet, $\|f\|_p$ tends to a limit, which must be $\|f\|_\infty$.

Question 6: We know that $(\ell^1)^* = \ell^\infty$, so it might be tempting to believe that $(\ell^\infty)^* = \ell^1$. This is impossible, as ℓ^∞ is not separable, while ℓ^1 is. However, let us give a more direct argument.

Treat c_0 as a (closed) subspace of ℓ^∞. Let $A \subseteq \mathbb{N}$ be infinite, so $\chi_A \in \ell^\infty$, but $\chi_A \notin c_0$. Show that

$$d(\chi_A, c_0) := \inf \left\{ \|\chi_A - x\|_\infty : x \in c_0 \right\} = 1.$$

Answer: If $x \in c_0$ then for $\epsilon > 0$, there exists N such that $|x_n| < \epsilon$ when $n \geq N$. Then, as A is infinite, there exists $n \in A$ with $n \geq N$, so that

$$|\chi_A(n) - x_n| = |1 - x_n| \geq 1 - \epsilon.$$

Hence $\|\chi_A - x_n\|_\infty \geq 1 - \epsilon$, and so as $\epsilon > 0$ was arbitrary, $\|\chi_A - x_n\|_\infty \geq 1$. Conversely, as $\|\chi_A\|_\infty = 1$, taking $x = 0$ gives $d(\chi_A, c_0) = 1$.

Question continued: Show that the linear map defined by

$$\phi : c_0 + \mathbb{K}\chi_A = \{x + t\chi_A : x \in c_0, t \in \mathbb{K}\} \to \mathbb{K}, \quad \phi(x + t\chi_A) = t,$$

is well-defined, and that $\|\phi\| = 1$. Hence, by the Hahn-Banach Theorem, show that there exists $\psi \in (\ell^\infty)^*$ such that

$$\psi(\chi_A) = 1, \qquad \psi(x) = 0 \qquad (x \in c_0).$$

Answer: If $x_1 + t_1\chi_A = x_2 + t_2\chi_A$, then either $t_1 = t_2$, or otherwise, $\chi_A = (t_1 - t_2)^{-1}(x_2 - x_1) \in c_0$, a contradiction. So ϕ is well-defined. If $t = 0$, then $\chi(x + t\chi_A) = 0 \leq \|x + t\chi_A\|$. For $t \neq 0$, from above, we have

$$1 \leq \|t^{-1}x + \chi_A\|_\infty = |t^{-1}|\|x + t\chi_A\|_\infty,$$

and so $|\phi(x + t\chi_A)| = |t| \leq \|x + t\chi_A\|_\infty$, showing that $\|\phi\| \leq 1$. As $\|\phi(\chi_A)\| = 1 = \|\chi_A\|$, we have $\|\phi\| = 1$. So let ψ be a Hahn-Banach extension to a member of $(\ell^\infty)^*$. Clearly ψ has the stated properties.

Question continued: Show that there cannot exist $(a_n) \in \ell^1$ such that

$$\psi(x) = \sum_{n=1}^{\infty} a_n x_n \qquad (x = (x_n) \in \ell^\infty).$$

Answer: Suppose there does exist such an (a_n). Then let $x_n = \overline{a_n}$ for each n, so as $\sum_n |a_n| < \infty$, clearly $(x_n) \in c_0$, and yet

$$\psi(x) = \sum_n a_n x_n = \sum_n |a_n|^2,$$

so we must have $a_n = 0$ for all n, giving that

$$1 = \psi(\chi_A) = \sum_{n \in A} a_n = 0,$$

a contradiction.

So ψ is not a member of ℓ^1.

Linear Analysis I: Worked Solutions 9

Question 1: Let K be a compact space. Let (f_n) be a sequence of positive functions in $C_{\mathbb{R}}(K)$, and let $f \in C_{\mathbb{R}}(K)$ be such that for each $x \in K$,

$$f_1(x) \leq f_2(x) \leq \cdots, \qquad f(x) = \lim_n f_n(x).$$

Show that

$$\lambda(f) = \lim_n \lambda(f_n) \qquad (\lambda \in C_{\mathbb{R}}(K)^*).$$

Hint: Use the Riesz Representation Theorem, Hahn-Decomposition, and Monotone Convergence.

Answer: By the Riesz Representation Theorem, there exists a finite, regular, Borel signed measure μ on K such that

$$\lambda(g) = \int_K g \ d\mu \qquad (g \in C_{\mathbb{R}}(K)).$$

By the Hahn-Decomposition, we can write $\mu = \mu_+ - \mu_-$ for some positive measures μ_+ and μ_-. By the conditions on (f_n) and f, the Monotone Convergence Theorem implies that

$$\int_K f \ d\mu_+ = \lim_n \int_K f_n \ d\mu_+, \quad \int_K f \ d\mu_- = \lim_n \int_K f_n \ d\mu_-.$$

Hence

$$\begin{aligned}\lambda(f) = \int_K f \ d\mu = \int_K f \ d\mu_+ - \int_K f \ d\mu_- = \lim_n \int_K f_n \ d\mu_+ - \int_K f_n \ d\mu_- \\ = \lim_n \int_K f_n \ d\mu = \lim_n \lambda(f_n),\end{aligned}$$

as required.

Question 2: Let K be a compact space, let (f_n) be a sequence in $C_{\mathbb{C}}(K)$, let $f \in C_{\mathbb{C}}(K)$ and let $M > 0$ be such that

$$\|f_n\|_\infty \leq M \quad (n \in \mathbb{N}), \qquad f(x) = \lim_n f_n(x) \qquad (x \in K).$$

Show that

$$\lambda(f) = \lim_n \lambda(f_n) \qquad (\lambda \in C_{\mathbb{C}}(K)^*).$$

Hint: Use the Riesz Representation Theorem, Hahn-Decomposition, Dominated Convergence, and take positive and negative parts.

Answer: This is similar to Question 1. By the Riesz Representation Theorem for complex numbers, there exists a complex, regular, finite, Borel measure μ on K which induces λ. Split μ up as $\mu_r + i\mu_i$ for signed measures μ_r and μ_i. Then split these up as $\mu_r = \mu_+^{(r)} - \mu_-^{(r)}$ and $\mu_r = \mu_+^{(i)} - \mu_-^{(i)}$ for positive measures $\mu_+^{(r)}, \mu_-^{(r)}, \mu_+^{(i)}$ and $\mu_-^{(i)}$. By the conditions on (f_n), as the constant function M is integrable on K (as all our measures are finite) we can apply the dominated convergence theorem to see that

$$\int_K f \ d\mu_+^{(r)} = \lim_n \int_K f_n \ d\mu_+^{(r)},$$

and for $\mu_-^{(r)}$, $\mu_+^{(i)}$ and $\mu_-^{(i)}$. The result then follows.

Question 3: Let $K = [0,1]$ and for each n, define $f_n \in C_{\mathbb{R}}(K)$ by

$$f_n(x) = \begin{cases} n^2 x & : 0 \le x \le 1/n, \\ 2n - n^2 x & : 1/n \le x \le 2/n, \\ 0 & : x > 2/n. \end{cases}$$

Show that $f_n(x) \to 0$ for each $x \in K$, but that there exists $\mu \in C_{\mathbb{R}}(K)^*$ such that $\mu(f_n) \not\to 0$.

Answer: We have that $f_n(0) = 0$ for all n, while, if $t > 0$, then for n sufficiently large, $f_n(t) = 0$, as eventually $t > 2/n$. Hence $f_n \to 0$ pointwise.

However, define $\lambda \in C_{\mathbb{R}}(K)^*$ by integrating against Lebesgue Measure μ, say

$$\lambda(f) = \int_{[0,1]} f \; d\mu \qquad (f \in C_{\mathbb{R}}([0,1])).$$

Then, for each n,

$$\begin{aligned} \lambda(f_n) &= \int_0^{1/n} n^2 x \; dx + \int_{1/n}^{2/n} 2n - n^2 x \; dx \\ &= \left[\frac{n^2x^2}{2}\right]_{x=0}^{1/n} + \left[2nx - \frac{n^2x^2}{2}\right]_{x=1/n}^{2/n} = \frac{1}{2} + 4 - 2 - 2 + \frac{1}{2} = 1. \end{aligned}$$

Question 4: Let K be a topological space. We shall define the *Borel σ-algebra* on K to be the σ-algebra generated by open sets in K; again we write $\mathcal{B}(K)$ for this. In particular, we get $\mathcal{B}(\mathbb{K})$.

Given two topological spaces K and L, we shall say that a map $f : K \to L$ is *Borel* if $f^{-1}(E) \in \mathcal{B}(K)$ for each $E \in \mathcal{B}(L)$.

Now let K be a compact space, and consider K with the Borel σ-algebra $\mathcal{B}(K)$. Show that $f : K \to \mathbb{K}$ is measurable if and only if f is Borel.

Answer: If f is measurable, then by definition, if $U \subseteq \mathbb{K}$ is open, then $f^{-1}(U) \in \mathcal{B}(K)$. But we need to show this for all Borel sets, for which a little trick is required. Define

$$\mathcal{S} = \{A \subseteq \mathbb{K} : f^{-1}(A) \in \mathcal{B}(K)\}.$$

We claim that this is a σ-algebra on K. Then it will contain all the open sets, and hence contains the σ-algebra generated by the open sets, that is, $\mathcal{B}(\mathbb{K}) \subseteq \mathcal{S}$, showing that f is Borel.

So how to prove the claim? Well, clearly $\emptyset, \mathbb{K} \in \mathcal{S}$. If $A \in \mathcal{S}$, then

$$f^{-1}(\mathbb{K} \setminus A) = K \setminus f^{-1}(A) \in \mathcal{B}(K),$$

so $\mathbb{K} \setminus A \in \mathcal{S}$. If $(A_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{S}$, then

$$f^{-1}\Big(\bigcup_n A_n\Big) = \bigcup_n f^{-1}(A_n) \in \mathcal{B}(K),$$

so $\bigcup_n A_n \in \mathcal{S}$. So $\mathcal{S}$ is a σ-algebra.

Conversely, let f be Borel. Then every open set is Borel in $\mathbb{K}$, and so automatically f is measurable.

Question 5: Let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$. Show that there exists $S \in \mathcal{B}(F^*, E^*)$ with the following property: for $\phi \in F^*$, we have that $S(\phi) = \psi \in E^*$, where

$$\psi(x) = \phi\big(T(x)\big) \qquad (x \in E).$$

We call S the *adjoint* of T, and write $S = T^*$.

Answer: As T and ϕ is linear, the map

$$E \to \mathbb{K}, \quad x \mapsto \phi(T(x))$$

is linear, and so ψ is linear. Then, for $x \in E$,

$$|\psi(x)| = \big|\phi(T(x))\big| \leq \|\phi\|\|T(x)\| \leq \|\phi\|\|T\|\|x\|.$$

As x was arbitrary, it follows that $\|\psi\| \leq \|\phi\|\|T\|$. So $\psi \in E^*$ as claimed.

It is easy to see that the map $\phi \mapsto \psi$ is linear, and so $S : F^* \to E^*$ is defined and linear. Then, for $\phi \in F^*$,

$$\|S(\phi)\| = \|\psi\| \leq \|\phi\|\|T\|,$$

so S is bounded, and $\|S\| \leq \|T\|$.

If you wish, try to use the Hahn-Banach theorem to show that actually $\|S\| = \|T\|$ (this is a bit tricky: ask if you are interested).

Question 6: Let $(X, \mathcal{R}, \mu)$ be a measure space. We say that $E \in \mathcal{R}$ is an *atom* if $\mu(E) \neq 0$, and if $F \in \mathcal{R}$ with $F \subseteq E$ then either $\mu(F) = \mu(E)$ or $\mu(F) = 0$.

Suppose that for some $x \in X$, we have that $\{x\} \in \mathcal{R}$. Show that $\{x\}$ is an atom if and only if $\mu(\{x\}) \neq 0$.

Answer: If $\mu(\{x\}) \neq 0$ then if $F \subseteq \{x\}$, either $F = \{x\}$, so $\mu(F) = \mu(\{x\})$, or $F = \emptyset$, so $\mu(\emptyset) = 0$. Hence $\{x\}$ is an atom. Conversely, if $\{x\}$ is an atom, then by definition, $\mu(\{x\}) \neq 0$.

Question continued: Let $E \in \mathcal{R}$ be an atom. Let $(E_n)_{n=1}^\infty$ be a *partition* of E; that is, $E_n \in \mathcal{R}$ and $E_n \subseteq E$ for each n, for $n \neq m$ we have $E_n \cap E_m = \emptyset$, and finally $\bigcup_n E_n = E$. If μ is finite, show that there exists a unique n_0 with E_{n_0} being an atom.

Answer: Suppose that no E_n is an atom, so by definition, for each n, we can find $F_n \in \mathcal{R}$ with $F_n \subseteq E_n$, and with $0 < \mu(F_n) < \mu(E_n)$. Let $F = \bigcup_n F_n \in \mathcal{R}$, so that

$$0 < \sum_n \mu(F_n) = \mu(F) = \sum_n \mu(F_n) < \sum_n \mu(E_n) = \mu(E),$$

so $0 < \mu(F) < \mu(E)$, which contradicts E being an atom.

So there exists n_0 with E_{n_0} being an atom. In particular, $\mu(E_{n_0}) \neq 0$. Then $E_{n_0} \in \mathcal{R}$ and $E_{n_0} \subseteq E$, so as E is an atom, $\mu(E_{n_0}) = \mu(E)$. Thus

$$0 = \mu(E \setminus E_{n_0}) = \sum_{n \neq n_0} \mu(E_n),$$

showing that no other E_n can an atom (as they all have zero measure).

Question continued: Is this still true if μ is not finite?

Answer: Where did we use that E is finite? We actually used it in the final displayed equation! Indeed,

$$\mu(E) = \mu(E_{n_0}) + \mu(E \setminus E_{n_0}) = \mu(E) + \mu(E \setminus E_{n_0})$$

for any measure, but we can only conclude that $\mu(E \setminus E_{n_0}) = 0$ if $\mu(E) < \infty$.

A silly example is given by the following: let X be an infinite set, let $\mathcal{R}$ be power set of X, and define μ on $\mathcal{R}$ by $\mu(\emptyset) = 0$ and $\mu(A) = \infty$ for any non-empty $A \subseteq X$. Then μ is a measure, and every non-empty set is an atom!

Question 7: This follows on from Question 6. Let K be a compact Hausdorff space, and let μ be a finite, regular (positive) Borel measure. Let $E \in \mathcal{B}(K)$ be an atom. Show that there exists a closed set $F \subseteq E$ which is an atom.
Answer: As μ is regular,

$$\mu(E) = \sup\big\{\mu(F) : F \subseteq E \text{ is compact}\big\}$$

As E is an atom, $\mu(E) > 0$. So we can find $F \subseteq E$ compact with $\mu(F) > 0$. As E is an atom, we must have that $\mu(F) = \mu(E)$. If F is not an atom, then we can find $G \in \mathcal{B}(K)$ with $G \subseteq F$ and $0 < \mu(G) < \mu(F)$. Then $G \subseteq E$ and $\mu(G) < \mu(E)$, which contradicts E being an atom.

Question continued: Suppose, towards a contradiction, that $x \in F$ implies that $\{x\}$ is not an atom. Show that for each $x \in F$ there exists an open set U_x with $x \in U_x$ and $\mu(U_x) < \mu(F)$.

As F is compact, and $\{U_x : x \in F\}$ is an open cover, there exist $x_1, \cdots, x_n$ in F with $U_{x_1} \cup \cdots \cup U_{x_n} \supseteq F$. Let $A_j = U_{x_j} \cap F$ for $1 \leq j \leq n$, let $B_1 = A_1$ and $B_j = A_j \setminus (A_1 \cup \cdots \cup A_{j-1})$ for $j \geq 2$. Why is $(B_j)_{j=1}^n$ a partition of F? Show that $\mu(B_j) < \mu(F)$ for each j, and hence derive a contradiction (think about Question 6 here).
Answer: By the above, this is equivalent to $\mu(\{x\}) = 0$ for all $x \in F$. As μ is regular,

$$0 = \mu(\{x\}) = \inf\big\{\mu(U) : \{x\} \subseteq U \text{ is open }\big\}.$$

So we can find U_x and open set with $x \in U_x$ and $\mu(U_x)$ as small as we like, certainly $\mu(U_x) < \mu(F)$.

Following the hint, we find $x_1, \cdots, x_n \in F$ with $F \subseteq U_{x_1} \cup \cdots \cup U_{x_n}$. By definition,

$$F = \bigcup_j U_{x_j} \cap F = \bigcup_j A_j = \bigcup_j B_j$$

and clearly the (B_j) are pairwise disjoint. Then

$$\mu(B_j) \leq \mu(A_j) \leq \mu(U_{x_j}) < \mu(F).$$

By the previous question, this is a contradiction, as one B_j must be an atom.

This contradiction shows that for some $x \in F$, we have that $\{x\}$ is an atom.

Question continued: Hence show that if $E \in \mathcal{B}(K)$ is an atom, then there exists a unique $x \in E$ with $\{x\}$ being an atom, and $\mu(E \setminus \{x\}) = 0$.
Answer: We have shown that if E is an atom, then there exists $x \in E$ with $\{x\}$ an atom. If $\mu(E) \neq \mu(\{x\})$, then $0 < \mu(E \setminus \{x\}) < \mu(E)$, contradicting E being an atom.

Question 8: Let K be a compact space. Given a Borel map $\psi : K \to K$ and $\mu \in M_{\mathbb{C}}(K)$, show (carefully) that

$$\psi(\mu) : \mathcal{B}(K) \to \mathbb{C}, \quad A \mapsto \mu(\psi^{-1}(A)) \qquad (A \in \mathcal{B}(K))$$

defines a measure on $\mathcal{B}(K)$.
Answer: As ψ is Borel, for $A \in \mathcal{B}(K)$, we have that $\psi^{-1}(A) \in \mathcal{B}(K)$, and so $\mu(\psi^{-1}(A))$ is defined. Clearly $\psi(\mu)(\emptyset) = 0$. Let (A_n) be a sequence of pairwise disjoint sets in $\mathcal{B}(K)$. Then, as inverse images behave very nicely with respect to disjoint unions, we have

$$\psi(\mu)\Big(\bigcup_n A_n\Big) = \mu\psi^{-1}\Big(\bigcup_n A_n\Big) = \mu\Big(\bigcup_n \psi^{-1}(A_n)\Big) = \sum_n \mu(\psi^{-1}(A_n)) = \sum_n \psi(\mu)(A_n).$$

So $\psi(\mu)$ is a measure.

Question continued: Do you think that $\psi(\mu)$ need be regular? What if ψ is even continuous?

Answer: There appears to no reason why $\psi(\mu)$ should be regular, as we know very little about what ψ^{-1} will do to compact sets, say.

If ψ is continuous, however, then we can argue as follows. Let $A \in \mathcal{B}(K)$. If $B \subseteq A$ then $\psi^{-1}(B) \subseteq \psi^{-1}(A)$, so automatically

$$\mu(\psi^{-1}(A)) \geq \sup\left\{\mu(\psi^{-1}(B)) : B \subseteq A \text{ is compact}\right\}.$$

As μ is regular, we know that

$$\mu(\psi^{-1}(A)) = \sup\left\{\mu(C) : C \subseteq \psi^{-1}(A) \text{ is compact}\right\}.$$

For $\epsilon > 0$, pick $C \subseteq \psi^{-1}(A)$ compact with $\mu(C) > \mu(\psi^{-1}(A)) - \epsilon$. Then $\psi(C)$ is also compact[1] and as $C \subseteq \psi^{-1}(A)$, we have that $\psi(C) \subseteq A$. Then let $D = \psi^{-1}(\psi(C))$ so that $C \subseteq D$. Then

$$\mu(\psi^{-1}(\psi(C))) = \mu(D) \geq \mu(C) > \mu(\psi^{-1}(A)) - \epsilon.$$

As $\epsilon > 0$ was arbitrary, we conclude that

$$\mu(\psi^{-1}(A)) \leq \sup\left\{\mu(\psi^{-1}(B)) : B \subseteq A \text{ is compact}\right\},$$

and so we actually have that equality. So $\psi(\mu)$ is *inner regular*.

We now use a trick which we saw a couple of sheets ago. Let $A' = K \setminus A$, so $A' \in \mathcal{B}(K)$, and hence

$$\mu(\psi^{-1}(A')) = \sup\left\{\mu(\psi^{-1}(B)) : B \subseteq A' \text{ is compact}\right\}.$$

For $\epsilon > 0$, we can hence find $B \subseteq A'$ compact (hence closed) with $\mu(\psi^{-1}(B)) > \mu(\psi^{-1}(A')) - \epsilon$. Let $U = K \setminus B$, so that U is open, and $A \subseteq U$. Then

$$\begin{aligned}\mu(\psi^{-1}(U)) &= \mu(\psi^{-1}(K)) - \mu(\psi^{-1}(B)) < \mu(K) - \mu(\psi^{-1}(A')) + \epsilon \\ &= \mu(K) + \epsilon - \mu(K \setminus \psi^{-1}(A)) = \mu(K) + \epsilon - \mu(K) + \mu(\psi^{-1}(A)) \\ &= \mu(\psi^{-1}(A)) + \epsilon.\end{aligned}$$

As $\epsilon > 0$ was arbitrary, we conclude that

$$\mu(\psi^{-1}(A)) = \inf\left\{\mu(\psi^{-1}(U)) : A \subseteq U \text{ is open}\right\}.$$

So $\psi(\mu)$ is regular in the case that ψ is continuous.

Question 9: This uses the notation of Question 5, and continued from Question 8. Let $\psi : K \to K$ be a continuous map. Show that we can define $S_\psi : C_{\mathbb{K}}(K) \to C_{\mathbb{K}}(K)$ by

$$S_\psi(f) = f \circ \psi \qquad (f \in C_{\mathbb{K}}(K)).$$

Show that S_ψ is bounded. What is $\|S_\psi\|$?

Answer: As ψ is continuous, for $f \in C_{\mathbb{K}}(K)$, we have that $f \circ \psi \in C_{\mathbb{K}}(K)$. Obviously S_ψ is linear. Then

$$\|f \circ \psi\|_\infty = \sup_{t \in K} |f(\psi(t))| \leq \sup_{s \in K} |f(s)| = \|f\|_\infty.$$

So $\|S_\psi(f)\| \leq \|f\|_\infty$, so S_ψ is bounded with $\|S_\psi\| \leq 1$. Notice that if 1 denotes the constant function, then $S_\psi(1) = 1$, and so actually $\|S_\psi\| = 1$.

[1]This is a lemma from Topology: the image of a compact set under a continuous map is always compact.

Question continued: Calculate what S_ψ^* is: you will need to use the proof of the Riesz-representation theorem.

Answer: Well, S_ψ^* should map from $M_{\mathbb{K}}(K)$ to $M_{\mathbb{K}}(K)$. So let $\mu \in M_{\mathbb{K}}(K)$ and let $\lambda = S_\psi^*(\mu)$. Then

$$\int_K f \, d\lambda = S_\psi^*(\mu)(f) = \int_K f \circ \psi \, d\mu \qquad (f \in C_{\mathbb{K}}(K)).$$

Following the vague hint, we might hope that $\lambda = \psi(\mu)$. Let's prove this!

Let's suppose that μ is positive! By (the proof of) the Riesz Representation Theorem, for $U \subseteq K$ open

$$\begin{aligned}\lambda(U) &= \sup\big\{\lambda(f) : f \in C_{\mathbb{K}}(K), 0 \leq f \leq \chi_U, \operatorname{supp}(f) \subseteq U\big\} \\ &= \sup\Big\{\int_K f \circ \psi \, d\mu : f \in C_{\mathbb{K}}(K), 0 \leq f \leq \chi_U, \operatorname{supp}(f) \subseteq U\Big\}.\end{aligned}$$

Now, if $0 \leq f \leq \chi_U$ and $\operatorname{supp}(f) \subseteq U$, then

$$0 \leq f(\psi(s)) \leq \chi_U(\psi(s)) = \chi_{\psi^{-1}(U)}.$$

If $t \in \operatorname{supp}(f \circ \psi)$ then there exists (t_n) with $t_n \to t$ and $f(\psi(t_n)) \neq 0$ for each n. Then $\psi(t_n) \to \psi(t)$, so $\psi(t) \in \operatorname{supp}(f)$, that is, $t \in \psi^{-1}(\operatorname{supp}(f)) \subseteq \psi^{-1}(U)$. [2] So, setting $g = f \circ \psi$, we see that

$$\begin{aligned}\lambda(U) &\leq \sup\Big\{\int_K g \, d\mu : g \in C_{\mathbb{K}}(K), 0 \leq g \leq \chi_{\psi^{-1}(U)}, \operatorname{supp}(g) \subseteq \psi^{-1}(U)\Big\} \\ &= \mu(\psi^{-1}(U)).\end{aligned}$$

Conversely, let $0 \leq g \leq \chi_{\psi^{-1}(U)}$ with $\operatorname{supp}(g) \subseteq \psi^{-1}(U)$. We cannot expect to find $f \in C_{\mathbb{K}}(K)$ with $g = f \circ \psi$. But we only need to find f with $f \circ \psi \geq g$ (as ultimately we take an supremum), and of course with f continuous, $0 \leq f \leq \chi_U$ and $\operatorname{supp}(f) \subseteq U$. For the moment, let's assume that we can do this!

So we have $f \in C_{\mathbb{K}}(K)$ with $0 \leq f \leq \chi_U$, $\operatorname{supp}(f) \subseteq U$, and $f \circ \psi \geq g$. Thus

$$\lambda(U) \geq \sup\Big\{\int_K g(t) \, d\mu(t) : g \in C_{\mathbb{K}}(K), 0 \leq g \leq \chi_{\psi^{-1}(U)}, \operatorname{supp}(g) \subseteq \psi^{-1}(U)\Big\}.$$

So we conclude that $\lambda(U) = \mu(\psi^{-1}(U))$ for open U.

By the previous question, we know that $\psi(\mu)$ is a regular measure. We now also know that $\psi(\mu)(U) = \lambda(U)$ for all open sets U, and that λ is regular. So, for any $E \in \mathcal{B}(K)$, we have

$$\lambda(E) = \inf\big\{\lambda(U) : E \subseteq U \text{ is open}\big\} = \inf\big\{\mu(\psi^{-1}(U)) : E \subseteq U \text{ is open}\big\} = \psi(\mu)(E).$$

So $\psi(\mu) = \lambda$, as required.

I *think*[3] that the general case follows by taking real and imaginary parts, and then using the Hahn-Decomposition.

Okay, so it remains to prove that we can construct such a g. The following is very much off syllabus, but if you are interested, it is hopefully interesting!

[2]This assumes a metric space: a more tedious argument works for a general topological space.

[3]Which means: I haven't checked the details

Recall the setup of the Riesz Representation theorem. We have a compact space K, the Borel σ-algebra $\mathcal{B}(K)$, and a *positive* $\lambda \in C_{\mathbb{K}}(K)^*$. We define an outer measure μ^* by, for $U \subseteq K$ open,

$$\mu^*(U) = \sup\Big\{\lambda(f) : f \in C_{\mathbb{K}}(K), 0 \leq f \leq \chi_U, \operatorname{supp}(f) \subseteq U\Big\}.$$

We had a lemma in the lectures which, vaguely, justified this definition; the weird condition on the support of f is, basically, because it makes a certain proof work! Then for arbitrary $E \subseteq K$, we define

$$\mu^*(E) = \inf\big\{\mu^*(U) : K \subseteq U, U \text{ is open}\big\}.$$

Then μ^* *is* an outer measure, and every member of $\mathcal{B}(K)$ is μ^*-measurable, so if we let μ be the restriction of μ^* to $\mathcal{B}(K)$, then μ is a measure.

Let's think about this, and apply Urysohn's Lemma repeatedly. Let $U \subseteq K$ be open, and let f be continuous with $0 \leq f \leq \chi_U$ and $\operatorname{supp}(f) \subseteq U$.

Then, immediately, Urysohn, applied to the closed sets $\operatorname{supp}(f)$ and $K \setminus U$, yields a continuous function $g : K \to [0, 1]$ with $g \equiv 0$ on $K \setminus U$ and $g \equiv 1$ on $\operatorname{supp}(f)$. Then clearly $0 \leq f \leq g \leq \chi_U$, but we *do not* have that $\operatorname{supp}(g) \subseteq U$, because $\operatorname{supp}(g)$ involves a closure. So g is not a "valid test function".

We have to study the proof of Urysohn. Recall that the key idea is that K, being compact, is *normal*, so given disjoint closed sets E and F, we can find disjoint open sets W and V with $E \subseteq W$ and $F \subseteq V$. We apply this to $\operatorname{supp}(f)$ and $K \setminus U$ to find disjoint open sets W and V with $\operatorname{supp}(f) \subseteq W$ and $K \setminus U \subseteq V$. Let $\overline{V}$ be the closure of V. As $V \subseteq K \setminus W$ which is closed, $\overline{V} \subseteq K \setminus W$, and so $\overline{V}$ is disjoint from $\operatorname{supp}(f)$.

Applying Urysohn to the disjoint closed sets $\operatorname{supp}(f)$ and $\overline{V}$, we find a continuous $g : K \to [0, 1]$ with $g \equiv 1$ on $\operatorname{supp}(f)$ and $g \equiv 0$ on $\overline{V}$. In particular, $g \equiv 0$ on V, and so $\{x : g(x) \neq 0\} \subseteq K \setminus V$, a closed set. Hence $\operatorname{supp}(g) \subseteq K \setminus V$, so as $K \setminus U \subseteq V$, it follows that $K \setminus V \subseteq U$, and so $\operatorname{supp}(g) \subseteq U$.

In summary, given any continuous f with $0 \leq f \leq \chi_U$ and $\operatorname{supp}(f) \subseteq U$, we can find a continuous g with $g \equiv 1$ on $\operatorname{supp}(f)$, $0 \leq g \leq \chi_U$ and $\operatorname{supp}(g) \subseteq U$.

In fact, we have proved more. Given any closed set F contained in U, we can find a continuous g with $g \equiv 1$ on F, $0 \leq g \leq \chi_U$ and $\operatorname{supp}(g) \subseteq U$. Call this g_F. It follows immediately that

$$\mu(U) = \sup\big\{\lambda(g_F) : F \subseteq U \text{ is closed}\big\}.$$

So why not *define* μ^* in this way? I think because it is hard to motivate, and because it makes life very difficult later on: when showing that μ^* *is* an outer measure, I think the proof really uses the freedom to use arbitrary continuous functions f, and not just these special functions g_F.

However, now we can complete the proof above. Recall that we have $0 \leq g \leq \chi_{\psi^{-1}(U)}$ with $\operatorname{supp}(g) \subseteq \psi^{-1}(U)$. We seek f with $0 \leq f \leq \chi_U$, $\operatorname{supp}(f) \subseteq U$ and with $f \circ \psi \geq g$. If you play with this for a while, it seems natural to define

$$F = \text{closure of } \{\psi(x) : g(x) > 0\}.$$

If $F \subseteq U$, then can let $f = g_F$. Then if $g(x) > 0$ then $\psi(x) \in F$ so $f(\psi(x)) = 1$, showing that $f \circ \psi \geq g$, as required.

So it remains to show that $F \subseteq U$. We again assume that K is a metric space. If $F \not\subseteq U$, then we can find $x \in F$ with $x \notin U$. Hence there exists a sequence (x_n) with $\psi(x_n) \to x$ and $g(x_n) > 0$ for each n. So (x_n) is a sequence in the compact set $\operatorname{supp}(g)$, so we may suppose, by moving to a subsequence, that x_n converges, say to y. Then $\psi(y) = \lim_n \psi(x_n) = x$. As $y \in \operatorname{supp}(g) \subseteq \psi^{-1}(U)$, it follows that $x = \psi(y) \in U$, a contradiction as required.

Linear Analysis I: Worked Solutions 10

Question 1: Let E and G be Banach spaces, and let $F \subseteq E$ be a subspace which is dense. Let $T : F \to G$ be a bounded linear map. Show that we can extend T to give a bounded linear map $E \to G$. Show that such an extension must be unique.

Answer: First we show uniqueness. Let $T_1, T_2 : E \to G$ be extensions. Let $x \in E$, so as F is dense, we can find a sequence (x_n) in F with $\lim_n x_n = x$. Then as T_1 and T_2 are continuous,

$$T_1(x) = \lim_n T_1(x_n) = \lim_n T(x_n) = \lim_n T_2(x_n) = T_2(x).$$

As x was arbitrary, $T_1 = T_2$.

Now to existence. We extend T be continuity. Let $x \in E$, so we can find a sequence (x_n) in F with $x_n \to x$. In particular, (x_n) is Cauchy, so for $\epsilon > 0$, there exists N_ϵ such that $\|x_n - x_m\| < \epsilon$ if $n, m \geq N_\epsilon$. Then

$$\|T(x_n) - T(x_m)\| = \|T(x_n - x_m)\| \leq \|T\|\|x_n - x_m\| < \epsilon\|T\| \qquad (n, m \geq N_\epsilon).$$

Hence $(T(x_n))$ is a Cauchy sequence in G, which is a Banach space, and so $T(x_n) \to \hat{T}(x)$ say. Notice that

$$\|\hat{T}(x)\| = \lim_n \|T(x_n)\| \leq \|T\| \lim_n \|x_n\| = \|T\|\|x\|.$$

Firstly, we note that if $x \in F$ to start with, then $\hat{T}(x) = \lim_n T(x_n) = T(x)$, so $\hat{T}$ and T agree on F. If now $x \in E$ is arbitrary, and (y_n) is another sequence converging to x, then for $\epsilon > 0$, there exists N such that both $\|x_n - x\| < \epsilon$, and $\|y_n - x\| < \epsilon$, for $n \geq N$. Hence $\|x_n - y_n\| < 2\epsilon$ for $n \geq N$, and so $(x_n - y_n)$ is a sequence converging to 0. Hence $T(x_n - y_n) \to 0$, and so $\lim_n T(x_n) = \lim_n T(y_n)$. Hence $\hat{T}$ is well-defined.

Finally, if $x, y \in E$ and $\alpha \in \mathbb{K}$, then if $x_n \to x$ and $y_n \to y$ with (x_n) and (y_n) sequences in F, then $\alpha x_n + y_n \to \alpha x + y$, and so

$$\hat{T}(\alpha x + y) = \lim_n T(\alpha x_n + y_n) = \lim_n \alpha T(x_n) + \lim_n T(y_n) = \alpha\hat{T}(x) + \hat{T}(y),$$

showing that $\hat{T}$ is linear. We showed above that $\hat{T}$ was bounded. So $\hat{T}$ is our extension.

Actually, this argument would also show the following: if X and Y are metric spaces, $X_0 \subseteq X$ is dense, $f : X_0 \to Y$ is continuous, and Y is *complete*, then f has a unique extension to all of X. I thought that you would have seen this in the Topology course, but apparently not.

Question 2: Define $f : [0, 1] \to \mathbb{C}$ by

$$f(t) = \begin{cases} \exp(t) & : 0 \leq t \leq 1/2, \\ \exp(1-t) & : 1/2 \leq t \leq 1. \end{cases}$$

Thus f is periodic. Calculate the Fourier transform of f.

By using Fejer's Theorem, and evaluating at $t = 0$ and $t = 1/2$, show that

$$\sum_{k=1}^{\infty} \frac{1}{1 + 16\pi^2 k^2} = \frac{1}{4(e^{1/2} - 1)} - \frac{3}{8}.$$

Answer: Notice that $f(1-t) = f(t)$ for $1/2 \le t \le 1$. So

$$\begin{aligned}
\hat{f}(n) &= \int_0^1 f(t)e^{2\pi int}\,dt = \int_0^{1/2} f(t)e^{2\pi int}\,dt + \int_{1/2}^1 f(1-t)e^{2\pi int}\,dt \\
&= \int_0^{1/2} f(t)e^{2\pi int}\,dt + \int_0^{1/2} f(s)e^{2\pi in(1-s)}\,ds \\
&= \int_0^{1/2} f(t)\big(e^{2\pi int} + e^{-2\pi int}\big)\,dt
\end{aligned}$$

Now,

$$\int_0^{1/2} e^t e^{2\pi int}\,dt = \left[\frac{e^{t(1+2\pi in)}}{1+2\pi in}\right]_{t=0}^{1/2} = \frac{e^{1/2+\pi in}-1}{1+2\pi in} = \frac{e^{1/2}(-1)^n - 1}{1+2\pi in}.$$

Putting these together, we get

$$\hat{f}(n) = \frac{e^{1/2}(-1)^n - 1}{1+2\pi in} + \frac{e^{1/2}(-1)^{-n} - 1}{1-2\pi in} = \frac{2((-1)^n e^{1/2} - 1)}{1+4\pi^2 n^2}.$$

You could also do the integral directly, of course!

We consider the partial sums at 0,

$$\sum_{k=-n}^{n} \hat{f}(k)e^{-2\pi ik0} = \sum_{k=-n}^{n} \hat{f}(k) = \sum_{k=-n}^{n} 2\frac{(-1)^k e^{1/2} - 1}{1+4\pi^2 k^2}.$$

This is (absolutely) convergent, so the Cesaro sums converge to the same limit, and hence by Fejer's Theorem,

$$1 = f(0) = 2\sum_{k=-\infty}^{\infty} \frac{(-1)^k e^{1/2} - 1}{1+4\pi^2 k^2} = 2(e^{1/2}-1) + 4\sum_{k=1}^{\infty} \frac{(-1)^k e^{1/2} - 1}{1+4\pi^2 k^2}.$$

Re-arrange, and we get

$$3 - 2e^{1/2} = 4\sum_{k=1}^{\infty} \frac{(-1)^k e^{1/2} - 1}{1+4\pi^2 k^2}.$$

If we evaluate at 1/2 instead, we get

$$\begin{aligned}
f(1/2) = e^{1/2} &= 2\sum_{k=-\infty}^{\infty} (-1)^k \frac{(-1)^k e^{1/2} - 1}{1+4\pi^2 k^2} = 2\sum_{k=-\infty}^{\infty} \frac{e^{1/2} - (-1)^k}{1+4\pi^2 k^2} \\
&= 2(e^{1/2}-1) + 4\sum_{k=1}^{\infty} \frac{e^{1/2} - (-1)^k}{1+4\pi^2 k^2},
\end{aligned}$$

and so

$$2 - e^{1/2} = 4\sum_{k=1}^{\infty} \frac{e^{1/2} - (-1)^k}{1+4\pi^2 k^2}.$$

Adding these two, and taking even parts, we get

$$5 - 3e^{1/2} = 4\sum_{k=1}^{\infty} \frac{(-1)^k e^{1/2} - 1 + e^{1/2} - (-1)^k}{1+4\pi^2 k^2} = 8\sum_{k=1}^{\infty} \frac{e^{1/2} - 1}{1+16\pi^2 k^2}.$$

We conclude

$$\sum_{k=1}^{\infty} \frac{1}{1+16\pi^2k^2} = \frac{5-3e^{1/2}}{8(e^{1/2}-1)} = \frac{1}{4(e^{1/2}-1)} - \frac{3}{8}.$$

At least, if I haven't made a mistake!

Question 3: Let $f(t) = e^t$ for $0 \leq t \leq 1$; show that $f \in \mathcal{L}^2([0,1])$ and compute $\|f\|_2$. Find $\mathcal{F}(f)$, and hence deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1+4\pi^2n^2} = \frac{3-e}{4(e-1)}.$$

Answer: We have that

$$\|f\|_2^2 = \int_0^1 |e^t|^2 \, dt = \int_0^1 e^{2t} \, dt = \Big[\frac{e^{2t}}{2}\Big]_{t=0}^1 = \frac{e^2-1}{2}.$$

So $f \in \mathcal{L}^2([0,1])$. Also

$$\hat{f}(n) = \int_0^1 e^t e^{2\pi i n t} \, dt = \Big[\frac{\exp(t(1+2\pi i n))}{1+2\pi i n}\Big]_{t=0}^1 = \frac{\exp(1+2\pi i n)-1}{1+2\pi i n} = \frac{e-1}{1+2\pi i n}.$$

So by Parseval (that is, the Fourier transform is an isometry $\mathcal{L}^2([0,1]) \to \ell^2(\mathbb{Z})$),

$$\frac{e^2-1}{2} = \|f\|_2^2 = \|\mathcal{F}(f)\|_2^2 = \sum_{n=-\infty}^{\infty} \frac{(e-1)^2}{1+4\pi^2n^2} = (e-1)^2 + 2\sum_{n=1}^{\infty} \frac{(e-1)^2}{1+4\pi^2n^2}.$$

And so we see that

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{1+4\pi^2n^2} &= \frac{1}{2(e-1)^2}\Big(\frac{e^2-1}{2} - (e-1)^2\Big) = \frac{(e-1)(e+1)}{4(e-1)^2} - \frac{1}{2} \\ &= \frac{e+1}{4(e-1)} - \frac{1}{2} = \frac{e+1-2(e-1)}{4(e-1)} = \frac{3-e}{4(e-1)}\end{aligned}$$

Question 4: Show that $C_{\mathbb{C}}(\mathbb{T})$ is dense in $\mathcal{L}^2(\mathbb{T}) = \mathcal{L}^2([0,1])$.

Proof: We know that $C_{\mathbb{C}}([0,1])$ is dense in $\mathcal{L}^2([0,1])$. So for $f \in \mathcal{L}^2([0,1])$ and $\epsilon > 0$, there exists $g \in C_{\mathbb{C}}([0,1])$ with $\|f-g\|_2 < \epsilon$. Pick $\delta > 0$ small and define $h : [0,1] \to \mathbb{C}$ by

$$h(t) = \begin{cases} g(t) & : 0 \leq t \leq 1-\delta, \\ g(t)\frac{1-t}{\delta} + g(0)\frac{t-1+\delta}{\delta} & : 1-\delta \leq t \leq 1. \end{cases}$$

In words, h is g, except that we chop it off at $1-\delta$ and linearly interpolate between $g(1-\delta)$ and $g(0)$ to get a periodic function.

Then h is continuous, and $h(1) = g(0) = h(0)$, so $h \in C_{\mathbb{C}}(\mathbb{T})$. Furthermore, for $1-\delta \leq t \leq 1$, we see that

$$\begin{aligned}|h(t)-g(t)| &= \Big|g(t)\frac{1-t}{\delta} + g(0)\frac{t-1+\delta}{\delta} - g(t)\Big| \\ &\leq |g(t)|\frac{\delta-(1-t)}{\delta} + |g(0)|\frac{t-1+\delta}{\delta} \leq |g(t)| + |g(0)| \leq 2\|g\|_\infty.\end{aligned}$$

Hence

$$\|h-g\|_2 = \Big(\int_{1-\delta}^1 |h(t)-g(t)|^2 \, d\mu(t)\Big)^{1/2} \leq \Big(4\|g\|_\infty^2\delta\Big)^{1/2} = 2\sqrt{\delta}\|g\|_\infty,$$

which is $\leq \epsilon$ if δ is sufficiently small. Hence

$$\|f - h\|_2 \leq 2\epsilon,$$

as required.

Question 5: Let (f_n) be a sequence in $C_{\mathbb{C}}([0,1])$ converging to f with respect to the $\|\cdot\|_\infty$ norm. Suppose each f_n is differentiable (to be precise, on $(0,1)$, or suppose each f_n is periodic) with a continuous derivative, and $f_n' \to g \in C_{\mathbb{C}}([0,1])$ with respect to the $\|\cdot\|_\infty$ norm. Show that f is differentiable with derivative g.
Answer: For $0 \leq t \leq 1$, define

$$h(t) = \int_0^t g(x)\ dx + f(0).$$

Hence $h \in C_{\mathbb{C}}([0,1])$. For $\epsilon > 0$, there exists N such that both $\|f_n' - g\|_\infty < \epsilon$ and $\|f_n - f\|_\infty < \epsilon$, when $n \geq N$. Hence for $0 \leq t \leq 1$ and $n \geq N$,

$$\begin{aligned}|h(t) - f_n(t)| = \left|h(t) - \int_0^t f_n'(x)\ dx - f_n(0)\right| &\leq \left|\int_0^t g(x) - f_n'(x)\ dx\right| + |f(0) - f_n(0)| \\ &\leq t\|g - f_n'\|_\infty + \|f - f_n\|_\infty < (1+t)\epsilon \leq 2\epsilon.\end{aligned}$$

Hence $\|h - f_n\|_\infty \leq 2\epsilon$ for $n \geq N$. So $h = \lim_n f_n = f$, and clearly h is differentiable with derivative g, as required.

Question 6: For $n \geq 1$ let $x_n = (x_m^{(n)})_{m\in\mathbb{Z}} \in c_0(\mathbb{Z})$ be defined by

$$x_m^{(n)} = \begin{cases} 1 & : |m| \leq n, \\ 0 & : |m| > n. \end{cases}$$

Then $x_n \in \ell^1(\mathbb{Z})$ so that $\mathcal{F}^{-1}(x_n)$ makes sense. Show that $\|\mathcal{F}^{-1}(x_n)\|_1$ is large.

Hence, by using a result from lectures that $\mathcal{F}$ is injective, and assuming the Open Mapping Theorem, show that $\mathcal{F}$ does *not* map $\mathcal{L}^1([0,1])$ onto $c_0(\mathbb{Z})$.
Answer: We calculate that for $0 \leq t \leq 1$,

$$\begin{aligned}\mathcal{F}^{-1}(x_n)(t) &= \sum_{k=-n}^{n} e^{-2\pi ikt} = e^{2\pi int}\big(1 + z + \cdots + z^{2n}\big) = e^{2\pi int}\frac{1 - z^{2n+1}}{1 - z} \\ &= \frac{z^{-n} - z^{n+1}}{1 - z} = \frac{z^{-n-1/2} - z^{n+1/2}}{z^{-1/2} - z^{1/2}} = \frac{2i\sin(2\pi(n+1/2)t)}{2i\sin(2\pi(1/2)t)} = \frac{\sin((2n+1)\pi t)}{\sin(\pi t)},\end{aligned}$$

where $z = e^{-2\pi it}$. Of course, $\mathcal{F}^{-1}(x_n)(0) = 2n+1$.

I now copy Korner.[1] We know that (or we can prove that)

$$0 \leq s \leq \pi/2 \implies \frac{2s}{\pi} \leq \sin(s) \leq s.$$

So letting $s = \pi t$, we see that

$$0 \leq t \leq 1/2 \implies 2t \leq \sin(\pi t) \leq \pi t \implies 2 \leq \frac{\sin(\pi t)}{t} \leq \pi.$$

[1]See chapter 18; I don't understand Korner's proof, so this is a little different!

Thus

$$\begin{aligned}
\int_{[0,1]} |\mathcal{F}^{-1}(x_n)|\ d\mu &= \int_0^1 \Big|\frac{\sin((2n+1)\pi t)}{\sin(\pi t)}\Big|\ dt = 2\int_0^{1/2} \Big|\frac{\sin((2n+1)\pi t)}{\sin(\pi t)}\Big|\ dt \\
&\geq 2\int_0^{1/2} \Big|\frac{\sin((2n+1)\pi t)}{\pi t}\Big|\ dt \\
&= 2\sum_{r=0}^{2n} \int_{r/(4n+2)}^{(r+1)/(4n+2)} \frac{\big|\sin((2n+1)\pi t)\big|}{\pi t}\ dt \\
&= 2\sum_{r=0}^{2n} \int_0^{1/(4n+2)} \frac{\big|\sin((2n+1)\pi t + \pi r/2)\big|}{\pi t + r\pi/(4n+2)}\ dt.
\end{aligned}$$

For $0 \leq t \leq 1/(4n+2)$, by our previous inequality, with $s = (2n+1)\pi t$, we get

$$(4n+2)t \leq \sin((2n+1)\pi t) \leq (2n+1)\pi t.$$

So, when $r = 0$, we see

$$\int_0^{1/(4n+2)} \frac{\big|\sin((2n+1)\pi t)\big|}{\pi t}\ dt \geq \int_0^{1/(4n+2)} \frac{4n+2}{\pi}\ dt = \frac{1}{\pi}.$$

When $r > 0$, as also $0 \leq t \leq 1/(4n+2)$, we use the simple inequality

$$\frac{1}{\pi t + r\pi/(4n+2)} \geq \frac{1}{(r+1)\pi/(4n+2)} = \frac{4n+2}{(r+1)\pi}.$$

So we get an new estimate for our integral,

$$\begin{aligned}
&\geq \frac{2}{\pi} + 2\sum_{r=1}^{2n} \frac{4n+2}{(r+1)\pi} \int_0^{1/(4n+2)} \big|\sin((2n+1)\pi t + \pi r/2)\big|\ dt \\
&= \frac{2}{\pi} + 2\sum_{r=1}^{2n} \frac{1}{(r+1)\pi} \int_0^1 \big|\sin(\pi t/2 + \pi r/2)\big|\ dt \\
&= \frac{2}{\pi} + 2\sum_{r=1}^{2n} \frac{1}{(r+1)\pi} \int_0^1 \sin(\pi t/2)\ dt \quad \text{(draw a picture!)} \\
&= \frac{2}{\pi} + 2\sum_{r=1}^{2n} \frac{1}{(r+1)\pi}\frac{2}{\pi} \geq \frac{4}{\pi^2}\sum_{r=0}^{2n} \frac{1}{r+1}.
\end{aligned}$$

This of course is the harmonic series, which diverges! So we conclude that

$$\lim_{n\to\infty} \big\|\mathcal{F}^{-1}(x_n)\big\|_1 = \infty.$$

Of course, $\|x_n\|_\infty = 1$ for all n. So let $f_n = \mathcal{F}^{-1}(x_n)$ for each n. As $x_n \in c_0(\mathbb{Z}) \cap \ell^1(\mathbb{Z})$, we see that $\mathcal{F}(f_n) = x_n$.

Suppose that $\mathcal{F} : \mathcal{L}^1([0,1]) \to c_0(\mathbb{Z})$ is surjective. By a result from the lectures, $\mathcal{F}$ is injective. By the Open Mapping Theorem, there exists a *bounded* map $T : c_0(\mathbb{Z}) \to \mathcal{L}^1([0,1])$ such that $T\mathcal{F}$ is the identity on $\mathcal{L}^1([0,1])$. Then

$$n < \|f_n\|_1 = \|T\mathcal{F}(f_n)\|_1 \leq \|T\|\|\mathcal{F}(f_n)\|_\infty = \|T\|\|x_n\|_\infty = \|T\|.$$

This contradicts $\|T\|$ being finite. So $\mathcal{F}$ is not surjective.[2]

[2]To be honest, this is the *only* way which I can think of to show this result. But maybe it is possible to simply write down something in $c_0(\mathbb{Z})$ and show, directly, that it cannot be the image of something $\mathcal{L}^1([0,1])$, but I don't see it. Let me know if you find an example!

Thinking more about Riesz Representation

Question i: For a compact (Hausdorff) space K let $M_{\mathbb{C}}(K)$ be the space of finite, complex, regular Borel measures on K. For $\mu \in M_{\mathbb{C}}(K)$ define $\phi_\mu \in C_{\mathbb{C}}(K)^*$ by

$$\phi_\mu(f) = \int_K f \ d\mu \qquad (f \in C_{\mathbb{C}}(K)).$$

Let $g : K \to \mathbb{C}$ be a simple function (of course, not assumed continuous!) with $\|g\|_\infty \leq 1$. Show that

$$\Big| \int_K g \ d\mu \Big| \leq \|\mu\|.$$

Now let $f \in C_{\mathbb{C}}(K)$ with $\|f\|_\infty \leq 1$. Show that we can find a sequence (g_n) of simple functions with $g_n \to f$ pointwise, and with $|g_n| \leq |f|$ everywhere for each n. (*Hint:* Apply our "canonical" method for getting simple functions, but taking account of real and imaginary parts, etc.) Conclude, by using the Dominated Convergence Theorem, that $|\phi_\mu(f)| \leq \|\mu\|$. Conclude that $\|\phi_\mu\| \leq \|\mu\|$.

Answer: Let $g = \sum_n a_n \chi_{A_n}$. As $\|g\|_\infty \leq 1$, we have that $|a_n| \leq 1$, or $\mu(A_n) = 0$, for each n. Of course, we may suppose that the (A_n) are pairwise disjoint. Thus

$$\Big| \int_K g \ d\mu \Big| = \Big| \sum_n a_n \mu(A_n) \Big| \leq \sum_n |a_n||\mu(A_n)| \leq \sum_n |\mu(A_n)| \leq \|\mu\|,$$

by the definition of $\|\mu\|$.

If $f \geq 0$ then we can let

$$g_n = \min(n, 2^{-n}\lfloor 2^n f \rfloor),$$

as usual. If f is real-valued, let

$$g_n = \min(n, 2^{-n}\lfloor 2^n f_+ \rfloor) - \min(n, 2^{-n}\lfloor 2^n f_- \rfloor).$$

If f is complex-valued, take real and imaginary parts (which is tedious to type). Clearly we have that $|g_n| \leq |f|$ everywhere, and that $g_n \to f$ pointwise. As $|f|$ is integrable for μ, Dominated Convergence shows that

$$\Big| \int_K f \ d\mu \Big| = \lim_n \Big| \int_K g_n \ d\mu \Big| \leq \|\mu\|,$$

as $|g_n| \leq 1$ everywhere. So $|\phi_\mu(f)| \leq \|\mu\|$. Taking the supremum over such f, we conclude that $\|\phi_\mu\| \leq \|\mu\|$.

Question ii: Firstly, prove the following useful lemma. Let τ be a positive Borel measure. Show that τ is regular if and only if, for each $E \in \mathcal{B}(K)$ and $\epsilon > 0$, we can find an open set U and a closed set C with $C \subseteq E \subseteq U$ and with $\tau(U \setminus C) < \epsilon$.

Proof: If τ is regular, then we can find such U and C with $\tau(C) > \tau(E) - \epsilon/2$ and $\tau(U) < \tau(E) + \epsilon/2$. Then $\tau(U \setminus C) = \tau(U) - \tau(C) = \tau(U) - \tau(E) + \tau(E) - \tau(C) < \epsilon$. Conversely, if we can find U and C, then $\tau(U) - \tau(E) \leq \tau(U) - \tau(C) = \tau(U \setminus C) < \epsilon$ so $\tau(U) < \tau(E) + \epsilon$. Similarly, $\tau(C) > \tau(E) - \epsilon$, and so τ is regular.

Question continued: For a signed measure τ, we defined $|\tau| = \tau_+ + \tau_-$, where τ_+ and τ_- are defined by way of a Hahn-Decomposition for τ. Show that

$$|\tau|(E) = \sup\big\{\tau(U) - \tau(V) : U, V \in \mathcal{B}(K), U \cap V = \emptyset, U \cup V = E\big\} \qquad (E \in \mathcal{B}(K)).$$

So we don't actually need a Hahn-Decomposition to define $|\tau|$ (and this works for any measure on any σ-algebra).

Answer: Let (A, B) be a Hahn-Decomposition for τ, so that

$$|\tau|(E) = \tau_+(E) + \tau_-(E) = \tau(E \cap A) - \tau(E \cap B).$$

If $U = E \cap A$ and $V = E \cap B$, then $E = U \cup V$ is a disjoint union, and $|\tau|(E) = \tau(U) - \tau(V)$.

Conversely, let $U \cup V = E$ be a pairwise disjoint union. Then

$$\tau(U) - \tau(V) = \tau(U \cap A) + \tau(U \cap B) - \tau(V \cap A) - \tau(V \cap B).$$

Now, as B is a negative set, $\tau(U \cap B) \leq 0$. Similarly, $-\tau(V \cap A) \leq 0$. So

$$\begin{aligned}\tau(U) - \tau(V) &\leq \tau(U \cap A) - \tau(V \cap B) = \tau_+(U) + \tau_-(V) \\ &\leq \tau_+(E) + \tau_-(E) = |\tau|(E).\end{aligned}$$

So $|\tau|(E)$ does equal the supremum (and the supreumu is obtained!)

Question continued: Now prove a third useful lemma. Let $\tau \in M_{\mathbb{R}}(K)$. Show that τ is regular (defined to mean that τ_+ and τ_- are regular) if and only if $|\tau|$ is regular.

Answer: We use the condition given by the first lemma. If τ is regular, then as τ_+ and τ_- are regular, by our first lemma, given E and $\epsilon > 0$, we can find closed sets C_+ and C_- and open sets U_+ and U_- with $C_\pm \subseteq E \subseteq U_\pm$, and with $\tau_\pm(U_\pm \setminus C_\pm) < \epsilon$. Let $U = U_+ \cap U_-$ and $C = C_+ \cup C_-$, so that $U \setminus C \subseteq U_\pm \setminus C_\pm$, and hence both $\tau_+(U \setminus C) < \epsilon$ and $\tau_-(U \setminus C) < \epsilon$. Thus $|\tau|(U \setminus C) < 2\epsilon$.

Conversely, if we have $C \subseteq E \subseteq U$ with $|\tau|(U \setminus C) < \epsilon$, then certainly both $\tau_+(U \setminus C) < \epsilon$ and $\tau_-(U \setminus C) < \epsilon$. Thus τ_+ and τ_- are regular.

Question continued: Let $\mu, \lambda \in M_{\mathbb{R}}(K)$, and let $\tau = \mu + \lambda$. Using the 2nd lemma, show that $|\tau| \leq |\mu| + |\lambda|$. Deduce, using the 3rd lemma, that τ is regular.

Answer: For $E \in \mathcal{B}(K)$, we have that

$$\begin{aligned}|\tau|(E) &= \sup\big\{\tau(U) - \tau(V) : E = U \cup V, U \cap V = \emptyset\big\} \\ &= \sup\big\{\mu(U) - \mu(V) + \lambda(U) - \lambda(V) : E = U \cup V, U \cap V = \emptyset\big\} \\ &\leq \sup\big\{\mu(U) - \mu(V) : E = U \cup V, U \cap V = \emptyset\big\} \\ &\qquad + \sup\big\{\lambda(U) - \lambda(V) : E = U \cup V, U \cap V = \emptyset\big\} \\ &= |\mu|(E) + |\lambda|(E).\end{aligned}$$

So $|\tau| \leq |\mu| + |\lambda|$.

So, for $E \in \mathcal{B}(K)$ and $\epsilon > 0$, we can find open sets U, V which contain E, and we can find closed sets C, D contained in E, with

$$|\mu|(U \setminus C) < \epsilon, \quad |\lambda|(V \setminus D) < \epsilon.$$

Let $U' = U \cap V$ and $C' = C \cup D$, so $U \setminus C \supseteq U' \setminus C'$, and $V \setminus D \supseteq U' \setminus C'$, and still $C' \subseteq E \subseteq U'$. Then

$$|\tau|(U' \setminus C') \leq |\mu|(U' \setminus C') + |\lambda|(U' \setminus C') < 2\epsilon.$$

This show that $\tau = \mu + \lambda$ is regular, as required.

Question continued: Show the same for complex measures: this is easier, as we can directly take real and imaginary parts.

Answer: This is easy: if $\mu, \lambda \in M_{\mathbb{C}}(K)$, then by definition, μ_r, μ_i, λ_r and λ_i are regular. So $(\mu + \lambda)_r = \mu_r + \lambda_r$ is regular, as is $(\mu + \lambda)_i$. So $\mu + \lambda$ is regular.

Question iii: Let K be compact and Hausdorff, and let $\lambda \in C_{\mathbb{C}}(K)$ with $\|\lambda\| = 1$. It is possible[3] to construct a positive $\Phi \in C_{\mathbb{R}}(K)^*$ with the property that for any $f \in C_{\mathbb{C}}(K)$,

$$|\lambda(f)| \leq \Phi(|f|) \leq \|f\|_\infty,$$

where $|f|(x) = |f(x)|$ for each $x \in K$. Show that $\|\Phi\| = 1$.
Answer: As $\|\lambda\| = 1$, for each $\epsilon > 0$ we can find $f \in C_{\mathbb{C}}(K)$ with $\|f\|_\infty = 1$ and $|\lambda(f)| > 1 - \epsilon$. Then clearly $\|\,|f|\,\|_\infty = 1$ as well, so that as

$$1 - \epsilon < |\lambda(f)| \leq \Phi(|f|) \leq \|f\|_\infty = 1,$$

we see that $\|\Phi\| > 1 - \epsilon$. So $\|\Phi\| \geq 1$, but by assumption, also $\|\Phi\| \leq 1$.
Question continued: We can then apply Riesz representation to find some a regular, positive Borel measure μ_0 with

$$\Phi(g) = \int_K g \; d\mu_0 \qquad (g \in C_{\mathbb{R}}(K)).$$

As $\|\Phi\| = 1$, we have that $\mu_0(K) = 1$.

We can hence form that space $\mathcal{L}^1(\mu_0)$. There is a natural map $C_{\mathbb{C}}(K) \to \mathcal{L}^1(\mu_0)$; let X be the image, so that X is a subspace of $\mathcal{L}^1(\mu_0)$. Show that the map

$$\phi : X \to \mathbb{C}; \quad f \mapsto \lambda(f)$$

is linear and bounded. What is $\|\phi\|$? Using that $\mathcal{L}^1(\mu_0)^* \cong \mathcal{L}^\infty(\mu_0)$ (and Hahn-Banach), show that there exists $h \in \mathcal{L}^\infty(\mu_0)$ with

$$\lambda(f) = \int_K fh \; d\mu_0 \qquad (f \in C_{\mathbb{C}}(K)).$$

Answer: Let us write $\iota : C_{\mathbb{C}}(K) \to \mathcal{L}^1(\mu_0)$ be the map; notice that ι need not be injective. So ϕ is really defined by $\iota(f) \mapsto \lambda(f)$. This is well-defined, for if $\iota(f) = \iota(g)$, then $f - g = 0$ in $\mathcal{L}^1(\mu_0)$, so $f - g = 0$ almost everywhere (with respect to μ_0). Hence also $|f - g| = 0$ almost everywhere. So

$$\Phi(|f-g|) = \int_K |f-g| \; d\mu_0 = 0.$$

Thus $|\lambda(f-g)| \leq \Phi(|f-g|) = 0$, so $\lambda(f) = \lambda(g)$.

It is easy to see that ϕ is linear. Then, for $f \in C_{\mathbb{C}}(K)$,

$$|\phi(\iota(f))| = |\lambda(f)| \leq \Phi(|f|) = \int_K |f| \; d\mu_0 = \|\iota(f)\|_1,$$

from which it follows that $\|\phi\| \leq 1$. Conversely,

$$|\lambda(f)| = |\phi(\iota(f)| \leq \|\phi\|\|\iota(f)\|_1 = \|\phi\| \int_K |f| \; d\mu_0 = \|\phi\|\Phi(|f|) \leq \|\phi\|\|f\|_\infty.$$

As we can find f with $\|f\|_\infty = 1$ and $|\lambda(f)|$ as close as we like to 1, we must have that $\|\phi\| = 1$.

So ϕ is a norm one functional defined on a subspace of $\mathcal{L}^1(\mu)$. By Hahn-Banach, we extend ϕ to a norm one functional defined on all of $\mathcal{L}^1(\mu)$. So there exists some $h \in \mathcal{L}^\infty(\mu)$ with $\|h\|_\infty = 1$ and with

$$\int_K fh \; d\mu_0 = \phi(\iota(f)) = \lambda(f) \qquad (f \in C_{\mathbb{C}}(K)).$$

[3] See Rudin's book; the construction is very similar to how we defined λ_+ given $\lambda \in C_{\mathbb{R}}(K)$.

Question continued: Let $\mu = h\mu_0$, so μ is the complex measure with

$$\mu(E) = \int_K \chi_E h \; d\mu_0.$$

This *is* regular: this isn't too hard to show, if you adopt the philosophy of question ii. We immediately see that

$$\lambda(f) = \int_K f \; d\mu \qquad (f \in C_{\mathbb{C}}(K)).$$

Finally, show that $\|h\|_\infty = 1$ (hint: what is $\|\phi\|$?) Deduce that $\|\mu\| = 1 = \|\lambda\|$ (hint: Use Question i).

Answer: Formally, to show that μ is a measure, we need to show countable additivity, which would require the Dominated Convergence Theorem (or take real+imaginary, and positive+negative parts, and use Monotone Convergence). Now we should regularity.

As μ_0 is regular, for $E \in \mathcal{B}(K)$ and $\epsilon > 0$, we can find an open set U and a closed set C with $C \subseteq E \subseteq U$, and with $\mu_0(U \setminus C) < \epsilon$. By Question ii, to show that μ is regular, it is enough to show that $|\mu_r|$ and $|\mu_i|$ are regular. But clearly $|\mu_r| = |\Re h|\mu_0$, so

$$|\mu_r|(U \setminus C) = \int_{U\setminus C} |\Re h| \; d\mu_0 \leq \int_{U\setminus C} 1 \; d\mu_0 = \mu_0(U \setminus C) < \epsilon,$$

and similarly $|\mu_i|(U \setminus C) < \epsilon$. This establishes that μ is indeed regular.

As $\|h\|_\infty = 1$, we see that if (A_n) is a partition of K, then

$$\sum_n |\mu(A_n)| = \sum_n \Big| \int_K \chi_{A_n} h \; d\mu_0 \Big| \leq \sum_n \int_K \chi_{A_n} |h| \; d\mu_0 = \int_K |h| \; d\mu_0 \leq \mu_0(K) = 1.$$

So $\|\mu\| \leq 1$. By Question 1, $1 = \|\lambda\| \leq \|\mu\|$, so we must have equality.

Question A: Let $(a_n) \in \ell^1(\mathbb{Z})$ be a sequence such that $(na_n) \in \ell^1(\mathbb{Z})$ as well. Let $f = \mathcal{F}^{-1}((a_n))$. Show that f is differentiable.
Answer: We let

$$f_n(t) = \sum_{k=-n}^{n} a_k e^{-2\pi ikt},$$

so as $(a_n) \in \ell^1(\mathbb{Z})$, by Fejer's Theorem, we have that $f_n \to f$ in $C_{\mathbb{C}}([0,1])$. Then

$$f_n'(t) = \sum_{k=-n}^{n} (-2\pi ik)a_k e^{-2\pi ikt} = -2\pi i \sum_{k=-n}^{n} ka_k e^{-2\pi ikt}.$$

As $(ka_k) \in \ell^1(\mathbb{Z})$, we see that f_n' converges to $g \in C_{\mathbb{C}}([0,1])$ defined by

$$g(t) = -2\pi i \sum_{k\in\mathbb{Z}} ka_k e^{-2\pi ikt}.$$

Thus by Question 5, f is differentiable with derivative g.

Question C: Let X be the subspace of $C_{\mathbb{C}}(\mathbb{T})$ spanned by functions of the form $t \mapsto e^{2\pi int}$, for $n \in \mathbb{Z}$. We saw in lectures that, because of Fejer's Theorem, X is dense in $C_{\mathbb{C}}(\mathbb{T})$.

Now let $f : [0,1] \to \mathbb{R}$ be continuous (but not necessarily periodic) and define $g \in C_{\mathbb{C}}(\mathbb{T})$ by

$$g(t) = \begin{cases} f(2t) & : 0 \leq t \leq 1/2, \\ f(2-2t) & : 1/2 \leq t \leq 0. \end{cases}$$

Fix $\epsilon > 0$. Then we can find $h \in X$ with $\|g - h\|_\infty < \epsilon$. We know that on the interval $[0,1]$ and for $n \in \mathbb{Z}$, we have that

$$\sum_{k=0}^{K} \frac{(2\pi int)^k}{k!}$$

converges uniformly to $e^{2\pi int}$, as $K \to \infty$. Use this to approximate h by a complex polynomial in t.

By taking real parts, and thinking about the definition of g, show that we have approximated f be a real polynomial.

This is the Weierstrauss Approximation Theorem, see "Fourier Analysis", Chapter 4.
Answer: Notice that g is periodic, so we can certainly find $h \in X$ with $\|g - h\|_\infty < \epsilon$. Say that

$$h(t) = \sum_{k=-n}^{n} a_k e^{2\pi ikt} \qquad (t \in \mathbb{T}).$$

Then for each k with $|k| \leq n$, we can find $L(k)$ such that

$$\left| e^{2\pi ikt} - \sum_{l=0}^{L(k)} \frac{(2\pi ikt)^l}{l!} \right| < \epsilon \Big(\sum_{|k|\leq n} |a_k| \Big)^{-1} \qquad (0 \leq t \leq 1).$$

Let

$$G(t) = \sum_{k=-n}^{n} a_k \sum_{l=0}^{L(k)} \frac{(2\pi ikt)^l}{l!} \qquad (t \in \mathbb{T}),$$

so that G is a complex polynomial in t. Then

$$|G(t) - h(t)| \leq \sum_{k=-n}^{n} |a_k| \epsilon \Big(\sum_{|k| \leq n} |a_k| \Big)^{-1} = \epsilon \qquad (t \in \mathbb{T}),$$

so that $\|g - G\|_\infty \leq \|G - h\|_\infty + \|h - g\|_\infty < 2\epsilon$.

For a complex number z let $\Re(z)$ and $\Im(z)$ be the real and imaginary parts of z, respectively. Then

$$\begin{aligned} \Re G(t) &= \sum_{k=-n}^{n} \sum_{l=0}^{L(k)} \frac{\Re(a_k i^l (2\pi k t)^l)}{l!} \\ &= \sum_{k=-n}^{n} \sum_{l=0}^{L(k)} \frac{(2\pi k t)^l)}{l!} \Re(a_k i^l) \\ &= \sum_{k=-n}^{n} \Big(\sum_{0 \leq l \leq L(k), l \text{ even}} \frac{(2\pi k t)^l)}{l!} \Re(a_k) i^l - \sum_{0 \leq l \leq L(k), l \text{ odd}} \frac{(2\pi k t)^l)}{l!} \Im(a_k) i^{l-1} \Big), \end{aligned}$$

which is a real polynomial in t. As g is real valued, clearly $\|g - \Re G\|_\infty < 2\epsilon$.

By definition, $f(t) = g(t/2)$ for $0 \leq t \leq 1$. Hence if

$$F(t) = \Re G(t/2) \qquad (0 \leq t \leq 1),$$

then F is a real-valued polynomial, and $\|f - F\|_\infty \leq \|g - \Re G\|_\infty < 2\epsilon$.