Introduction to Bases in Banach Spaces

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Abstract

We introduce the notion of Schauder bases in Banach spaces, aiming to be able to give a statement of, and make sense of, the Gowers Dichotomy Theorem.

Contents

1 Introduction

When working with finite dimensional vector spaces, it is often convenient to take a basis, and then work with the co-ordinate system which this basis gives us. In an infinite dimensional vector space, the axiom of choice shows that we can still find a basis: however, because it is now infinite, its use is often less. This is especially true in a Banach space, because such an algebraic basis takes no account of the extra structure induced by the norm.

This leads to the notion of a *Schauder basis*. Let E be a Banach space and let (e_n) be a sequence of vectors in E. Then (e_n) is a Schauder basis (or, from now on, simply a *basis*) if every $x \in E$ admits an expansion of the form

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

for some *unique* sequence of scalars (x_n).

Note. As someone interested in algebraic questions, I always work over the complex numbers. However, it is common (for simplicity) to work with real scalars in Banach space theory. Often, a similar result for complex numbers follows easily, sometimes we have to work somewhat harder, and occasionally, the result is true only for real scalars. We shall try to give proofs which work with either choice of scalar, but occasionally the reader should assume the result is only certainly known for real scalars.

Example 1.1. Let $E = l^p$ for $1 \leq p < \infty$, or $E = c_0$. Note that some authors write l_p, ℓ^p, or ℓ_p instead. For $n \geq 1$, let $e_n \in E$ be the sequence which is 0 except with a 1 in the nth position. Then (e_n) is a Schauder basis for E, called the "standard unit-vector basis" of E.

Example 1.2. Let H be a separable Hilbert space with an orthonormal basis (e_n). Then (e_n) is a basis (in fact, an unconditional basis: see later) for H.

2 Schauder bases

We first give some general results on Schauder bases. Let E be a Banach space with a basis (e_n). Clearly, we are free to multiply each e_n by a non-zero scalar without changing the basis property. Hence we may assume that (and will henceforth do so) (e_n) is a *normalised basis*, that is, $\|e_n\| = 1$ for each n. Then, for $x \in E$, we can uniquely write $x = \sum x_n e_n$, and hence define

$$\|x\|_0 = \sup_{N \geq 1} \Big\| \sum_{n=1}^{N} x_n e_n \Big\|.$$

This is well-defined, as $\sum_{n=1}^{N} x_n e_n \to x$ as $N \to \infty$, and every convergent sequence in a Banach space is clearly bounded. We also easily see that $\|x\| \leq \|x\|_0$ for each $x \in E$, and that $\|x\|_0 = 0$ if and only if $x = 0$.

Theorem 2.1. *Let E be a Banach space with a basis (e_n), and let $\|\cdot\|_0$ be as defined above. Then $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent norms on E.*

Proof. Let F be the normed space which is E together with the norm $\|\cdot\|_0$ (it is easily checked that F is indeed a normed-space). Let $\iota : F \to E$ be the formal inclusion map. Then ι is norm-decreasing, and is a bijection. Suppose that F is Banach space, that is, F is complete. Then the Open Mapping Theorem implies that ι has a continuous inverse, which implies that $\|\cdot\|_0$ is equivalent to $\|\cdot\|$.

So, we wish to show that F is complete. Firstly, given $x = \sum_{n=1}^{\infty} x_n e_n$, and $m \geq 1$, we notice that

$$\begin{aligned} |x_m| = \|x_m e_m\| \|e_m\|^{-1} &= \|e_m\|^{-1} \Big\| \sum_{n=1}^{m} x_n e_n - \sum_{n=1}^{m-1} x_n e_n \Big\| \\ &\leq \|e_m\|^{-1} \Big(\Big\| \sum_{n=1}^{m} x_n e_n \Big\| + \Big\| \sum_{n=1}^{m-1} x_n e_n \Big\| \Big) \leq 2\|e_m\|^{-1} \|x\|_0, \end{aligned}$$

by the definition of $\|\cdot\|_0$.

Now let (x_n) be a Cauchy-sequence in F, and let

$$x_n = \sum_{m=1}^{\infty} x_{n,m} e_m \qquad (n \geq 1).$$

Then, for $\epsilon > 0$, there exists N_ϵ so that for $r, s \geq N_\epsilon$,

$$\|x_r - x_s\|_0 = \sup_{N \geq 1} \Big\| \sum_{m=1}^{N} (x_{r,m} - x_{s,m}) e_m \Big\| < \epsilon.$$

By the calculation above, we hence see that for each $m \geq 1$, the sequence of scalars $(x_{n,m})_{n=1}^{\infty}$ is a Cauchy-sequence, with limit y_m say. We now show that $y = \sum y_m e_m$

converges in F. As $\|\cdot\|_0 \geq \|\cdot\|$, this implies that the sum converges in E, and hence that y exists, as E is complete. Towards this end, let $n \geq N_\epsilon$, so that for each $N \geq 1$, we have that

$$\Big\| \sum_{m=1}^{N} (y_m - x_{n,m}) e_m \Big\| = \Big\| \sum_{m=1}^{N} \lim_{r\to\infty} (x_{r,m} - x_{n,m}) e_m \Big\| = \lim_{r\to\infty} \Big\| \sum_{m=1}^{N} (x_{r,m} - x_{n,m}) e_m \Big\| \leq \epsilon,$$

by the definition of N_ϵ. Notice that this shows that y is indeed the limit of (x_n), supposing of course that y exists. Then $\sum_{m=1}^{\infty} x_{n,m} e_m$ converges in E, so there exists M_ϵ such that if $M_\epsilon \leq r < s$, then $\big\| \sum_{m=r}^{s} x_{n,m} e_m \big\| < \epsilon$. Thus we have that

$$\begin{aligned}
\Big\| \sum_{m=r}^{s} y_m e_m \Big\|_0 &= \sup_{t \geq r} \Big\| \sum_{m=r}^{\min(t,s)} y_m e_m \Big\| \\
&\leq \sup_{r \leq t \leq s} \Big(\Big\| \sum_{m=r}^{t} (y_m - x_{n,m}) e_m \Big\| + \Big\| \sum_{m=r}^{t} x_{n,m} e_m \Big\| \Big) \\
&< \sup_{r \leq t \leq s} \Big(\Big\| \sum_{m=1}^{t} (y_m - x_{n,m}) e_m \Big\| + \Big\| \sum_{m=1}^{r-1} (y_m - x_{n,m}) e_m \Big\| + \epsilon \Big) \leq 3\epsilon,
\end{aligned}$$

so that $\sum_m y_m e_m$ does converge in F.

□

We can now give a "finite" characterisation of a basis.

Theorem 2.2. *Let E be Banach space, and let (e_n) be a sequence in E. Then (e_n) is a basis for E if and only if:*

1. *each e_n is non-zero;*
2. *the linear space of (e_n) is dense in E;*
3. *there exists a constant K such that for every sequence of scalars (x_n), and each $N < M$, we have that*
$$\Big\| \sum_{n=1}^{N} x_n e_n \Big\| \leq K \Big\| \sum_{n=1}^{M} x_n e_n \Big\|.$$

Proof. Suppose that (e_n) is a basis, and form the norm $\|\cdot\|_0$, as above, so that by the above theorem, there exists K such that

$$\|x\| \leq \|x\|_0 \leq K\|x\| \qquad (x \in E).$$

Clearly we have condition (1) by the uniqueness of the expansion of $0 = \sum_{n=1}^{\infty} 0 e_n$; condition (2) is trivial. Then, for (3), let (y_n) be the sequence of scalars defined by $y_n = x_n$ for $n \leq M$, and $y_n = 0$ otherwise. Then we see that

$$\Big\| \sum_{n=1}^{N} x_n e_n \Big\| = \Big\| \sum_{n=1}^{N} y_n e_n \Big\| \leq \Big\| \sum_{n=1}^{\infty} y_n e_n \Big\|_0 \leq K \Big\| \sum_{n=1}^{\infty} y_n e_n \Big\| = K \Big\| \sum_{n=1}^{M} x_n e_n \Big\|,$$

as required.

Conversely, let F be the linear span of (e_n), a dense subspace of E. For each $n \geq 1$, define a linear map $P_n : F \to F$ by

$$P_n\Big(\sum_{k=1}^{N} x_k e_k \Big) = \sum_{k=1}^{n} x_k e_k \qquad \Big(N \geq n, \sum_{k=1}^{N} x_k e_k \in F \Big).$$

By condition (3), we see that $\|P_n\| \leq K$. It is then clear that P_n is a bounded projection on F, and that P_n thus extends to a bounded projection on E. For each $n \geq 1$, define a linear functional e_n^* on F by the formula

$$e_n^*(x)e_n = P_n(x) - P_{n-1}(x) \qquad (x \in F),$$

where $P_0 = 0$. Then e_n^* is well-defined upon F (by condition (1)) and $\|e_n^*\| \leq 2K$, so that e_n^* extends by continuity to a bounded linear functional on E. It is clear that

$$P_n(x) = \sum_{k=1}^{n} e_k^*(x)e_k \qquad (x \in F),$$

so by continuity, this formula holds for $x \in E$ as well.

Now let $x \in E$. We claim that

$$x = \sum_{n=1}^{\infty} e_n^*(x)e_n.$$

Let (x_n) be a sequence in F which converges to x. Then, for $\epsilon > 0$, let $M \geq 1$ be such that $\|x - x_M\| < \epsilon$, and let $N \geq 1$ be sufficiently large so that $P_N(x_M) = x_M$, which we can do, as $x_M \in F$. Then we see that

$$\begin{aligned}\Big\| \sum_{n=1}^{N} e_n^*(x)e_n - x \Big\| &= \|P_N(x) - x\| \\ &\leq \|P_N(x) - P_N(x_M)\| + \|P_N(x_M) - x_M\| + \|X_M - x\| \\ &\leq K\|x - x_M\| + \|x_M - x\| < (K+1)\epsilon.\end{aligned}$$

Finally, we note that if

$$x = \sum_{n=1}^{\infty} x_n e_n = \sum_{n=1}^{\infty} e_n^*(x)e_n,$$

then applying e_N^*, we conclude that $x_N = e_N^*(x)$ for each N, so that such an expansion is unique. □

The smallest constant which can arise in (3) is the *basis constant* of the basis (e_n). Theorem 2.1 shows that we can always renorm E to give a basis on 1. Such bases are called *monotonic*.

It is immediate that a Banach space with a basis is separable, so that, for example, l^∞ does not have a basis. Most common separable Banach spaces do have bases, although they are often non-obvious to find. For a while, it was thought that all separable Banach spaces would have a basis, but Enflo produced a counter-example in [Enflo, 1973]. There is, however, a weaker notion where we do have a positive result.

3 Basic sequences

Let E be a Banach space, and let (x_n) a sequence in E such that (x_n) is a basis for its closed linear span. Then (x_n) is a *basic sequence* in E. It is easy to show that *every* Banach space has a basic sequence; that is, every Banach space E contains a closed, infinite-dimensional subspace F with a basis.

Lemma 3.1. *Let E be an infinite-dimensional Banach space, let F be a finite-dimensional subspace of E, and let $\epsilon > 0$. Then there exists $x \in E$ such that $\|x\| = 1$ and*

$$\|y\| \leq (1+\epsilon)\|y + ax\|,$$

for all $y \in F$ and all scalars a.

Proof. We may suppose that $\epsilon < 1$. As the unit ball of F is compact, there is a finite set $\{y_1, \ldots, y_n\}$ in F such that

$$\|y_k\| = 1 \quad (1 \leq k \leq n), \qquad \min_{1\leq k\leq n} \|y - y_k\| < \epsilon/2 \quad (y \in F, \|y\| = 1).$$

Pick $y_1^*, \ldots, y_n^*$ norm-one vectors in E', the dual of E, such that $y_k^*(y_k) = 1$ for each k. Then there exists $x \in E$ with $\|x\| = 1$ and $y_k^*(x) = 0$ for each k. For any norm-one $y \in F$, pick y_k such that $\|y_k - y\| < \epsilon/2$. For a scalar a, we have that

$$\begin{aligned}\|y + ax\| &\geq \|y_k + ax\| - \|y - y_k\| > \|y_k + ax\| - \epsilon/2 \\ &\geq |y_k^*(y_k + ax)| - \epsilon/2 = 1 - \epsilon/2 \geq (1+\epsilon)^{-1},\end{aligned}$$

as required. □

Theorem 3.2. *Every Banach space E contains a basic sequence.*

Proof. We use induction to pick a sequence of norm-one vectors (x_n) such that condition (3) of Theorem 2.2 always holds, with $K = 2$ say (the proof works for any $K > 1$). Suppose we have chosen $x_1, \ldots, x_n$ and $\epsilon > 0$ such that

$$\Big\| \sum_{k=1}^m a_k x_k \Big\| \leq (2-\epsilon) \Big\| \sum_{k=1}^n a_k x_k \Big\|,$$

for any $m \leq n$ and any scalars $(a_k)_{k=1}^n$. Note that we can clearly do this for $n = 1$. We now try to find x_{n+1}. We need to ensure that $\|x_{n+1}\| = 1$ and that for some $\epsilon_0 > 0$, we have that

$$\Big\| \sum_{k=1}^m a_k x_k \Big\| \leq (2-\epsilon_0) \Big\| \sum_{k=1}^{n+1} a_k x_k \Big\|,$$

for any $m \leq n$ and any scalars $(a_k)_{k=1}^{n+1}$.

Let F_n be the linear span of $x_1, \ldots, x_n$, a finite-dimensional subspace of E. Use the above lemma to find a norm-one vector x_{n+1} such that $\|y\| \leq (1+\delta)\|y + a_{n+1}x_{n+1}\|$ for each $y \in F_n$ and each scalar a_{n+1}, where $\delta > 0$ is chosen so that $(2-\epsilon)(1+\delta) = 2 - \epsilon/2$, that is, $\delta = \epsilon/2(2-\epsilon)$. Then, for a sequence of scalars $(a_k)_{k=1}^{n+1}$, let $y = \sum_{k=1}^n a_k x_k \in F_n$, so that for $m \leq n$, we see that

$$\begin{aligned}\Big\| \sum_{k=1}^m a_k x_k \Big\| &\leq (2-\epsilon) \Big\| \sum_{k=1}^n a_k x_k \Big\| = (2-\epsilon)\|y\| \leq (2-\epsilon)(1+\delta)\|y + a_{n+1}x_{n+1}\| \\ &= (2-\epsilon/2) \Big\| \sum_{k=1}^{n+1} a_k x_k \Big\|,\end{aligned}$$

as required. □

4 Unconditional bases

A basis for a Banach space is not, in many cases, of a huge amount of use, as the convergence properties are rather weak. A more useful notion is that of an *uncondtional basis*.

A series (x_n) in a Banach space E is said to *sum unconditionally* if, for each permutation σ on $\mathbb{N}$, the sum

$$\sum_{n=1}^{\infty} x_{\sigma(n)}$$

converges, and converges to the same limit (although, see below, this is automatic), independently of σ. Recall that in a finite-dimensional Banach space (or just in $\mathbb{R}$ or $\mathbb{C}$) this notion is equivalent to $\sum_{n=1}^{\infty} \|x_n\| < \infty$. This is not true in the infinite-dimensional case (indeed, it characterises finite-dimensional Banach spaces).

Then a basis (e_n) for a Banach space E is an unconditional basis if, for each $x \in E$, there exists a unique expansion of the form

$$x = \sum_{n=1}^{\infty} x_n e_n,$$

where the sum converges unconditionally.

Firstly, we need to explore some properties of unconditional convergence in Banach spaces.

Proposition 4.1. *Let E be a Banach space, and let (x_n) be a series in E. Then the following are equivalent:*

1. *the sum $\sum_{n=1}^{\infty} x_n$ is unconditional;*
2. *for each permutation σ of $\mathbb{N}$, the sum $\sum_{n=1}^{\infty} x_{\sigma(n)}$ converges in E;*
3. *for each $\epsilon > 0$, there exists a finite subset $A_\epsilon \subseteq \mathbb{N}$ such that if $B \subseteq \mathbb{N}$ is finite with $A_\epsilon \cap B = \emptyset$, then $\| \sum_{n\in B} x_n\| < \epsilon$.*

Proof. Clearly (1) implies (2). To show that (2) implies (3), suppose towards a contradiction that (3) does not hold, so that we can find $\epsilon > 0$ such that for each finite $A \subseteq \mathbb{N}$, there exists a finite $B_A \subseteq \mathbb{N}$ with $A \cap B_A = \emptyset$ and $\| \sum_{n\in B_A} x_n\| \geq \epsilon$. So let $B_1 = B_{\{1\}}$, then let $A_2 = \{1, 2, \dots, \max B_1\}$, and let $B_2 = B_{A_2}$. Continue in this manner to find a sequence of finite sets (B_n) such that $\max B_n < \min B_{n+1}$ for each $n \geq 1$. We can easily construct a permutation σ of $\mathbb{N}$ such that each B_n is the image under σ of some interval, say $B_n = \{\sigma(r_n), \sigma(r_n + 1), \dots, \sigma(s_n)\}$, so that

$$\epsilon \leq \Big\| \sum_{k\in B_n} x_k \Big\| = \Big\| \sum_{k=r_n}^{s_n} x_{\sigma(k)} \Big\| \qquad (n \geq 1).$$

However, this implies that $\sum_k x_{\sigma(k)}$ does not converge, giving the required contradiction.

Now suppose that (3) holds, and that σ is a permutation of $\mathbb{N}$. For $\epsilon > 0$, choose $A_\epsilon \subseteq \mathbb{N}$ and let $N_\epsilon > 0$ be such that if $n \geq N_\epsilon$, then $\sigma(n) \notin A_\epsilon$, $\sigma^{-1}(n) \notin A_\epsilon$, and $n \notin A_\epsilon$. Then, for $N_\epsilon \leq r < s$, let $B = \{\sigma(n) : r \leq n \leq s\}$ and let $C = \{n : r \leq n \leq s\}$, so that B and C are finite, and $B \cap A_\epsilon = C \cap A_\epsilon = \emptyset$. Thus we have that

$$\Big\| \sum_{n=r}^{s} x_{\sigma(n)} \Big\| = \Big\| \sum_{n\in B} x_n \Big\| < \epsilon, \quad \Big\| \sum_{n=r}^{s} x_n \Big\| = \Big\| \sum_{n\in C} x_n \Big\| < \epsilon,$$

which shows that $x = \sum_n x_n$ and $y = \sum_n x_{\sigma(n)}$ exist. Furthermore, let $K = \{1 \leq n \leq N_\epsilon : \sigma(n) > N_\epsilon\}$ and $L = \{\sigma(n) : 1 \leq n \leq N_\epsilon, \sigma(n) > N_\epsilon\}$, so that $L \cap A_\epsilon = \emptyset$. Also, if $n \in K$, then $m = \sigma(n) > N_\epsilon$ so that $\sigma^{-1}(m) = n \notin A_\epsilon$, that is, $K \cap A_\epsilon = \emptyset$. Then we see that

$$\Big\| \sum_{n=1}^{N_\epsilon} x_n - \sum_{n=1}^{N_\epsilon} x_{\sigma(n)} \Big\| = \Big\| \sum_{n\in K} x_n - \sum_{n\in L} x_n \Big\| < 2\epsilon,$$

which easily shows that $\|x - y\| \leq 4\epsilon$. Thus we see that (1) holds. □

Notice that condition (3) shows that if $\sum_n x_n$ converges unconditionally, then given any $A \subseteq \mathbb{N}$, the sum $\sum_{n\in A} x_n$ converges unconditionally. Indeed, we can say more.

Proposition 4.2. *Let E be a Banach space, and let (x_n) be a series in E. If (x_n) sums unconditionally, then for each sequence (z_n) in $\{+1, -1\}$, the sum $\sum_n z_n x_n$ converges uniformly, independently of the choice of (z_n). Furthermore, the following are equivalent:*

1. *the sum $\sum_{n=1}^\infty x_n$ is unconditional;*
2. *for any bounded sequence of scalars (z_n), the sum $\sum_{n=1}^\infty z_n x_n$ converges in E.*

Proof. We use condition (3) from the above proposition to characterise when $\sum_n x_n$ converges unconditionally. So suppose that $x = \sum_n x_n$ converges unconditionally. By the above comment, for each $A \subseteq \mathbb{N}$, we may define $P_A(x) = \sum_{n\in A} x_n$. Then suppose that (z_n) is a sequence in $\{+1, -1\}$, so that if $A = \{n : z_n = 1\}$, then clearly $\sum_n z_n x_n = P_A(x) - P_{\mathbb{N}\setminus A}(x)$ converges.

Suppose that there exists $\epsilon > 0$ such that for each $r \geq 1$, there exists a sequence (a_n) in $\{+1, -1\}$ and $s > r$ such that $\| \sum_{n=r}^s a_n x_n \| \geq \epsilon$. Then we find can sequences (A_n) and (B_n) of finite subsets of $\mathbb{N}$ such that $A_n \cap B_n = A_n \cap A_m = B_n \cap B_m = \emptyset$ for all $n \neq m$, and

$$\Big\| \sum_{k\in A_n} x_k - \sum_{k\in B_n} x_k \Big\| \geq \epsilon.$$

Let σ and τ be permutations of $\mathbb{N}$ such that each A_n occurs as the image under σ of an interval, and each B_n occurs as the image under τ of an interval. As $\sum_n x_{\sigma(n)}$ and $\sum_n x_{\tau(n)}$ converge, we again conclude that for n sufficiently large, $\| \sum_{k\in A_n} x_k \| < \epsilon/3$ and $\| \sum_{k\in B_n} x_k \| < \epsilon/3$, which is a contradiction. This shows that $\sum_n z_n x_n$ converges uniformly, independently of the choice of the (z_n).

Suppose now we are working over the real numbers, and let (z_n) be a bounded sequence in $\mathbb{R}$. For $1 \leq r < s$, pick μ in the dual space to E with $\|\mu\| = 1$ and $\sum_{n=r}^s z_n \mu(x_n) = \| \sum_{n=1}^N z_n x_n \|$. For each n, let $a_n = 1$ if $\mu(x_n) \geq 0$ or $a_n = -1$ otherwise, so that

$$\Big\| \sum_{n=r}^s z_n x_n \Big\| \leq \sum_{n=r}^s |z_n||\mu(x_n)| = \sum_{n=r}^s |z_n| a_n \mu(x_n) \leq \|(z_n)\|_\infty \Big\| \sum_{n=r}^s a_n x_n \Big\|.$$

By the above, we know that we can make $\| \sum_{n=r}^s a_n x_n \|$ arbitrarily small, independently of the choice of (a_n). Thus we conclude that $\sum_n z_n x_n$ converges.

We have already essentially shown that (2) implies (1), just by taking sequences (z_n) in $\{+1, -1\}$. □

We are now in a position to prove a version of Theorem 2.2 for unconditional bases: this is left as an exercise to the reader. Instead, we prove the hard parts of such a theorem.

Theorem 4.3. *Let E be a Banach space, and let (e_n) be an unconditional basis for E. For $x = \sum_n x_n e_n$, define*

$$\|x\|_0 = \sup_{\|(z_n)\|_\infty \leq 1} \Big\| \sum_{n=1}^\infty z_n x_n e_n \Big\|.$$

Then $\|\cdot\|_0$ is an equivalent norm on E.

Proof. As (e_n) is a basis for E, we may assume that (e_n) is actually a monotone, normalised basis. Firstly, note that the above proposition shows that $\|\cdot\|_0$ is indeed well-defined. This follows as for $\epsilon > 0$, there exists N_ϵ, independent of (z_n), such that if $N_\epsilon \leq r < s$, then $\|\sum_{n=r}^s z_n x_n e_n\| \leq \epsilon \|(z_n)\|_\infty$. Thus we have that $\|\sum_n z_n x_n e_n\| \leq \|(z_n)\|_\infty \big(\epsilon + \sum_{n=1}^{N_\epsilon} |x_n| \|e_n\|\big)$.

We proceed as in the proof of the preceeding proposition. For $x = \sum_n x_n e_n$, define

$$\|x\|_1 = \sup_{(z_n) \in \{+1,-1\}^{\mathbb{N}}} \Big\| \sum_{n=1}^\infty z_n x_n e_n \Big\|,$$

again, this is well-defined. If we can show that $\|\cdot\|_1$ is an equivalent norm on E, then the argument used above shows that $\|\cdot\|_0$ is also an equivalent norm on E.

Let Λ be the collection of sequences in $\{+1, -1\}$, which is a group under pointwise multiplication. For each $z = (z_n) \in \Lambda$, let $T_z : E \to E$ be the linear map

$$T_z\Big(\sum_{n=1}^\infty x_n e_n\Big) = \sum_{n=1}^\infty z_n x_n e_n.$$

Suppose that (x_n) is a sequence in E tending to 0, that $x_n = \sum_m x_{n,m} e_m$ for each n, and that $T_z(x_n) \to \sum_m y_m e_m$ as $n \to \infty$. That is,

$$\lim_{n\to\infty} \Big\| \sum_{m=1}^\infty (z_m x_{n,m} - y_m) e_m \Big\| = 0.$$

As (e_n) is a monotone basis, we may apply the co-ordinate functional e_m^* to conclude that

$$\lim_{n\to\infty} |z_m x_{n,m} - y_m| = 0 \qquad (m \geq 1).$$

However, as $x_{n,m} = e_m^*(x_n) \to 0$ as $n \to \infty$, we see that $y_m = 0$ for each m, so that $y = 0$. The Closed Graph Theorem then implies that T_z is bounded.

The above proof shows that, for a fixed $x \in E$, the family $\{T_z(x) : z \in \Lambda\}$ is bounded. It follows from the Principle of Uniform Boundedness that the family of operators $\{T_z : z \in \Lambda\}$ is bounded. However, this is simply the statement that $\|\cdot\|_1$ is equivalent to $\|\cdot\|$, as required. □

Example 4.4. In c, the closed subspace of l^∞ consisting of convergent sequences, let $f_n = (0, \dots, 0, 1, 1, \dots) \in c$, where the 1 appears in the nth position. Then, for $x = (x_n) \in c$, let $y_1 = x_1$, and $y_n = x_n - x_{n-1}$ for $n \geq 2$, so that

$$\begin{aligned}\sum_{n=1}^N y_n f_n &= \big(y_1, y_1 + y_2, \dots, y_1 + y_2 + \dots + y_n, \dots\big) \\ &= \big(x_1, x_2, \dots, x_n, x_n, \dots\big),\end{aligned}$$

which tends to x as $N \to \infty$. That is, (f_n) is a basis for c, called the *summing basis*.

However, we notice that

$$\Big\|\sum_{n=1}^{N} f_n\Big\| = N, \qquad \Big\|\sum_{n=1}^{N}(-1)^n f_n\Big\| = 1,$$

so that the above theorem implies that (f_n) is a *conditional* basis.

Let E be a Banach space with an unconditional basis (e_n). Then we may renorm E so that there is a norm-decreasing algebra homomorphism $l^\infty \to \mathcal{B}(E)$, from the Banach algbera l^∞ under pointwise multiplication to the algebra of operators on E.

In particular, for $A \subseteq \mathbb{N}$, define an operator $P_A : E \to E$ by $P_A(\sum_n x_n e_n) = \sum_{n\in A} x_n e_n$. Then $\{P_A : A \subseteq \mathbb{N}\}$ is a bounded family of projections on E; by renorming, we may suppose that each P_A is actually norm-decreasing.

5 The Gowers Dichotomy Theorem

Let E be a Banach space, and let P be a bounded projection on E. Then let F be the kernal of P, a closed subspace of E, and let G be the image of P, which is equal to the kernal of $I - P$, and is hence a closed subspace. It is trivial to see that $E = F \oplus G$.

Conversely, if $E = F \oplus G$ for some closed subspaces F and G, then for each $x \in E$, we can write $x = f + g$ for unique $f \in F$ and $g \in G$. It is easy to see that the map $x \to g$ defines a bounded projection on E with kernel F and image G.

If F is a finite-dimensional subspace of E, then by taking a finite basis of F, we can easily define a bounded projection $E \to F$. Hence E always admits some projections: however, need there exist a projection with an infinite-dimensional image?

An infinite-dimensional Banach space E is *indecomposable* if whenever we can write E as the direct sum of two closed subspaces, $E = F \oplus G$, then one of F or G is finite-dimensional. That is, the only bounded projections on E are the trivial ones. A Banach space E is *hereditarily indecomposable* if, furthermore, every infinite-dimensional closed subspace F of E is indecomposable.

That such Banach spaces exist is somewhat surprising, but in [GM, 1993], Gowers and Maurey constructed an example of such a space. Notice that by the comments at the end of the last section, a hereditarily indecomposable Banach space E cannot contain an unconditional basic sequence, or we would have an infinite dimensional closed subspace F which would admit many non-trivial bounded projections. Indeed, this was the problem which Gowers and Maurey were trying to find a counter-example to.

Hence we see that being hereditarily indecomposable and having an unconditional basic sequence are somehow the opposite of each other. Remarkably, this statement can be made precise, in the Gowers Dichotomy Theorem.

Theorem 5.1. *Let E be a Banach space. Then there exists a subspace F of E such that either F is hereditarily indecomposable, or F has an unconditional basis.*

Proof. See [Gowers, 1996] for a discussion, and [Gowers, 2002] for the full proof. □

References

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