Corrections and updates to papers

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January 7, 2021

Abstract

Here collected are various updates, minor, and major corrections to papers. Mostly I am just correcting typos or adding references, but where major problems have occurred, I have added ♡ to the title.

Contents

1 "Arens regularity of the algebra of operators on a Banach space"

Thanks to Volker Runde, I now know that the argument I use to prove Proposition 5 is not new. The paper [4] showed this result for a representation of a Banach algebra on a Banach space; furthermore, they use almost exactly the same argument. My apologies to the authors of this paper for not having done a better literature search.

The construction of the map π in Section 3 is, of course, the *Arens representation*: see [16, Section 1.4].

The proof of Theorem 2 contains a number of calculation errors, although the conclusions remain the same. In particular, at a number of points, I assume that $\theta = 1$, which is not correct. The final line should read "Then $\langle \lambda, T_M T_N \rangle = \theta^3$ if $M \leq N \ldots$".

2 “Semisimplicity of $\mathcal{B}(E)''$”

In the proof of Theorem 3.7, we should define

$$\mu = \sum_{n=1}^{\infty} \hat{\mu}_n \|\hat{\mu}_n\|^{-1+1/q},$$

that is, replace the p by a q.

3 “Dual Banach algebras: representations and injectivity”

My thanks to Garth Dales for the following comments, which apply to the printed version of this paper, which is in Studia Mathematica. Lemma 2.3 should be attributed to John Pym. The statement after this lemma, that the intersection of the topological centres is an ideal, is false. For example, in the extreme case that the intersection is the original algebra, it does not follow that the algebra is an ideal in its bidual! Consider a discrete group algebra, for example. It may help readers to know that the first condition of Proposition 2.4 is often referred to as saying that X is *left-introverted*. This section of the paper is annoying: as I state a number of times, most of the results can be found in some form in the literature, and they are surely known to experts. However, I couldn't find a source which would be understandable without a large amount of explanation, and hence I decided to simply sketch all the results I needed.

4 ♡“Can $\mathcal{B}(\ell^p)$ ever be amenable?”

The following has been submitted to Studia Math. as an erratum.

Some of the results of section 3 of [9] are incorrect; in particular, we claimed that the implication (i)⇒(ii) of Lemma 3.3 was “routine”, whereas it appears to be false, or at least difficult to prove.

The main result of this section, Theorem 3.2, claims that a separable Banach algebra $\mathfrak{A}$ is ultra-amenable (that is, all ultrapowers of $\mathfrak{A}$ are amenable) if and only if $\ell^\infty(\mathfrak{A})$ is amenable. However, if we let $\mathfrak{A} = \mathbb{C}$, then any ultrapower of $\mathfrak{A}$ is also $\mathbb{C}$, and hence trivially amenable. While ℓ^∞ is amenable, this is not trivial, and in no sense do our arguments reduce to this special case.

Furthermore, the motivation for Section 3 was [6, Section 5], where the first named author studied similar ideas for ultra-amenability. The proof of [6, Proposition 5.4], (ii)⇒(i), also needs further justification, as currently the map ψ_0 is implicitly assumed to be at least bounded below. However, in this case, in light of [6, Proposition 4.7], it seems possible that this could be true, at least for certain well-behaved spaces $\mathfrak{A}$.

Let us restate Theorem 3.2. In light of the example of $\mathfrak{A} = \mathbb{C}$, it seems unlikely that Lemma 3.3, even suitably adjusted, could be true, and so fully correcting Theorem 3.2 seems out of reach.

Theorem 4.1. *Let $\mathfrak{A}$ be a Banach algebra, and consider the conditions:*

(i) $\ell^\infty(\mathfrak{A})$ is amenable;

(ii) $\ell^\infty(\mathbb{I}, \mathfrak{A})$ is amenable for every index set $\mathbb{I}$;

(iii) $\mathfrak{A}$ is ultra-amenable.

Then (ii)⇒(i) and (ii)⇒(iii).

Proof. Clearly (ii)⇒(i), and as an ultrapower of $\mathfrak{A}$ is a quotient of $\ell^\infty(\mathbb{I}, \mathfrak{A})$ for a suitable $\mathbb{I}$, and amenability passes to quotients, it follows that (ii)⇒(iii). □

Corollary 4.4 uses Theorem 3.2, but only implication (ii)⇒(iii), and hence remains true. The rest of [9] is unaffected. In particular, the tentative approach, outlined in Section 6, to showing that $\mathcal{B}(\ell^p)$ is not amenable, is not affected. We remark that the second named author has recently shown in [17] that, in particular, $\mathcal{B}(\ell^p)$ is not amenable for any $p \in [1, \infty)$; the arguments only rely upon Section 2 of [9] and are hence unaffected by this erratum.

Acknowledgments: We wish to thank Seytek Tabaldyev who brought this problem to our attention.

5 ♡"Amenability of ultrapowers of Banach algebras"

The following will shortly be submitted to Proc. Edinb. Math. Soc. as an erratum.

Some of the results of Section 5 of [6] are incorrect. The claim (ii)⇒(i) of Proposition 5.4 implicitly assumes that ψ_0 is bounded below, but this is unproven. Hence also the claim, in Corollary 5.5, that if a Banach algebra $\mathcal{A}$ is contractible then it is ultra-amenable, is unproven. Similarly (ii)⇒(i) of Theorem 5.6 requires ψ_0 to be bounded below. The rest of the paper is unaffected. We used some of these ideas in [9, Section 4], and so this is also incorrect; an erratum has been submitted.

Firstly, we deal with correcting Theorem 5.7. We say that a C*-algebra $\mathcal{A}$ is *subhomogeneous* if there exists $n \in \mathbb{N}$ such that every irreducible representation of $\mathcal{A}$ has dimension at most n. Subhomogeneous von Neumann algebras have the special form claimed in Theorem 5.7, but this is not true for C*-algebras, see [1, Section IV.1.4] for examples. This circle of ideas was considered in [14, Theorem 2.5] but we have been unable to follow some of the proofs (in particular, the claim that (A4)⇒(R5)) so we provide details here.

Theorem 5.1. *Let $\mathcal{A}$ be a C^*-algebra. Then the following are equivalent:*

1. *$\mathcal{A}$ is ultra-amenable;*
2. *$\mathcal{A}''$ is amenable;*
3. *$\ell^\infty(\mathcal{A}, I)$ is amenable for any index set I;*
4. *$\mathcal{A}$ is subhomogeneous.*

Proof. The original argument using the approximation property in [6] is correct and shows (1)⇒(2). Similarly, as argued in [6] (see also [14]) if (2) holds, then $\mathcal{A}''$ has the form

$$\mathcal{A}'' = \sum_{k=1}^{n} L^\infty(X_k) \otimes \mathbb{M}_{n_k},$$

where for each k, X_k is a measure space, and $n_k \in \mathbb{N}$. Notice that if $\mathcal{A}''$ is of this form, then following [14], it is elementary to see that so is $l^\infty(\mathcal{A}'', I)$ for any index set I. This does imply that $\mathcal{A}$ is subhomogeneous (see [1, Proposition IV.1.4.6]) but not that $\mathcal{A}$ has the form originally claimed in [6, Theorem 5.7].

However, there is an algebraic characterisation of when C^*-algebras are subhomogeneous, see [11, Section 3.6] or [1, Section IV.1.4.5]. The algebra $\mathbb{M}_n$ is of dimension n^2 and so for any $r > n^2$, we have

$$\sum_{\sigma \in S_r} \epsilon_\sigma x_{\sigma(1)} \cdots x_{\sigma(r)} = 0, \tag{1}$$

for any $x_1, \cdots, x_r \in \mathbb{M}_n$ (this is readily seen by taking a basis). Here S_r is the symmetric group and $\epsilon : S_r \to \{\pm 1\}$ the signature. Let $r(n)$ be the smallest r for which this holds for $\mathbb{M}_n$. Then [11, Lemma 3.6.2] shows that $r(n) \geq r(n-1) + 2$ (see [1, Section IV.1.4.5] and references therein for better estimates). As irreducible representations separate the points of a C*-algebra $\mathcal{A}$, we conclude that the following are equivalent:

i. any irreducible representation of $\mathcal{A}$ is of dimension at most n;

ii. for any $x_1, \cdots, x_{r(n)} \in \mathcal{A}$, identity (1) holds for $r = r(n)$.

Indeed, the only unclear issue is if $\pi : \mathcal{A} \to \mathcal{B}(H)$ is irreducible, with H infinite dimensional, why cannot (ii) hold? However, then $\pi(\mathcal{A})$ is strongly dense in $\mathcal{B}(H)$, and $\mathbb{M}_{n+1}$ is a subalgebra of $\mathcal{B}(H)$, which is enough to show that (ii) fails.

It is clear that the second condition passes to subalgebras, and with a little thought, it is seen to pass to ultrapowers as well. Thus, if (2) holds, then $\ell^\infty(\mathcal{A}, I)$ is subhomogeneous and $\mathcal{A}$ is subhomogeneous, showing (4). It is reasonably easy to show that $\ell^\infty(\mathcal{A}, I)$ is thus nuclear (see [2, Proposition 2.7.7]), or follow [14, Theorem 2.5] for a direct argument that $\ell^\infty(\mathcal{A}, I)$ is thus amenable. As amenability passes to quotients, (3)⇒(1) is clear. Finally, if (4) holds then any ultrapower of $\mathcal{A}$ is subhomogeneous and hence amenable, showing (1). □

We erroneously claimed in [9] that (1) and (3) are equivalent for any Banach algebra $\mathcal{A}$. It would be interesting to know if this is true.

We shall now improve [6, Proposition 4.7], and show that the map ψ_0 is indeed bounded below for a wide class of Banach algebras $\mathcal{A}$. We leave open whether this holds for all $\mathcal{A}$ (which seems unlikely). It seems possible that similar, but stronger, conditions could characterise when $\mathcal{A}$ is ultra-amenable, but we shall not pursue this here.

Let E and F be Banach spaces, and let $\mathcal{U}$ be an ultrafilter on an index set I. As in [6], we shall suppose that $\mathcal{U}$ is countably incomplete. Recall from [6, Section 4] the map $\psi_0 : (E)_\mathcal{U} \widehat{\otimes} (F)_\mathcal{U} \to (E \widehat{\otimes} F)_\mathcal{U}$, defined on elementary tensors by

$$\psi_0((x_i) \otimes (y_i)) = (x_i \otimes y_i) \qquad ((x_i) \in (E)_\mathcal{U}, (y_i) \in (F)_\mathcal{U}).$$

For the following, we recall that [13, Theorem 9.1] characterises, in terms of local properties, when an ultrapower has the (bounded) approximation property.

Theorem 5.2. *If $(E)_\mathcal{U}$ has the approximation property, then ψ_0 is an injection for any F.*

Proof. Let $\tau \in (E)_\mathcal{U} \widehat{\otimes} (F)_\mathcal{U}$ have representation $\tau = \sum_{n=1}^\infty x_n \otimes y_n$ with $\sum_n \|x_n\| \|y_n\| < \infty$. If $(E)_\mathcal{U}$ has the approximation property then, by [19, Proposition 4.6], if $\tau \in (E)_\mathcal{U} \widehat{\otimes} (F)_\mathcal{U}$ is non-zero, then there exist $\mu \in (E)'_\mathcal{U}$ and $\lambda \in (F)'_\mathcal{U}$ with

$$0 \neq \langle \mu \otimes \lambda, \tau \rangle = \sum_{n=1}^\infty \langle \mu, x_n \rangle \langle \lambda, y_n \rangle.$$

As we only care about the value of μ on the countable set $\{x_n\}$, by [13, Corollary 7.5], we may suppose that $\mu \in (E')_\mathcal{U}$, and similarly, that $\lambda \in (F')_\mathcal{U}$, say $\mu = (\mu_i)$ and $\lambda = (\lambda_i)$. Pick representatives $x_n = (x_n^{(i)})$ and $y_n = (y_n^{(i)})$, so that by absolute convergence,

$$\langle \mu \otimes \lambda, \tau \rangle = \lim_{i \to \mathcal{U}} \sum_{n=1}^\infty \langle \mu_i, x_n^{(i)} \rangle \langle \lambda_i, y_n^{(i)} \rangle = \langle (\mu_i \otimes \lambda_i), \psi_0(\tau) \rangle.$$

Hence we must have that $\psi_0(\tau) \neq 0$. □

Consequently, [6, Corollary 5.5] correctly shows that if $\mathcal{A}$ is a contractible Banach algebra with the approximation property, then $\mathcal{A}$ is ultra-amenable. However, a result of Selivanov, see [18, Theorem 4.1.5], shows that under these conditions, $\mathcal{A}$ is already the finite sum of full matrix algebras!

It is worth pointing out what can go wrong here (and hence the exact mistake in the proof of [6, Proposition 5.4]). If $\mathcal{A}$ is contractible, then we can find $\tau \in \mathcal{A} \widehat{\otimes} \mathcal{A}$ with $a \cdot \tau = \tau \cdot a$ and $\Delta(\tau) a = a$ for $a \in \mathcal{A}$. We can then treat τ as a member of $(\mathcal{A})_\mathcal{U} \widehat{\otimes} (\mathcal{A})_\mathcal{U}$, and we have that $a \cdot \psi_0(\tau) = \psi_0(\tau) \cdot a$ for $a \in (\mathcal{A})_\mathcal{U}$. As ψ_0 is an $(\mathcal{A})_\mathcal{U}$-module homomorphism, $\psi_0(a \cdot \tau - \tau \cdot a) = 0$ for any $a \in (\mathcal{A})_\mathcal{U}$. However, if ψ_0 might fail to be injective, then this is not useful.

The following improves [6, Proposition 4.7], as a result of Grothendieck, see [19, Corollary 5.51], shows that a reflexive Banach space with the approximation property automatically has the metric approximation property.

Theorem 5.3. *If $(E)_\mathcal{U}$ has the bounded approximation property, then ψ_0 is bounded below.*

Proof. Let $(E)_\mathcal{U}$ have the bounded approximation property with bound M, so by (the obvious generalisation of) [19, Theorem 4.14], the embedding $(E)_\mathcal{U}\widehat{\otimes}(F)_\mathcal{U} \to \mathcal{F}((E)_\mathcal{U}, (F)'_\mathcal{U})$ is bounded below by M^{-1}. Here $\mathcal{F}((E)_\mathcal{U}, (F)'_\mathcal{U})$ is the collection of finite-rank operators $(E)_\mathcal{U} \to (F)'_\mathcal{U}$, given the operator norm.

Let $\tau \in (E)_\mathcal{U}\widehat{\otimes}(F)_\mathcal{U}$ have representative $\tau = \sum_{n=1}^\infty x_n \otimes y_n$. For $\epsilon > 0$, we can find $T \in \mathcal{F}((E)_\mathcal{U}, (F)'_\mathcal{U})$ with $\|T\| \leq M + \epsilon$ and $|\langle T, \tau\rangle| \geq \|\tau\|$. Pick a representative

$$T = \sum_{k=1}^N \mu_k \otimes \lambda_k,$$

for some $(\mu_k) \subseteq (E)'_\mathcal{U}$ and $(\lambda_k) \subseteq (F)'_\mathcal{U}$, so that

$$\langle T, \tau\rangle = \sum_{n=1}^\infty \sum_{k=1}^N \langle \mu_k, x_n\rangle\langle\lambda_k, y_n\rangle.$$

Let G be the closed span of $\{x_n\}$, so by [13, Corollary 7.5], we can find a contraction ϕ : $\mathrm{lin}\{\mu_k\} \to (E')_\mathcal{U}$ such that

$$\langle \phi(\mu_k), x\rangle = \langle \mu_k, x\rangle \qquad (1 \leq k \leq N, x \in G).$$

It's not hard to see that then

$$T_0 = \sum_{k=1}^N \phi(\mu_k) \otimes \lambda_k$$

satisfies $\|T_0\| \leq M + \epsilon$ and $\langle T_0, \tau\rangle = \langle T, \tau\rangle$. In other words, we can assume that $\mu_k \in (E')_\mathcal{U}$ for each k; analogously, we may also assume that $\lambda_k \in (F')_\mathcal{U}$ for each k.

So, pick representatives $\mu_k = (\mu_k^{(i)})$ and $\lambda_k = (\lambda_k^{(i)})$, and for each i, let

$$T_i = \sum_{k=1}^N \mu_k^{(i)} \otimes \lambda_k^{(i)}.$$

As $\mathcal{U}$ is countably incomplete, we can find a sequence (ϵ_i) of strictly positive reals such that $\lim_{i\to\mathcal{U}} \epsilon_i = 0$. For each i, pick $y_i \in E$ with $\|y_i\| \geq 1$ and $\|T_i(y_i)\| \geq \|T_i\| - \epsilon_i$. Let $y = (y_i)$ so $\|y\| = 1$ and $T(y) = (T_i(y_i))$ so that

$$\lim_{i\to\mathcal{U}} \|T_i\| = \lim_{i\to\mathcal{U}} \|T_i\| - \epsilon_i \leq \lim_{i\to\mathcal{U}} \|T_i(y_i)\| = \|T(y)\| \leq (M + \epsilon).$$

Finally, a calculation shows that

$$\langle T, \tau\rangle = \langle (T_i), \psi_0(\tau)\rangle,$$

where $(T_i) \in (\mathcal{B}(E, F^*))_\mathcal{U} \subseteq (E\widehat{\otimes}F)'_\mathcal{U}$. We conclude that $\|\psi_0(\tau)\| \geq \|\tau\|(M + \epsilon)^{-1}$, so that ψ_0 is bounded below by M^{-1}. □

This shows that [6, Theorem 5.6] does give a correct characterisation of ultra-amenability for Banach algebras $\mathcal{A}$ whose ultrapowers have the bounded approximation property. This includes, for example, algebras of the form $L^1(G)$ for a locally compact group G.

Acknowledgments: I wish to thank: Seytek Tabaldyev who brought the problems in [9] (and hence also in [6]) to my attention; Volker Runde for pointing out the error in Theorem 5.7; and both Volker Runde and the authors of [14] for suggesting to look at polynomial identities as a way to fix Theorem 5.7.

6 "Reiter's properties (P_1) and (P_2) for locally compact quantum groups"

My thanks to Kenny De Commer for pointing out that the proof of [10, Proposition 3.2] needs further justification. Indeed, we first just give a short direct proof of this proposition (but combining it with the proof of the preceding lemma).

Let us recall some notation. We have a locally compact quantum group $(L^\infty(\mathbb{G}),\Gamma)$. For $g\in L^1(\mathbb{G})$ let $(\Gamma|g)$ be the map $L^\infty(\mathbb{G})\to L^\infty(\mathbb{G})$ given by $x\mapsto(\mathrm{id}\otimes g)\Gamma(x)$. For $a,b\in C_0(\mathbb{G})$, let $M_{a,b}$ be the map $L^\infty(\mathbb{G})\to L^\infty(\mathbb{G})$ given by $x\mapsto axb$.

Proposition 6.1. *Let $\mathbb{G}$ be a locally compact quantum group, let $g\in L^1(\mathbb{G})$, and let $a,b\in C_0(\mathbb{G})$. Then $M_{a,b}\circ(\Gamma|g)$ is a completely bounded operator from $L^\infty(\mathbb{G})$ to $C_0(\mathbb{G})$ that lies in the cb-norm closure of the finite rank operators form $L^\infty(\mathbb{G})$ to $C_0(\mathbb{G})$.*

Proof. Let W be the multiplicative unitary for $L^\infty(\mathbb{G})$, so that $\Gamma(x)=W^*(1\otimes x)W$ for $x\in L^\infty(\mathbb{G})$. Observe that for $x\in L^\infty(\mathbb{G})$,

$$\begin{aligned}M_{a,b}\circ(\Gamma|g)(x)&=a\big((\mathrm{id}\otimes g)\Gamma(x)\big)b=a\big((\mathrm{id}\otimes g)W^*(1\otimes x)W\big)b\\&=(\mathrm{id}\otimes g)\big((a\otimes 1)W^*(1\otimes x)W(b\otimes 1)\big).\end{aligned}$$

Let $g=\omega_{\xi,\eta}$ for some $\xi,\eta\in L^2(\mathbb{G})$, and pick compact operators L and K with $L(\xi)=\xi$ and $K^*(\eta)=\eta$. Then we have that

$$M_{a,b}\circ(\Gamma|g)(x)=(\mathrm{id}\otimes g)\big((a\otimes K^*)W^*(1\otimes x)W(b\otimes L)\big).$$

As $W\in M(C_0(\mathbb{G})\otimes K(L^2(\mathbb{G})))$ it follows that $W(b\otimes L)$ and $(a\otimes K^*)W^*$ are members of $C_0(\mathbb{G})\otimes K(L^2(\mathbb{G}))$ (the minimal C*-tensor product). So we can norm approximate $W(b\otimes L)$ and $(a\otimes K^*)W^*$ by $\tau,\sigma\in C_0(\mathbb{G})\odot K(L^2(\mathbb{G}))$ (algebraic tensor product), and it follows that $M_{a,b}\circ(\Gamma|g)$ is cb-norm approximated by the map

$$T:x\mapsto(\mathrm{id}\otimes g)\big(\sigma(1\otimes x)\tau\big).$$

Now, let $c,d\in C_0(\mathbb{G})$ and $A,B\in K(L^2(\mathbb{G}))$. Then, for $x\in L^\infty(\mathbb{G})$,

$$(\mathrm{id}\otimes g)\big((c\otimes A)(1\otimes x)(d\otimes B)\big)=\langle x,BgA\rangle cd=\langle x,\omega_{B\xi,A^*\eta}\rangle cd.$$

So the map $x\mapsto(\mathrm{id}\otimes g)\big((c\otimes A)(1\otimes x)(d\otimes B)\big)$ is finite rank. By linearity, it follows that $x\mapsto(\mathrm{id}\otimes g)\big(\sigma(1\otimes x)\tau\big)$ is also finite rank, as required. $\square$

Now let us turn to the problem with the original proof. For operator spaces E and F, write $E\check{\otimes}F$ for the operator space injective tensor norm. In particular, $C_0(\mathbb{G})\check{\otimes}L^\infty(\mathbb{G})^*$ is the closure of the finite rank operators in $\mathcal{CB}(L^\infty(\mathbb{G}),C_0(\mathbb{G}))$. At the end of [10, Page 357], we claimed without proof that if $T\in C_0(\mathbb{G})\check{\otimes}L^\infty(\mathbb{G})^*$ is such that $T^*(C_0(\mathbb{G})^*)\subseteq L^1(\mathbb{G})$, then we can identify T with a member of $C_0(\mathbb{G})\check{\otimes}L^1(\mathbb{G})$. This is actually true, but requires further justification, because analogous claims (for more general operator spaces) are false. We now explain this.

Proposition 6.2. *Let M be a von Neumann algebra, let A be a C*-algebra, and let $T:M\to A$ be a completely bounded map which can be cb-norm approximated by finite ranks. If $T^*(A^*)\subseteq M_*$, then we can identify T with a member of $A\check{\otimes}M_*$ (and not just $A\check{\otimes}M^*$).*

Proof. The main tool is the fact that M_* is complemented in M^*. Indeed, M_* is an invariant submodule of M^*, so there is a central projection $p\in M^{**}$ with $M_*=M^*p$. (See, for example, [20, Chapter III, Section 2]). It is easy to see that the map $P:M^*\to M_*,\phi\mapsto\phi p$ is a complete contraction; also P leaves M_* fixed.

So for $\epsilon > 0$, we can find $S : M \to A$ finite rank, with $\|T - S\|_{cb} < \epsilon$. Let $S = \sum_{i=1}^n a_i \otimes \phi_i \in A \odot M^*$, and let $R = \sum_{i=1}^n a_i \otimes P(\phi_i) \in A \odot M_*$. We also treat R as a finite rank map $M \to A$. Then, for $\mu \in A^*$,

$$S^*(\mu) = \sum_{i=1}^n \langle \mu, a_i \rangle \phi_i, \quad R^*(\mu) = \sum_{i=1}^n \langle \mu, a_i \rangle P(\phi_i) = PS^*(\mu).$$

Thus $R^* = PS^*$, and as $PT^* = T^*$, it follows that $\|T - R\|_{cb} = \|T^* - R^*\|_{cb} = \|PT^* - PS^*\|_{cb} \leq \|T^* - S^*\|_{cb} = \|T - S\|_{cb} < \epsilon$.

So we can cb-norm approximate T be elements of $A \odot M_*$, as required. $\square$

If we just work at the level of tensors, then for operator spaces E and F, we can consider "slices" of $E \check{\otimes} F$. That is, for $\mu \in E^*$, we have a completely bounded map $\mu \otimes \iota : E \check{\otimes} F \to F; x \otimes y \mapsto \langle \mu, y \rangle$. We can analogously define $\iota \otimes \lambda$ for $\lambda \in F^*$. The previous proposition thus states that if M is a von Neumann algebra, and A is a C*-algebra (actually, A could be any operator space) then if $\tau \in A \check{\otimes} M^*$ satisfies that $(\mu \otimes \iota)(\tau) \in M_*$ for all $\mu \in A^*$, then $\tau \in A \check{\otimes} M_*$.

Let A and B be C*-algebras, and let $X \subseteq B$ be a closed subspace. Following [2, Definition 12.4.3], we say that (A, B, X) satisfies the *slice map property* if, whenever $\tau \in A \check{\otimes} B$ is such that $(\mu \otimes \iota)\tau \in X$ for $\mu \in A^*$, then actually $\tau \in A \check{\otimes} X$. This is similar to the above, but there we had the dual and predual of a von Neumann algebra.

Then [2, Theorem 12.4.4] shows that a C*-algebra A has the Operator Approximation Property (OAP) if and only if $(A, \mathcal{K}(H), X)$ satisfies the slice map property, where H is a separable Hilbert space, $\mathcal{K}(H)$ is the compact operators on H, and X is any closed subspace of $\mathcal{K}(H)$. However, there are C*-algebras without the OAP, for example $\mathcal{B}(H)$ for a separable Hilbert space H.

It seems like there should be some link between this and an operator space being locally reflexive (see [12, Section 14.3]). This is true in the Banach space case (compare with the proof of [19, Proposition 5.55]) but the proof uses compactness in a way that doesn't seem to generalise to operator spaces.

7 "Multipliers, Self-Induced and Dual Banach Algebras"

I wrote this paper partly to argue that multipliers should be thought of in a more "algebraic" way– for example, I tried to show that in the presence of a bounded approximate identity, the Cohen Factorisation Theorem should do all the heavy lifting, and then most arguments could be made almost as trivial as in the unital case.

As ever, I am now somewhat unhappy about my approach– my referencing to the literature was fairly woeful. In particular, I should certainly have mentioned the book [3]. The introduction to this book spells out the philosophy which I was trying to articulate.[1]

I remain convinced that there should be some mileage in exploring "self-induced" algebras and modules in the completely bounded world– here the Fourier algebra is an obvious candidate.

7.1 Mathematical corrections

- Section 2, Page 7, the definition of "faithful" is wrong; I should have stuck to the term "non-degenerate", as in the C*-algebra world.
- Section 3, Page 16, I should probably have defined

  $$\langle \Phi \otimes_\Diamond \Psi, T \rangle = \langle T^{***} \kappa_A^{**}(\Psi), \Phi \rangle = \langle \Psi, \kappa_A^* T^{**}(\Phi) \rangle.$$

  This seems asymmetric, but it induces the correct 2nd Arens product using the definition in the following paragraph.

[1] My thanks to Yemon Choi for obliquely bringing this reference to my attention.

8 "Completely positive definite functions and Bochner's theorem for locally compact quantum groups"

The claim of [8, Lemma 3] is not shown; but if we take the linear span of the set D then the proof is valid, and all applications of this result continue to work. See also [5, Proposition 6.8].

The issue with the proof is that the set D does not appear to be a subspace, and so the "smearing" technical, the integral defining $\omega(r)$ need not define something in the closure of D. Simply replacing D by its linear span fixes these issues. However, as $L^\infty(\mathbb{G})$ is in standard position on $L^2(\mathbb{G})$, we know that every normal functional on $L^\infty(\mathbb{G})$ is given by the a "vector functional". We might hence conjecture that actually perhaps D is a linear subspace (despite not obviously being so), or alternatively that the lemma remains true (that D is a core) with a different proof. However, we see no way to provide such a proof at present.

9 "Ring-theoretic (in)finiteness in reduced products of Banach algebras"

After this paper, [7], went (very quickly) to press, Hannes Thiel indicated to us how to show the following. In [7, Remark 5.9] we remarked that we, in a roundabout way, showed that if A is a C^*-algebra of stable rank one, then we get a form of "norm control" on how to approximate elements by invertibles. In fact, this can be shown directly, using work of Pedersen, [15].

Fix a unital C^*-algebra A, and let A^{-1} be the collection of invertible elements. For $a \in A$ let $d(a, A^{-1}) = \inf\{\|a-b\| : b \in A^{-1}\}$. For $a \in A$, consider A embedded in a von Neumann algebra (for example, represent A on a Hilbert space H, so $A \subseteq \mathcal{B}(H)$) let $a = v|a|$ be the polar decomposition. Let E_δ be the spectral projection of $|a|$ onto the set $(\delta, \|a\|]$.

Theorem 9.1 ([15, Theorem 5]). *For $a = v|a| \in A$ and $\delta > d(a, A^{-1})$, there is a unitary $u \in A$ with $uE_\delta = vE_\delta$.*

Now fix $a \in A$ with polar decomposition $a = v|a|$, and let $\epsilon > d(a, A^{-1})$. Choose a unitary $u \in A$ with $uE_\epsilon = vE_\epsilon$. Consider the function

$$h(t) = (t-\epsilon)_+ = \begin{cases} t-\epsilon & : t \geq \epsilon, \\ 0 & : t < \epsilon. \end{cases}$$

Write $(|a|-\epsilon)_+ = h(|a|)$ using continuous functional calculus. Using the Borel functional calculus, we see that $E_\epsilon(|a|-\epsilon)_+ = (|a|-\epsilon)_+$.

Set $c = u(\epsilon + E_\epsilon(|a|-\epsilon)_+) \in A$. Then $c = \epsilon u + vE_\epsilon(|a|-\epsilon)_+$, and as $0 \leq (|a|-\epsilon)_+ \leq \|a\|-\epsilon$, we see that $\|c\| \leq \|a\|$. Furthermore, as $u^*c \geq \epsilon$, we see that c is invertible, with $\|c^{-1}\| \leq \epsilon^{-1}$. Also, from the Borel functional calculus,

$$a = v|a| = v\big(E_\epsilon|a| + (1-E_\epsilon)|a|\big) = v\big(E_\epsilon((|a|-\epsilon)_+ + \epsilon) + (1-E_\epsilon)|a|\big).$$

Thus

$$\begin{aligned} a &= c - \epsilon u + \epsilon v E_\epsilon + v(1-E_\epsilon)|a| \\ &= c - \epsilon u(1-E_\epsilon) + v(1-E_\epsilon)|a| \end{aligned}$$

and so as $\|(1-E_\epsilon)|a|\| \leq \epsilon$,

$$\|a-c\| \leq 2\epsilon.$$

Finally, suppose that A has real-rank one, so invertibles are dense in A, so we can choose any $\epsilon > 0$. Let a be a contraction. Then c is invertible, $\|a-c\| \leq 2\epsilon$, and $\|c\| \leq 1$, and finally $\|c^{-1}\| \leq \epsilon^{-1}$.

Pedersen's proof is "elementary", in the sense that it uses nothing more than functional calculus arguments. However, it is quite long, and our argument in [7] is also elementary, and quite long. The advantage of the argument here is that it is direct, without passing through the asymptotic sequence algebra.

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