1 Theorem 3.26

At the top of page 124, why is it "elementary to see" that $f(A)$ is open?

By a "topological vector space" I mean a vector space which has a topology making the vector space operations continuous. Any normed space, or locally convex space, is a topological vector space.

Lemma 1.1. *Let E be a topological vector space, and let $f : E \to \mathbb{R}$ be a continuous linear functional. Then either $f = 0$ or f is an open mapping.*

Proof. Suppose $f \neq 0$, so there is $x_0 \in E$ with $f(x_0) \neq 0$. Let $A \subseteq E$ be open, and let $a \in A$. As $f(x_0) \neq 0$ we can find some scalar λ such that $f(\lambda x_0) = \lambda f(x_0)$ does not equal $f(a)$. Set $x_1 = \lambda x_0$. Consider the function $\varphi : \mathbb{R} \to E; t \mapsto tx_1 + (1-t)a$. This is continuous (as it only involves the vector space operations) and $\varphi(0) = a \in A$. As A is open, there is $\epsilon > 0$ such that $\varphi(t) \in A$ if $-\epsilon < t < \epsilon$. Then consider the set $\{f(\varphi(t)) : -\epsilon < t < \epsilon\}$, which is the image of a line segment under a linear map, and is hence the open interval between $-\epsilon f(x_1) + (1+\epsilon)f(a)$ and $\epsilon f(x_1) + (1-\epsilon)f(a)$. The actual values are irrelevant– the point is that $f(x_1) \neq f(a)$, so this is a proper open interval containing $f(a)$. Thus every point in $f(A)$ has an open neighbourhood, and we conclude that $f(A)$ is open. □

In the proof of part (iii) (also on page 124) it's written "since A is compact, there is a convex open neighbourhood V of 0_E with $(A + V) \cap B = \emptyset$". Why is this? We first need a lemma.

Lemma 1.2. *Let E be a real locally convex space. For each $x \in E$ and each open set U containing x, there is a convex open neighbourhood W of 0_E with $x + W + W \subseteq U$, where $x + W + W = \{x + w + v : w, v \in W\}$.*

Proof. As the topology is translation invariant, $U = x + V$ for some open set V containing 0. As addition is continuous and $0_E + 0_E = 0_E$, there are neighbourhoods V_1, V_2 of 0_E with $V_1 + V_2 \subseteq V$. As the topology is locally convex, we can find a convex open neighbourhood W of 0_E with $W \subseteq V_1 \cap V_2$. Thus $W + W \subseteq V$, or equivalently, $x + W + W \subseteq U$. □

We now show the claimed result. For each $a \in A$, as B is closed and $a \notin B$, by the lemma, there is a convex open neighbourhood W_a of 0_E with $a + W_a + W_a \cap B = \emptyset$. Then the family $\{a + W_a : a \in A\}$ is an open cover for A, so as A is compact, there is a finite subcover, say $\{a_i + W_{a_i} : 1 \leq i \leq n\}$. Set $V = W_{a_1} \cap \cdots \cap W_{a_n}$, which is a convex open neighbourhood of 0_E. Then

$$A + V \subseteq \bigcup_{i=1}^{n} a_i + W_{a_i} + V \subseteq \bigcup_{i=1}^{n} a_i + W_{a_i} + W_{a_i},$$

which is disjoint from B, as required.

2 Lemma 3.38

Let $T : E \to F$ be a (bounded) linear map. The first part of this lemma suggests that it is obvious to see that if

$$\text{for all } y \in F \text{ there is } x \in E \text{ with } T(x) = y, \|x\| \leq K\|y\|,$$

then T is open. Why is this?

Well, let $U \subseteq E$ be open. Then let $y_0 \in T(U)$, so $y_0 = T(x_0)$ for some $x_0 \in U$. As U is open, $B(x_0, \epsilon) \subseteq U$ for some $\epsilon > 0$. Let $z \in F$ with $\|z\| < \epsilon/K$, so by assumption, there is $x' \in E$ with $T(x') = z$ and $\|x'\| \leq K\|z\| < \epsilon$. Then $x_0 + x' \in B(x_0, \epsilon) \subseteq U$ and $T(x_0 + x') = y_0 + z$. As z was arbitrary, we've shown that $B(y_0, \epsilon/K) \subseteq T(U)$. As y_0 are arbitrary, we've shown that $T(U)$ is open.

If you think about it for a moment, we have actually just proved the following lemma:

Lemma 2.1. *Let E, F be normed space and $T : E \to F$ be linear. Let $B = \{x \in E : \|x\| < 1\}$ be the open unit ball of E, and suppose that $T(B)$ contains an open neighbourhood of 0. Then T is open.*

3 Theorem 3.40

What is the "elementary argument" in the proof. I think it is the following.

We know that $\overline{T(B_N)}$ has non-empty interior, which means we can find $y \in \overline{T(B_N)}$ and $\epsilon > 0$ with $B(y, \epsilon) \subseteq \overline{T(B_N)}$. Thus we can find $(x_n) \subseteq E$ with $\|x_n\| < N$ for all n, and with $T(x_n) \to y$. Let $w \in F$ with $\|w\| < \epsilon/2$. Then $y + 2w \in B(y, \epsilon) \subseteq \overline{T(B_N)}$ and so we can find $(x'_n) \subseteq E$ with $\|x'_n\| < N$ for all n, and with $T(x'_n) \to y + 2w$. Then

$$\Big\| \frac{1}{2}(x'_n - x_n) \Big\| < N \text{ for all } n, \quad \text{and} \quad T\Big(\frac{1}{2}(x'_n - x_n)\Big) \to \frac{1}{2}(y + 2w - y) = w.$$

That is, $w \in \overline{T(B_N)}$. We've hence shown that $B(0, \epsilon/2) \subseteq \overline{T(B_N)}$, which is what we need.

4 Section 5.3: The Jacobson Radical

Let A be a unital algebra. For a left ideal L of A, define $L : A = \{a \in A : ab \in L \ (b \in A)\}$ which agrees with the kernel of the natural representation of A on A/L. An ideal is *primitive* if it equals $L : A$ for some maximal left ideal L; equivalently, the primitive ideals are the kernels of irreducible representations.

If A is a Banach algebra, then a maximal (left) ideal must be closed. If L is closed, then so is $L : A$. Thus primitive ideals are automatically closed.

If A is commutative and Banach, then the maximal ideals correspond to the kernels of characters. It's claimed in the book (page 231) that it's "obvious" that the primitive ideals are simply the maximal ideals.

Lemma 4.1. *In a commutative unital algebra A, the primitive ideals are the maximal ideals.*

Proof. Let $M \subseteq A$ be a maximal ideal. Then by a standard Zorn's lemma argument, there is a maximal left ideal L containing M. Set $P = L : A$, so P is primitive. If $m \in M$, then for $b \in A$, also $mb \in M \subseteq L$; it follows that $m \in L : A = P$ so $M \subseteq P$. As M is maximal, we conclude that $M = P$ is primitive (and we didn't use that A is commutative).

Conversely, let P be a primitive ideal, so $P = L : A$ for some maximal left ideal L. As A is commutative, L is a maximal (two-sided) ideal and so if $a \in L$ and $b \in A$, then $ab \in L$; this shows that $L \subseteq L : A = P$. By maximality, $L = P$ and so P is a maximal ideal. □

{With hindsight that was easy! But the book starts talking about characters, which seems misleading to me.}

5 Comments on various exercises

5.1 Exercise 2.9

To show that $C(\mathbb{I})$ is isomorphic to $C(\mathbb{I}) \oplus C(\mathbb{I})$ is pretty hard– this is proved in Banach's book, for example! I cannot see how to give a nice "hint".

5.2 Exercise 4.4

This is very hard. Here are some hints which make it (a bit) easier:

- Suppose that the exercise is true for Banach algebras of the form $\mathcal{B}(E)$. Let $A \to \mathcal{B}(A); a \mapsto L_a$ where $L_a(b) = ab$, the left-regular representation. Use this to show the result.
- Suppose that the exercise is true for Banach algebras of the form $\mathcal{B}(E^*)$. Show that it's true for $\mathcal{B}(E)$.
- Use the Krein-Milman theorem applied to the unit ball of E^*, to show that the exercise is true for Banach algebras of the form $\mathcal{B}(E^*)$.

5.3 Exercise 4.5

This appears to be false under any reasonable interpretation.

5.4 Exercise 4.9

Of course, this should ask you to show that $LR = I_H$ but $RL \neq I_H$.

5.5 Exercise 4.10

This is false, as stated. We have that $\|T\| \leq 1$ and that $\bigcap_n T^n(E) = \{0\}$ (notice the typo here in the question). It is true that $0 \in \mathrm{Sp}(T)$ and that T has no eigenvalues. You can follow the construction with the operators U_ζ to show correctly that $\mathrm{Sp}(T)$ is rotationally invariant. So $\mathrm{Sp}(T)$ is a union of circles, all inside the closed unit disc in $\mathbb{C}$. However, this does not mean that $\mathrm{Sp}(T)$ is itself a disc (or $\{0\}$).

It is easy to see that

$$T^n(x_1, x_2, x_3, \cdots) = (0, \cdots, 0, w_1 w_2 \cdots w_n x_1, w_2 w_3 \cdots w_{n+1} x_2, \cdots),$$

where there are n zeros. It follows that

$$\|T^n\| = \sup_{m \geq 1} \|w_m \cdots w_{m+n-1}\| \implies \rho(T) = \lim_n \sup_{m \geq 1} \|w_m \cdots w_{m+n-1}\|^{1/n}.$$

5.5.1 Counter-example

Define

$$w_n = \begin{cases} 1/k & : n = 2^k \text{ for some } k \in \mathbb{N}, \\ 1 & : \text{otherwise.} \end{cases}$$

Because the gaps between successive powers of 2 increase without bound, it follows from the above formula that $\rho(T) = 1$. By rotational invariance, $\mathrm{Sp}(T)$ must contain the unit circle.

We'll now show that $0 \in \mathrm{Sp}_{\mathrm{ap}}(T)$,

5.6 Exercise 4.11(i)

K is a compact Hausdorff space, $F \subseteq K$ is closed, we define

$$I(F) = \{f \in C(K) : f|F = 0\}.$$

This is a closed ideal in $C(K)$ (which is not too hard to show).

Why does every closed ideal arise in this way? I think that this is slightly tricky to answer. Let $J \subseteq C(K)$ be a closed ideal, and then set

$$F = \{k \in K : f(k) = 0 \ (f \in J)\}.$$

It's not too hard to show that this is a closed subset of K, and that $J \subseteq I(F)$. But why do we have equality?

Form the quotient algebra $A = C(K)/J$, so that A is a commutative Banach algebra.

Claim 5.1. *With notation as above, A is semi-simple, that is, the Gelfand transform $\mathcal{G} : A \to C(\Phi_A)$ is injective.*

If we believe this, then suppose that $J \neq I(F)$. Thus there is $g \in I(F)$ with $g \notin J$, and so $g + J \neq 0$ in A, and so $\mathcal{G}(g + J) \neq 0$. Thus there is a character φ on A with $\varphi(g + J) \neq 0$. Then $\phi : C(K) \to \mathbb{C}; f \mapsto \varphi(f + J)$ is a character on $C(K)$, and so there is $k \in K$ with $\phi(f) = f(k)$ for all f. Thus $g(k) \neq 0$. As $g \in I(F)$, we must have that $k \notin F$. However, for any $f \in J$ we have that $f(k) = \phi(f) = \varphi(f + J) = 0$, and so $k \in F$, contradiction. So $J = I(F)$.

How do we prove the claim? We could use that $C(K)$ is a C*-algebra, that J is $*$-closed, and that thus $C(K)/J$ is also a C*-algebra. Then use that the Gelfand transform of a commutative C*-algebra is always injective (actually, an isomorphism).

5.6.1 A direct proof

Again let $g \in I(F)$. For $\epsilon > 0$ let $U = \{k \in K : |g(k)| < \epsilon\}$ so U is an open set containing F. For each $x \notin U$, as $x \notin K$, we can find $f_x \in J$ with $f_x(x) = 1$ say (by definition of F we can find $f_x \in J$ with $f_x(x) \neq 0$, and then rescale). Then $U_x = \{k \in K : |f_x(k)| > 1/2\}$ is open and contains x. As $K \setminus U$ is closed, hence compact, we can find $x_1, \cdots, x_n$ with $U_{x_1} \cup \cdots \cup U_{x_n} \supseteq K \setminus U$.

Given $f \in J$, notice that $|f|^2 = f\overline{f} \in J$ as J is an ideal. Thus $h = |f_{x_1}|^2 + \cdots + |f_{x_n}|^2 \in J$. Then $h(k) = 0$ for each $k \in F$, while for each $x \notin U$, there is i with $x \in U_{x_i}$, and so $h(x) > (1/2)^2 = 1/4$.

Now consider[1] $g_n \in C(K)$ defined by

$$g_n(x) = g(x)\frac{nh(x)}{1 + nh(x)}.$$

Notice that $nh(x)/(1 + nh(x)) \in [0, 1)$ for all x and n. If $x \notin U$ then $h(x) > 1/4$ and so $nh(x)/(1 + nh(x)) \to 1$ as $n \to \infty$, *uniformly* for $x \notin U$. In particular, if n is large, then $|g_n(x) - g(x)| < \epsilon$ for all $x \notin U$. If $x \in U$ then $|g(x)| < \epsilon$ and so also $|g_n(x)| < \epsilon$, and so $|g_n(x) - g(x)| < 2\epsilon$. We conclude that for n large, g_n approximates g in the supremum norm. However, notice that

$$g_n = \frac{ng}{1 + nh}h,$$

and so as J is an ideal, $g_n \in J$. As J is closed, we conclude that $g \in J$, as required.

[1] Thanks to George Berkley for point this trick out.

5.7 An example

Consider $B = C^1([0,1])$ the continuous differentiable functions on $[0,1]$ with the norm $\|f\| = \|f\|_\infty + \|f'\|_\infty$. This is a natural Banach function algebra on $[0,1]$ (see Exercise 4.12).

Let J be the collection of functions with $f(1/2) = f'(1/2) = 0$. This is a linear subspace, and an ideal, as for any g,

$$(gf)(1/2) = 0, \qquad (gf)'(1/2) = g'(1/2)f(1/2) + g(1/2)f'(1/2) = 0.$$

It's easy to see that it's closed (thanks to the norm we used). However, I claim that the associated F must be $\{1/2\}$, and so $I(F) \neq J$. Indeed, the function $f(x) = (x - 1/2)^2$ is in J but vanishes only at $1/2$.

What goes wrong with the above proof? The problem is that while $\|g_n - g\|$ is small, we have no control over $\|g_n' - g'\|$.