1 Introduction

A compact quantum group is a unital C*-algebra A together with a coassociative map $\Delta : A \to A \otimes A$ such that $\Delta(A)(A \otimes 1)$ and $\Delta(A)(1 \otimes A)$ are linearly dense in $A \otimes A$. We get the Haar measure φ which is the unique state on A with $(\varphi \otimes \iota)\Delta(a) = (\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in A$.

As argued in my PAMS paper, we can find a maximal family of irreducible unitary corepresentations $\{v^\alpha = (v^\alpha_{ij})_{i,j=1}^{n_\alpha} : \alpha \in \mathbb{A}\}$ such that the associated "F-matrices" are all diagonal.

Firstly, if $\mathcal{A}$ is the linear span of $\{v^\alpha_{ij}\}$, then $\mathcal{A}$ is a Hopf-$*$-algebra and is dense in A. We have that

$$\Delta(v^\alpha_{ij}) = \sum_k v^\alpha_{ik} \otimes v^\alpha_{kj}, \quad S(v^\alpha_{ij}) = (v^\alpha_{ji})^*, \quad \epsilon(v^\alpha_{ij}) = \delta_{ij}, \quad \varphi(v^\alpha_{ij}) = \delta_{\alpha,\alpha_0},$$

where α_0 is the unique member of $\mathbb{A}$ with $v^\alpha_0 = 1$.

Then we have positive numbers $(\lambda^\alpha_i)_{i=1}^{n_\alpha}$ such that $\sum_i \lambda^\alpha_i = \sum_i (\lambda^\alpha_i)^{-1} = \Lambda_\alpha$ say. We have that

$$\varphi\big((v^\alpha_{ij})^* v^\beta_{kl}\big) = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l}\frac{1}{\Lambda_\alpha \lambda^\alpha_i}, \quad \varphi\big(v^\alpha_{ij}(v^\beta_{kl})^*\big) = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l}\frac{\lambda^\alpha_j}{\Lambda_\alpha}.$$

We define characters f_z, for $z \in \mathbb{C}$, on $\mathcal{A}$ by

$$f_z\big(v^\alpha_{ij}\big) = \delta_{i,j}(\lambda^\alpha_i)^z,$$

where of course $t^z = \exp(z \log t)$ for $t > 0$. Then the modular automorphism group for φ, restricted to $\mathcal{A}$, is given by

$$\sigma_z : v^\alpha_{ij} \mapsto \sum_{k,l} f_{iz}(v^\alpha_{ik}) v^\alpha_{kl} f_{iz}(v^\alpha_{lj}) = (\lambda^\alpha_i)^{iz}(\lambda^\alpha_j)^{iz} v^\alpha_{ij}.$$

For example, we can show that $\varphi(ba) = \varphi(a\sigma_{-i}(b))$ for all $a, b \in \mathcal{A}$. Also, as $J\Lambda(a) = \Lambda(\sigma_{i/2}(a)^*)$ for $a \in \mathcal{A}$, we see that

$$J\Lambda(v^\alpha_{ij}) = (\lambda^\alpha_i \lambda^\alpha_j)^{-1/2}\Lambda((v^\alpha_{ij})^*).$$

Similarly, the scaling group on $\mathcal{A}$ is given by

$$\tau_z : v^\alpha_{ij} \mapsto (\lambda^\alpha_i)^{iz}(\lambda^\alpha_j)^{-iz} v^\alpha_{ij}.$$

Thus in particular,

$$S(v^\alpha_{ij}) = (v^\alpha_{ji})^* = R\tau_{-i/2}(v^\alpha_{ij}) = (\lambda^\alpha_i)^{1/2}(\lambda^\alpha_j)^{-1/2} R(v^\alpha_{ij}) \implies R(v^\alpha_{ij}) = \sqrt{\frac{\lambda^\alpha_j}{\lambda^\alpha_i}}(v^\alpha_{ji})^*.$$

However, also $R(x) = \hat{J}x^*\hat{J}$, and so

$$\hat{J}v^\alpha_{ij}\hat{J} = \sqrt{\frac{\lambda^\alpha_j}{\lambda^\alpha_i}} v^\alpha_{ji}.$$

2 Reduced case and duality

Now suppose that φ is faithful. Let (H, Λ) be the GNS construction for φ.

For each $\alpha \in \mathbb{A}$, let H_α be the finite-dimensional subspace of H spanned by $\{\Lambda((v^\alpha_{ij})^*) : 1 \leq i, j \leq n_\alpha\}$. Notice that H_α is orthogonal to H_β for $\alpha \neq \beta$. As $\mathcal{A}$ is dense in H, it follows that

H is isomorphic to the Hilbert space direct sum of $\{H_\alpha : \alpha \in \mathbb{A}\}$. There is a bijective linear map $U_\alpha : H_\alpha \to \ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$ given by

$$U_\alpha : \Lambda((v^\alpha_{ij})^*) \mapsto \sqrt{\frac{\lambda^\alpha_j}{\Lambda_\alpha}} \delta_i \otimes \delta_j.$$

We have that U_α is unitary, because

$$\big(U_\alpha((v^\alpha_{ij})^*)\big|U_\alpha((v^\alpha_{kl})^*)\big) = \frac{\lambda^\alpha_j}{\Lambda_\alpha}\big(\delta_i \otimes \delta_j \big| \delta_k \otimes \delta_l\big) = \varphi\big(v^\alpha_{kl}(v^\alpha_{ij})^*\big) = \big(\Lambda((v^\alpha_{ij})^*)\big|\Lambda((v^\alpha_{kl})^*)\big).$$

From the general LCQG theory, we form the unitary operator W^* on $H \otimes H$ by

$$W^*\big(\Lambda(a) \otimes \Lambda(b)\big) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)) \qquad (a, b \in A).$$

Notice that it is very easy to show that W^* is unitary in the compact case. It follows that

$$W^*\big(\xi \otimes \Lambda((v^\alpha_{ij})^*)\big) = \sum_k (v^\alpha_{ik})^*(\xi) \otimes \Lambda((v^\alpha_{kj})^*).$$

Now we calculate

$$\begin{aligned}
\big(W(\xi \otimes \Lambda((v^\alpha_{ij})^*))\big|\eta \otimes \Lambda((v^\beta_{kl})^*)\big) &= \sum_p \big(\xi \otimes \Lambda((v^\alpha_{ij})^*)\big|(v^\beta_{kp})^*(\eta) \otimes \Lambda((v^\beta_{pl})^*)\big) \\
&= (v^\alpha_{ki}(\xi)|\eta)\delta_{\alpha,\beta}\delta_{j,l}\frac{\lambda^\alpha_j}{\Lambda_\alpha} \\
&= \sum_p \big(v^\alpha_{pi}(\xi) \otimes \Lambda((v^\alpha_{pj})^*)\big|\eta \otimes \Lambda((v^\beta_{kl})^*)\big).
\end{aligned}$$

It follows that for each $\alpha \in \mathbb{A}$, the unitary W restricts to $H \otimes H_\alpha$ and is the map

$$W(\xi \otimes \Lambda((v^\alpha_{ij})^*)) = \sum_p v^\alpha_{pi}(\xi) \otimes \Lambda((v^\alpha_{pj})^*).$$

In particular, $(1 \otimes U_\alpha)W(1 \otimes U^*_\alpha)$ makes sense on $H \otimes \ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$ and is

$$w_\alpha = (1 \otimes U_\alpha)W(1 \otimes U^*_\alpha) : \xi \otimes \delta_i \otimes \delta_j \mapsto \sum_p v^\alpha_{pi}(\xi) \otimes \delta_p \otimes \delta_j.$$

Thus actually

$$w_\alpha = \sum_{ij} v^\alpha_{ij} \otimes e_{ij} \otimes 1,$$

where e_{ij} is the usual matrix unit in $\mathbb{M}_{n_\alpha} \cong \mathcal{B}(\ell^2_{n_\alpha})$.

2.0.1 Positive cone

The positive cone of $L^2(\mathbb{G})^+$ is by definition the closure of $\{xJxJ\Lambda(1) : x \in L^\infty(\mathbb{G})\}$. If $x \in L^\infty(\mathbb{G})$ then there is a norm-bounded net (a_α) in $\mathcal{A}$ converging to x strongly. In particular $\Lambda(x) = x\Lambda(1) = \lim_\alpha a_\alpha \Lambda(1) = \lim_\alpha \Lambda(a_\alpha)$ where the limits are in the norm of $L^2(\mathbb{G})$. Then

$$xJxJ\Lambda(1) = xJ\Lambda(x) = \lim_\alpha a_\alpha J\Lambda(x) = \lim_\alpha a_\alpha J\Lambda(a_\alpha),$$

as $\|a_\alpha J\Lambda(x) - a_\alpha J\Lambda(a_\alpha)\| \leq \|a_\alpha\|\|J\Lambda(x) - J\Lambda(a_\alpha)\| \to 0$. Thus the positive cone is the closure of the set $\{aJ\Lambda(a) : a \in \mathcal{A}\}$. Recall that $aJ\Lambda(a) = a\Lambda(\sigma_{i/2}(a)^*) = \Lambda(a\sigma_{i/2}(a)^*)$. In particular,

$$P^{it}\Lambda(a\sigma_{i/2}(a)^*) = \Lambda\big(\tau_t(a\sigma_{i/2}(a)^*\big) = \Lambda\big(\tau_t(a)\sigma_{i/2}(\tau_t(a))^*\big),$$

which is in the positive cone (as $\tau_t\sigma_s = \sigma_s\tau_t$ for all $s, t \in \mathbb{R}$).

2.1 Further facts about the irreducible corepresentations

We refer to later results; from our choices (compare Proposition prop:fmatrices A.21) we have that F^α is diagonal, with entries $(\Lambda_\alpha \lambda_i^\alpha)^{-1}$. By (the comment after) Corollary cor:fmatrixkappa A.27, it follows that

$$\sum_k u_{k,i}^\alpha \frac{\lambda_k^\alpha}{\lambda_i^\alpha} (u_{k,j}^\alpha)^* = \delta_{i,j}, \qquad \sum_k (u_{i,k}^\alpha)^* \frac{\lambda_i^\alpha}{\lambda_k^\alpha} u_{j,k}^\alpha = \delta_{i,j}.$$

We could also prove these by writing down what it means for u^α to be unitary, and then applying the map R, given the form for this which we have established above (though is A is not reduced, we then have to argue a little about the uniqueness of the Hopf algebra.)

Below, we'll see that for each α, the contragradient representation $\overline{u^\alpha}$ is also irreducible (Lemma lem:contra_irrep A.18) and is equivalent to a unitary corepresentation (Proposition prop:conjunitary A.19). So there is an invertible scalar matrix T (which is unique, up to a scalar, by Schur's Lemma, Proposition prop:schur A.15) and some β, with $(1 \otimes T^{-1})\overline{u^\alpha}(1 \otimes T) = u^\beta$. In Lemma lem:fmatrixkappa A.26 it's shown that TT^* is a scalar multiple of $\overline{F^\alpha}$; by considering the traces of these positive definite matrices, this scalar multiple is a postive number. It follows that, by rescaling T, we may suppose that $T = (\overline{F^\alpha})^{1/2} U$ for some scalar unitary matrix U.

Thus we find that $(1 \otimes U^*(F^\alpha)^{-1/2})\overline{u^\alpha}(1 \otimes (F^\alpha)^{1/2} U) = u^\beta$, and so

$$\overline{u^\beta} = (1 \otimes U^T (F^\alpha)^{-1/2}) u^\alpha (1 \otimes (F^\alpha)^{1/2} \overline{U}) \implies (1 \otimes (F^\alpha)^{1/2}\overline{U})\overline{u^\beta}(1 \otimes U^T (F^\alpha)^{-1/2}) = u^\alpha.$$

However, by the same reasoning, there is a scalar unitary V with $(1 \otimes V^*(F^\beta)^{-1/2})\overline{u^\beta}(1 \otimes (F^\beta)^{1/2} V) = u^\alpha$. By Schur, $V^*(F^\beta)^{-1/2} = \overline{\mu}(F^\alpha)^{1/2}\overline{U}$ for some $\mu \in \mathbb{C}$. Thus $\mu(F^\alpha)^{1/2} U (F^\beta)^{1/2}$ is unitary, that is,

$$|\mu|^2 (F^\beta)^{1/2} U^* F^\alpha U (F^\beta)^{1/2} = I \quad \Leftrightarrow \quad |\mu|^2 F^\alpha U = U (F^\beta)^{-1}.$$

As $U(F^\beta)^{-1}U^* = |\mu|^2 F^\alpha$, taking the trace of both sides shows that $\Lambda_\beta^2 = |\mu|^2$. Thus $F^\alpha U = U\Lambda_\beta^{-2}(F^\beta)^{-1}$. Notice that both the matrices F^α and $\Lambda_\beta^{-2}(F^\beta)^{-1}$ are diagonal, with strictly positive diagonal entries, and with unit trace.

Lemma 2.1. *Let U be a unitary matrix, and let A, B be diagonal matrices with non-zero diagonal entries (a_i) and (b_i). For each diagonal entry a of A, let E_a^A be the eigenspace of a, which is $\operatorname{lin}\{e_i : a_i = a\}$. Similarly define E_b^B. Suppose that $AU = UB$. Then, counting multiplicies, the sequences $\{a_i\}$ and $\{b_i\}$ are the same, and U restricts to a unitary between $E_{a_i}^A$ and $E_{a_i}^B$.*

Proof. For each i, notice that $A(Ue_i) = UBe_i = b_i Ue_i$, so b_i is an eigenvalue of A, and hence there exists j with $a_j = b_i$. Similarly, for each j there is i with $b_i = a_j$, so the sets $\{a_i\}$ and $\{b_j\}$ agree.

Now observe that U maps $E_{b_i}^B$ into $E_{b_i}^A$, so as U is invertible, the dimensions of these eigenspaces agree. Thus, counting multiplicities, the sequences $\{a_i\}$ and $\{b_j\}$ agree, and the proof is complete. □

So in our case $\{(\Lambda_\alpha \lambda_i^\alpha)^{-1}\}$ and $\{\lambda_j^\beta / \Lambda_\beta\}$ agree counting multiplicity, and U has the stated simple form. Then $\Lambda_\alpha^2 = \sum_i \Lambda_\alpha \lambda_i^\alpha = \sum_i \Lambda_\beta / \lambda_i^\beta = \Lambda_\beta^2$, so $\Lambda_\alpha = \Lambda_\beta$. Hence $\{\lambda_i^\alpha\}$ and $\{1/\lambda_j^\beta\}$ biject according to multiplicity.

2.2 Duality

2.2.1 The involution on $L^1(A)$

From general LCQG theory we have the homomorphism $\lambda : L^1(A) \to \hat{A}$ given by $\omega \mapsto (\omega \otimes \iota)(W)$. Recall the involution $\sharp$ defined on $L^1_\sharp(A)$ which satisfies

$$\langle a, \omega^\sharp \rangle = \overline{\langle S(a)^*, \omega \rangle} \qquad (a \in \mathcal{A}, \omega \in L^1(A)_\sharp).$$

Then λ is a $*$-homomorphism when restricted to $L^1_\sharp(A)$.

For $a, b \in \mathcal{A}$ we define $\omega(a,b) = \omega_{\Lambda(a),\Lambda(b)} \in L^1(A)$. Then for $c \in \mathcal{A}$,

$$\begin{aligned}\overline{\langle S(c)^*, \omega(a,b)\rangle} &= \overline{\varphi(b^*S(c)^*a)} = \overline{\varphi(S(b^*S(c)^*a))} = \overline{\varphi(S(a)c^*S(b^*))} = \varphi(S(b^*)^*cS(a)^*) \\ &= \langle c, \omega(S(a)^*, S(b^*))\rangle.\end{aligned}$$

That φ is S-invariant follows immediately from the action of φ and S on the elements v^α_{ij}. Thus $\omega(a,b) \in L^1_\sharp(A)$ with $\omega(a,b)^\sharp = \omega(S(a)^*, S(b^*))$.

2.2.2 Identifying the dual

Define the linear functional on $\mathcal{A}$ by

$$\omega^\alpha_{ij} : v^\beta_{kl} \mapsto \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l}.$$

Notice that

$$\big(v^\beta_{kl}\Lambda((v^\alpha_{ij})^*)\big|\Lambda(1)\big)\frac{\Lambda_\alpha}{\lambda^\alpha_j} = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l} = \langle v^\beta_{kl}, \omega^\alpha_{ij}\rangle,$$

from which it follows that

$$\omega^\alpha_{ij} = \frac{\Lambda_\alpha}{\lambda^\alpha_j}\omega_{\Lambda((v^\alpha_{ij})^*),\Lambda(1)}.$$

From the discussion above, $\omega^\alpha_{ij} \in L^1_\sharp(A)$.

We now compute

$$\begin{aligned}\lambda(\omega^\alpha_{ij})\Lambda((v^\beta_{kl})^*) &= \frac{\Lambda_\alpha}{\lambda^\alpha_j}(\omega_{\Lambda((v^\alpha_{ij})^*),\Lambda(1)} \otimes \iota)(W)\Lambda((v^\beta_{kl})^*) \\ &= \frac{\Lambda_\alpha}{\lambda^\alpha_j}\sum_p \big(v^\beta_{pk}\Lambda((v^\alpha_{ij})^*)\big|\Lambda(1)\big)\Lambda((v^\beta_{pl})^*) = \delta_{\alpha,\beta}\delta_{j,k}\Lambda((v^\beta_{il})^*).\end{aligned}$$

Thus each H_β is an invariant subspace for $\lambda(\omega^\alpha_{ij})$, and $\lambda(\omega^\alpha_{ij}) = 0$ on H_β for $\alpha \neq \beta$. Furthermore,

$$U_\alpha\lambda(\omega^\alpha_{ij})U^*_\alpha(\delta_k \otimes \delta_l) = \delta_{j,k}\delta_i \otimes \delta_l.$$

Hence $U_\alpha\lambda(\omega^\alpha_{ij})U^*_\alpha = e_{ij}$ the (i,j)th matrix entry of $\mathbb{M}_{n_\alpha}$, which acts on the 1st component of $\ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$ in the canonical way.

Lemma 2.2. *The linear span of* $\{\omega^\alpha_{ij} : \alpha \in \mathbb{A}, 1 \leq i,j \leq n_\alpha\}$ *is dense in* $L^1(A)$.

Proof. As $\mathcal{A}$ is dense in A, it follows that $\{\omega_{\Lambda(a),\Lambda(b)} : a, b \in \mathcal{A}\}$ is linearly dense in $L^1(A)$. For $a, b, c \in \mathcal{A}$,

$$\langle c, \omega_{\Lambda(a),\Lambda(b)}\rangle = \varphi(b^*ca) = \varphi(\sigma_i(a)b^*c) = \langle c, \omega_{\Lambda(\sigma_i(a)b^*),\Lambda(1)}\rangle.$$

By continuity, this also holds when $c \in A$, and so we see that $\{\omega_{\Lambda(a),\Lambda(1)} : a \in \mathcal{A}\}$ is linearly dense in $L^1(A)$, from which the result follows. $\square$

We hence conclude that

$$\hat{A} = \bigoplus_\alpha \mathbb{M}_{n_\alpha}.$$

Here, for each $\alpha \in \mathbb{A}$, the copy of $\mathbb{M}_{n_\alpha}$ acts on the first factor of $\ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha} \cong H_\alpha$ and acts as 0 on H_β for $\beta \neq \alpha$, all this happening on $H \cong \bigoplus_\alpha H_\alpha$.

We know that $W \in M\overline{\otimes}\hat{M}$ and thus we can identify W as a member of $M\overline{\otimes}\prod_\alpha \mathbb{M}_{n_\alpha} = \prod_\alpha M\overline{\otimes}\mathbb{M}_{n_\alpha}$. The calculation in the previous section immediately shows that $W = (v^\alpha_{ij}) \in \mathbb{M}_{n_\alpha}(M) \cong M\overline{\otimes}M_{n_\alpha}$.

Henceforth, write $e^\alpha_{ij} \in \mathbb{M}_{n_\alpha}$ for the standard matrix units, acting on the α part of $H \cong \bigoplus H_\alpha$.

2.2.3 Scaling group

We know that $\lambda(\omega\circ\tau_{-t}) = \hat\tau_t\lambda(\omega)$. Firstly, we calculate that

$$\langle v^\beta_{kl}, \omega^\alpha_{ij}\circ\tau_{-t}\rangle = (\lambda^\beta_k)^{-it}(\lambda^\beta_l)^{it}\langle v^\beta_{kl}, \omega^\alpha_{ij}\rangle = (\lambda^\alpha_i)^{-it}(\lambda^\alpha_j)^{it}\langle v^\beta_{kl}, \omega^\alpha_{ij}\rangle.$$

Thus

$$\hat\tau_t\big(e^\alpha_{ij}\big) = \lambda\big(\omega^\alpha_{ij}\circ\tau_{-t}\big) = (\lambda^\alpha_i)^{-it}(\lambda^\alpha_j)^{it}\lambda\big(\omega^\alpha_{ij}\big) = (\lambda^\alpha_i)^{-it}(\lambda^\alpha_j)^{it}e^\alpha_{ij}.$$

2.2.4 The weight on $\hat M$

From LCQG theory, we have a GNS construction for $\hat M$ given by

$$\big(\hat\Lambda(\lambda(\omega))\big|\Lambda(a)\big) = \langle a^*,\omega\rangle \qquad (a\in A),$$

for a suitable, dense collection of $\omega\in L^1(A)$. Thus

$$\big(\hat\Lambda(e^\alpha_{ij})\big|\Lambda((v^\beta_{kl})^*)\big) = \langle v^\beta_{kl},\omega^\alpha_{ij}\rangle = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l} = \frac{\Lambda_\alpha}{\lambda^\alpha_j}\varphi\big(v^\beta_{kl}(v^\alpha_{ij})^*\big) = \frac{\Lambda_\alpha}{\lambda^\alpha_j}\big(\Lambda((v^\alpha_{ij})^*)\big|\Lambda((v^\beta_{kl})^*)\big).$$

Thus

$$\hat\Lambda(e^\alpha_{ij}) = \frac{\Lambda_\alpha}{\lambda^\alpha_j}\Lambda((v^\alpha_{ij})^*)\in H_\alpha \implies U_\alpha\hat\Lambda(e^\alpha_{ij}) = \sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}\delta_i\otimes\delta_j.$$

We now see that

$$\hat\varphi\big((e^\beta_{kl})^*e^\alpha_{ij}\big) = \big(\hat\Lambda(e^\alpha_{ij})\big|\hat\Lambda(e^\beta_{kl})\big) = \frac{\Lambda_\alpha^2}{\lambda^\alpha_j\lambda^\alpha_l}\big(\Lambda((v^\alpha_{ij})^*)\big|\Lambda((v^\beta_{kl})^*)\big) = \delta_{\alpha,\beta}\delta_{i,k}\delta_{j,l}\frac{\Lambda_\alpha}{\lambda^\alpha_l}.$$

In particular,

$$\hat\varphi\big(e^\alpha_{ij}\big) = \delta_{i,j}\frac{\Lambda_\alpha}{\lambda^\alpha_i}.$$

Let $\hat T$ be the Tomita map, $\hat T\hat\Lambda(a) = \hat\Lambda(a^*)$ for $a\in\hat M$; notice that this will respect the decomposition $\hat M = \prod_\alpha \mathbb{M}_{n_\alpha}$. Then, on $\mathbb{M}_{n_\alpha}$,

$$\begin{aligned}\big(\hat\nabla\hat\Lambda(e^\alpha_{ij})\big|\hat\Lambda(e^\alpha_{kl})\big) &= \big(\hat T\hat\Lambda(e^\alpha_{kl})\big|\hat T\hat\Lambda(e^\alpha_{ij})\big) = \big(\hat\Lambda(e^\alpha_{lk})\big|\hat\Lambda(e^\alpha_{ji})\big) = \hat\varphi\big(e^\alpha_{ij}e^\alpha_{lk}\big) = \delta_{j,l}\hat\varphi\big(e^\alpha_{ik}\big)\\ &= \delta_{j,l}\delta_{i,k}\Lambda_\alpha\lambda^\alpha_i = \frac{\lambda^\alpha_i}{\lambda^\alpha_j}\hat\varphi(e^\alpha_{lk}e^\alpha_{ij}) = \frac{\lambda^\alpha_i}{\lambda^\alpha_j}\big(\hat\Lambda(e^\alpha_{ij})\big|\hat\Lambda(e^\alpha_{kl})\big),\end{aligned}$$

and so

$$\hat\nabla\hat\Lambda(e^\alpha_{ij}) = \frac{\lambda^\alpha_i}{\lambda^\alpha_j}\hat\Lambda(e^\alpha_{ij}) \implies U_\alpha\hat\nabla U_\alpha^*(\delta_i\otimes\delta_j) = \frac{\lambda^\alpha_i}{\lambda^\alpha_j}\delta_i\otimes\delta_j.$$

By uniqueness of positive square-roots, it follows that

$$\hat J\hat\Lambda(e^\alpha_{ji}) = \hat J\hat T\hat\Lambda(e^\alpha_{ij}) = \hat\nabla^{1/2}\hat\Lambda(e^\alpha_{ij}) = \sqrt{\frac{\lambda^\alpha_i}{\lambda^\alpha_j}}\hat\Lambda(e^\alpha_{ij}).$$

This also shows that

$$\hat J\Lambda((v^\alpha_{ij})^*) = \sqrt{\frac{\lambda^\alpha_j}{\lambda^\alpha_i}}\Lambda((v^\alpha_{ji})^*) = \lambda^\alpha_j J\Lambda(v^\alpha_{ji}) \implies J\hat J\Lambda((v^\alpha_{ij})^*) = \lambda^\alpha_j\Lambda(v^\alpha_{ji}).$$

Finally, we also see that

$$U_\alpha\hat J U_\alpha^*(\delta_i\otimes\delta_j) = \sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}U_\alpha\hat J\Lambda((v^\alpha_{ij})^*) = \sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}\sqrt{\frac{\lambda^\alpha_j}{\lambda^\alpha_i}}U_\alpha\Lambda((v^\alpha_{ji})^*) = \sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_i}}U_\alpha\Lambda((v^\alpha_{ji})^*) = \delta_j\otimes\delta_i.$$

2.2.5 The antipode

We calculate that

$$\hat{R}(e^\alpha_{ij})\hat{\Lambda}(e^\beta_{kl}) = \frac{\Lambda_\beta}{\lambda^\beta_l} J e^\alpha_{ji} J \Lambda((u^\beta_{kl})^*) = \frac{\Lambda_\beta \sqrt{\lambda^\beta_k \lambda^\beta_l}}{\lambda^\beta_l} J e^\alpha_{ji} \Lambda(u^\beta_{kl}).$$

From above, there is some γ and a scalar unitary matrix U with $(1 \otimes U^*(F^\beta)^{-1/2})\overline{u^\beta}(1 \otimes (F^\beta)^{1/2}U) = u^\gamma$ and $\Lambda^2_\gamma F^\beta U = U(F^\gamma)^{-1}$. So $(1 \otimes (F^\beta)^{1/2}U)u^\gamma(1 \otimes U^*(F^\beta)^{-1/2}) = \overline{u^\beta}$ and thus $(1 \otimes (F^\beta)^{1/2}\overline{U})\overline{u^\gamma}(1 \otimes U^T(F^\beta)^{-1/2}) = u^\beta$. It follows that

$$\begin{aligned}
\hat{R}(e^\alpha_{ij})\hat{\Lambda}(e^\beta_{kl}) &= \frac{\Lambda_\beta \sqrt{\lambda^\beta_k \lambda^\beta_l}}{\lambda^\beta_l} \sum_{p,q} J e^\alpha_{ji} ((F^\beta)^{1/2}\overline{U})_{k,p} (U^T(F^\beta)^{-1/2})_{q,l} \Lambda((u^\gamma_{pq})^*) \\
&= \frac{\Lambda_\beta \sqrt{\lambda^\beta_k \lambda^\beta_l}}{\lambda^\beta_l} \sum_{p,q} \frac{\sqrt{\lambda^\beta_l}}{\sqrt{\lambda^\beta_k}} U_{k,p}\overline{U_{l,q}} J e^\alpha_{ji} \Lambda((u^\gamma_{pq})^*) \\
&= \Lambda_\beta \sum_{p,q} U_{k,p}\overline{U_{l,q}} \frac{\lambda^\gamma_q}{\Lambda_\gamma} J e^\alpha_{ji} \Lambda(e^\gamma_{pq}) = \delta_{\alpha,\gamma}\Lambda_\beta \sum_q U_{k,i}\overline{U_{l,q}} \frac{\lambda^\gamma_q}{\Lambda_\gamma} J\hat{\Lambda}(e^\gamma_{jq}) \\
&= \delta_{\alpha,\gamma}\Lambda_\beta \sum_q U_{k,i}\overline{U_{l,q}} J\Lambda((u^\gamma_{jq})^*) = \delta_{\alpha,\gamma}\Lambda_\beta \sum_q U_{k,i}\overline{U_{l,q}} \sqrt{\lambda^\gamma_j\lambda^\gamma_q}\Lambda(u^\gamma_{jq}) \\
&= \delta_{\alpha,\gamma}\Lambda_\beta \sum_q U_{k,i}\overline{U_{l,q}} \sqrt{\lambda^\gamma_j\lambda^\gamma_q} \sum_{s,t} (U^*(F^\beta)^{-1/2})_{j,s}((F^\beta)^{1/2}U)_{t,q}\Lambda((u^\beta_{st})^*) \\
&= \delta_{\alpha,\gamma}\Lambda_\beta \sum_q U_{k,i}\overline{U_{l,q}} \sqrt{\lambda^\gamma_j\lambda^\gamma_q} \sum_{s,t} \overline{U_{s,j}} U_{t,q} \frac{\sqrt{\lambda^\beta_s}}{\sqrt{\lambda^\beta_t}} \Lambda((u^\beta_{st})^*)
\end{aligned}$$

Now, we know that $\Lambda_\gamma U_{i,j} = U_{i,j}\Lambda_\beta \lambda^\gamma_j \lambda^\beta_i$, for each i,j. Similarly, as $\Lambda^2_\gamma U^* F^\beta U = (F^\gamma)^{-1}$, by the uniqueness of positive square-roots, also $\Lambda_\gamma U^*(F^\beta)^{1/2}U = (F^\gamma)^{-1/2}$, so $\sqrt{\Lambda_\gamma} U_{i,j} = \sqrt{\Lambda_\beta \lambda^\gamma_j \lambda^\beta_i} U_{i,j}$. So we get

$$\begin{aligned}
\hat{R}(e^\alpha_{ij})\hat{\Lambda}(e^\beta_{kl}) &= \delta_{\alpha,\gamma}\sqrt{\Lambda_\beta} \sum_{q,s,t} U_{k,i}\overline{U_{l,q}} \sqrt{\lambda^\gamma_j}\overline{U_{s,j}}\sqrt{\Lambda_\gamma} U_{t,q} \frac{\sqrt{\lambda^\beta_s}}{\lambda^\beta_t} \Lambda((u^\beta_{st})^*) \\
&= \delta_{\alpha,\gamma}\sqrt{\Lambda_\beta\Lambda_\gamma} \sum_s U_{k,i}\sqrt{\lambda^\gamma_j}\overline{U_{s,j}} \frac{\sqrt{\lambda^\beta_s}}{\lambda^\beta_l} \Lambda((u^\beta_{sl})^*) \\
&= \delta_{\alpha,\gamma}\sqrt{\Lambda_\beta\Lambda_\gamma} \sum_s U_{k,i}\sqrt{\lambda^\gamma_j}\overline{U_{s,j}} \frac{\sqrt{\lambda^\beta_s}}{\lambda^\beta_l} \frac{\lambda^\beta_l}{\Lambda_\beta} \hat{\Lambda}(e^\beta_{sl}) \\
&= \delta_{\alpha,\gamma}\sqrt{\frac{\Lambda_\gamma}{\Lambda_\beta}} U_{k,i}\sqrt{\lambda^\gamma_j} \sum_s \overline{U_{s,j}}\sqrt{\lambda^\beta_s}\hat{\Lambda}(e^\beta_{sl}) = \delta_{\alpha,\gamma}\frac{\Lambda_\gamma}{\Lambda_\beta} U_{k,i} \sum_s \overline{U_{s,j}}\hat{\Lambda}(e^\beta_{sl}).
\end{aligned}$$

It follows that, with β being the unique index such that $\overline{u^\alpha}$ is equivalent to u^β, and recalling that $\Lambda_\alpha = \Lambda_\beta$, we have that

$$\hat{R}(e^\alpha_{ij}) = \sum_{p,k} \frac{\Lambda_\alpha}{\Lambda_\beta} U_{k,i}\overline{U_{p,j}} e^\beta_{p,k} = (U^* e^\beta U)_{j,i}.$$

Hence indeed $\hat{R}$ is an isometry etc.

Next we calculate

$$\begin{aligned}
\hat{\tau}_{-i/2}(e^{\alpha}_{ij})\hat{\Lambda}(e^{\beta}_{kl}) &= \nabla^{1/2}e^{\alpha}_{ij}\nabla^{-1/2}\hat{\Lambda}(e^{\beta}_{kl}) = \frac{\Lambda_\beta}{\lambda^\beta_l}\nabla^{1/2}e^{\alpha}_{ij}\nabla^{-1/2}\Lambda((u^{\beta}_{kl})^*) \\
&= \frac{\Lambda_\beta}{\lambda^\beta_l}\nabla^{1/2}e^{\alpha}_{ij}\Lambda(\sigma_{i/2}((u^{\beta}_{kl})^*)) = \frac{\Lambda_\beta}{\lambda^\beta_l}\nabla^{1/2}e^{\alpha}_{ij}\sqrt{\lambda^\beta_k\lambda^\beta_l}\,\Lambda((u^{\beta}_{kl})^*) \\
&= \nabla^{1/2}e^{\alpha}_{ij}\sqrt{\lambda^\beta_k\lambda^\beta_l}\,\hat{\Lambda}(e^{\beta}_{kl}) = \delta_{j,k}\delta_{\alpha,\beta}\nabla^{1/2}\sqrt{\lambda^\beta_k\lambda^\beta_l}\,\hat{\Lambda}(e^{\beta}_{il}) \\
&= \delta_{j,k}\delta_{\alpha,\beta}\sqrt{\lambda^\beta_k\lambda^\beta_l}\frac{\Lambda_\beta}{\lambda^\beta_l}\Lambda(\sigma_{-i/2}((u^{\beta}_{il})^*)) = \delta_{j,k}\delta_{\alpha,\beta}\sqrt{\lambda^\beta_k\lambda^\beta_l}\frac{\Lambda_\beta}{\lambda^\beta_l}(\lambda^\beta_i\lambda^\beta_l)^{-1/2}\Lambda((u^{\beta}_{il})^*) \\
&= \delta_{j,k}\delta_{\alpha,\beta}\sqrt{\frac{\lambda^\beta_j}{\lambda^\beta_i}}\hat{\Lambda}(e^{\beta}_{il}) = \sqrt{\frac{\lambda^\beta_j}{\lambda^\beta_i}}e^{\alpha}_{ij}\hat{\Lambda}(e^{\beta}_{kl})
\end{aligned}$$

So in conclusion, with α, β linked as before,

$$\hat{S}(e^{\alpha}_{ij}) = \sqrt{\frac{\lambda^\beta_j}{\lambda^\beta_i}}(U^*e^\beta U)_{j,i}.$$

2.2.6 The coproduct

For $\omega \in L^1(\mathbb{G})$, we find that

$$\begin{aligned}
\hat{\Delta}(\lambda(\omega_{\xi,\eta})) &= \hat{\Delta}\big((\omega_{\xi,\eta}\otimes\iota)(W)\big) = (\omega_{\xi,\eta}\otimes\iota\otimes\iota)(W_{13}W_{12}) \\
&= \sum_i(\omega_{\xi,e_i}\otimes\iota)(W)\otimes(\omega_{e_i,\eta}\otimes\iota)(W) = \sum_i \lambda(\omega_{\xi,e_i})\otimes\lambda(\omega_{e_i,\eta}),
\end{aligned}$$

where (e_i) is an orthonormal basis for H.

We'll use the orthonormal basis $\{U^*_\alpha(\delta_i\otimes\delta_j) : \alpha\in\mathbb{A}, 1\le i,j\le n_\alpha\}$. Now,

$$(\omega_{\Lambda((v^\alpha_{ij})^*),U^*_\beta(\delta_k\otimes\delta_l)}\otimes\iota)(W) = \sum_{\gamma,s,t} e^\gamma_{st}\langle v^\gamma_{st}, \omega_{\Lambda((v^\alpha_{ij})^*),U^*_\beta(\delta_k\otimes\delta_l)}\rangle = \sum_{\gamma,s,t} e^\gamma_{st}\sqrt{\frac{\Lambda_\beta}{\lambda^\beta_l}}\varphi(v^\beta_{kl}v^\gamma_{st}(v^\alpha_{ij})^*),$$

and also

$$(\omega_{U^*_\beta(\delta_k\otimes\delta_l),\Lambda(1)}\otimes\iota)(W) = \sum_{\gamma,s,t}\sqrt{\frac{\Lambda_\beta}{\lambda^\beta_l}}e^\gamma_{st}\varphi(v^\gamma_{st}(v^\beta_{kl})^*) = \sqrt{\frac{\lambda^\beta_l}{\Lambda_\beta}}e^\beta_{kl}.$$

Thus

$$\hat{\Delta}(e^\alpha_{ij}) = \frac{\Lambda_\alpha}{\lambda^\alpha_j}\sum_{\beta,k,l}\sum_{\gamma,s,t}\varphi(v^\beta_{kl}v^\gamma_{st}(v^\alpha_{ij})^*)\, e^\gamma_{st}\otimes e^\beta_{kl}.$$

Then

$$\hat{\varphi}\big((e^{*\gamma}_{st}\otimes\iota)\hat{\Delta}(e^\alpha_{ij})\big) = \frac{\Lambda_\alpha}{\lambda^\alpha_j}\sum_{\beta,k,l}\varphi(v^\beta_{kl}v^\gamma_{st}(v^\alpha_{ij})^*)\hat{\varphi}(e^\beta_{kl}) = \frac{\Lambda_\alpha}{\lambda^\alpha_j}\sum_{\beta,k}\varphi(v^\beta_{kk}v^\gamma_{st}(v^\alpha_{ij})^*)\frac{\Lambda_\beta}{\lambda^\beta_k}$$

2.3 Aspects of the locally compact setting

Recall the operator P defined by $P^{it}\Lambda(a) = \Lambda(\tau_t(a))$ (the scaling constant is trivial). Thus

$$\begin{aligned}
U_\alpha P^{it}U^*_\alpha(\delta_i\otimes\delta_j) &= U_\alpha\sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}P^{it}\Lambda((v^\alpha_{ij})^*) = U_\alpha\sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}\Lambda(\tau_t(v^\alpha_{ij})^*) \\
&= U_\alpha\sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}(\lambda^\alpha_i)^{-it}(\lambda^\alpha_j)^{it}\Lambda(\tau_t(v^\alpha_{ij})^*) = (\lambda^\alpha_i)^{-it}(\lambda^\alpha_j)^{it}\delta_i\otimes\delta_j.
\end{aligned}$$

3 Using the right regular representation

It is more common to use the right regular representation, which we shall denote by V. This satisfies

$$V(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(a)(1 \otimes b)),$$

where of course in generality Λ is using the right Haar weight; in the compact case, this agrees with the left Haar weight, of course. Thus we see that

$$V(\Lambda(v^\alpha_{ij}) \otimes \xi) = \sum_k \Lambda(v^\alpha_{ik}) \otimes v^\alpha_{kj}(\xi).$$

For each $\alpha \in \mathbb{A}$, let H'_α be the subspace of H spanned by $\{\Lambda(v^\alpha_{ij}) : 1 \leq i, j \leq n_\alpha\}$. As $\mathcal{A}$ is dense in H, it follows that H is isomorphic to the Hilbert space direct sum of $\{H'_\alpha : \alpha \in \mathbb{A}\}$. We can construct a unitary $U'_\alpha : H_\alpha \to \ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$ given by

$$U'_\alpha : \Lambda(v^\alpha_{ij}) \mapsto (\Lambda_\alpha \lambda^\alpha_i)^{-1/2} \delta_i \otimes \delta_j.$$

This is clearly a linear bijection, and it is unitary because

$$\big(U'_\alpha(v^\alpha_{ij}) \big| U'_\alpha(v^\alpha_{kl})\big) = \frac{1}{\Lambda_\alpha \sqrt{\lambda^\alpha_i \lambda^\alpha_k}} \big(\delta_i \otimes \delta_j \big| \delta_k \otimes \delta_l\big) = \delta_{i,k}\delta_{j,l} \frac{1}{\Lambda_\alpha \lambda^\alpha_i} = \big(\Lambda(v^\alpha_{ij}) \big| \Lambda(v^\alpha_{kl})\big).$$

So again V restricts to an operator on $H_\alpha \otimes H$, and

$$(U'_\alpha \otimes 1) V ({U'_\alpha}^* \otimes 1) : \delta_i \otimes \delta_j \otimes \xi \mapsto \sum_k \delta_i \otimes \delta_k \otimes v^\alpha_{kj}(\xi).$$

Setting

$$\omega^\alpha_{ij} = \Lambda_\alpha \lambda^\alpha_i \omega_{\Lambda(1), \Lambda(v^\alpha_{ij})},$$

we see that

$$\langle v^\beta_{kl}, \omega^\alpha_{ij} \rangle = \Lambda_\alpha \lambda^\alpha_i \varphi((v^\alpha_{ij})^* v^\beta_{kl}) = \delta_{\alpha,\beta} \delta_{i,k} \delta_{j,l}.$$

Then

$$\begin{aligned} \rho(\omega^\alpha_{ij}) \Lambda(v^\beta_{kl}) = (\iota \otimes \omega^\alpha_{ij})(V) \Lambda(v^\beta_{kl}) &= \Lambda_\alpha \lambda^\alpha_i (\iota \otimes \omega_{\Lambda(1), \Lambda(v^\alpha_{ij})})(V) \Lambda(v^\beta_{kl}) \\ &= \Lambda_\alpha \lambda^\alpha_i \sum_p \Lambda(v^\beta_{kp}) \big(\Lambda(v^\beta_{pl}) \big| \Lambda(v^\alpha_{ij})\big) = \delta_{\alpha,\beta} \delta_{j,l} \Lambda(v^\alpha_{ki}). \end{aligned}$$

Thus $\rho(\omega^\alpha_{ij})$ restricts to the zero map on each H_β with $\beta \neq \alpha$, and

$$U'_\alpha \rho(\omega^\alpha_{ij}) {U'_\alpha}^* : \delta_k \otimes \delta_l \mapsto \delta_{j,l} \delta_k \otimes \delta_i \implies U'_\alpha \rho(\omega^\alpha_{ij}) {U'_\alpha}^* = 1 \otimes e_{ij}.$$

A Finding the unitary corepresentations

A.1 The left regular representation

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Definition A.1. A *(unitary) corepresentation* of (A, Δ) is a (unitary) element U of $M(A \otimes \mathcal{B}_0(H))$ such that $(\Delta \otimes \iota)U = U_{13}U_{23}$.

Let H have an orthonormal basis (e_n), and let $U_{n,m}$ be the matrix elements of U; this means that $U_{n,m} = (\iota \otimes \omega_{e_m,e_n})U \in M(A)$. Then U is a corepresentation if and only if

$$\Delta(U_{n,m}) = (\iota \otimes \iota \otimes \omega_{e_m,e_n})(U_{13}U_{23}) = \sum_k (\iota \otimes \iota \otimes \omega_{e_k,e_n})(U_{13})(\iota \otimes \iota \otimes \omega_{e_m,e_k})(U_{23})$$
$$= \sum_k U_{n,k} \otimes U_{k,m}.$$

Let φ be the Haar state on A and let $L^2(\varphi)$ be the GNS space, with cyclic vector ξ_0. Let K be some auxiliary Hilbert space upon which A acts non-degenerately, say with $*$-homomorphism $\pi : A \to \mathcal{B}(K)$. At this stage, we shall not assume that π is injective.

Proposition A.2. *There is a (unique) unitary operator U on $K \otimes L^2(\varphi)$ with $U^*(\xi \otimes a\xi_0) = (\pi \otimes \iota)\Delta(a)(\xi \otimes \xi_0)$ for $a \in A$ and $\xi \in K$.*

Proof. For $(a_i) \subseteq A$ and $(\xi_i) \subseteq K$, we have that

$$\Big\| \sum_i (\pi \otimes \iota)\Delta(a_i)(\xi_i \otimes \xi_0) \Big\|^2 = \sum_{i,j} \big((\pi \otimes \iota)\Delta(a_j^* a_i)\xi_i \otimes \xi_0 \big| \xi_j \otimes \xi_0 \big)$$
$$= \sum_{i,j} \big(\pi((\iota \otimes \varphi)\Delta(a_j^* a_i))\xi_i \big| \xi_j \big)$$
$$= \sum_{i,j} \varphi(a_j^* a_i)(\xi_i|\xi_j) = \Big\| \sum_i \xi_i \otimes a_i\xi_0 \Big\|^2.$$

This shows that U^* is an isometry; clearly U^* is densely defined, and so U^* extends to an isometry on all of $K \otimes L^2(\varphi)$. As $\Delta(A)(A \otimes 1)$ is linearly dense in $A \otimes A$, we see that the image of U^* contains the closed linear span of

$$\big\{ \pi(a)\xi \otimes b\xi_0 : a, b \in A, \xi \in K \big\}.$$

As A acts non-degenerately on K, this shows that U^* is a surjection, so U is unitary as required. □

escomult **Proposition A.3.** *The operator U is a member of $M(\pi(A) \otimes \mathcal{B}_0(L^2(\varphi)))$, and for $a \in A$, we have that $(\pi \otimes \iota)\Delta(a) = U^*(1 \otimes a)U$ in $\mathcal{B}(K \otimes L^2(\varphi))$.*

Proof. For $a, b \in A$ and $\xi \in K$, we have that $U^*(1 \otimes a)(\xi \otimes b\xi_0) = (\pi \otimes \iota)\Delta(ab)(\xi \otimes \xi_0) = (\pi \otimes \iota)\Delta(a)U^*(\xi \otimes b\xi_0)$ and so $U^*(1 \otimes a)U = (\pi \otimes \iota)\Delta(a)$.

Let $a, b \in A, \xi_1, \xi_2 \in L^2(\varphi)$ and $\xi \in K$. For $\epsilon > 0$ we can find $\sum_i a_i \otimes b_i \in A \otimes A$ with $\| \sum_i a_i \otimes b_i - \Delta(a)(b \otimes 1) \| < \epsilon$. Then

$$U^*(\pi(b) \otimes \theta_{a\xi_0,\xi_1})(\xi \otimes \xi_2) = (\xi_2|\xi_1)U^*(\pi(b)\xi \otimes a\xi_0) = (\xi_2|\xi_1)(\pi \otimes \iota)(\Delta(a)(b \otimes 1))(\xi \otimes \xi_0).$$

It follows that

$$\Big\| \Big(U^*(\pi(b) \otimes \theta_{a\xi_0,\xi_1}) - \sum_i \pi(a_i) \otimes \theta_{b_i\xi_0,\xi_1} \Big)(\xi \otimes \xi_2) \Big\|$$
$$= \Big\| (\xi_2|\xi_1)(\pi \otimes \iota)(\Delta(a)(b \otimes 1))(\xi \otimes \xi_0) - \sum_i \pi(a_i)\xi \otimes b_i\xi_0(\xi_2|\xi_1) \Big\|$$
$$\leq \epsilon \|\xi_2\| \|\xi_1\| \|\xi\| \|\xi_0\|.$$

As $\epsilon > 0$ was arbitrary, this shows that $U^*(\pi(b) \otimes \theta_{a\xi_0,\xi_1}) \in \pi(A) \otimes \mathcal{B}_0(L^2(\varphi))$. By linearity and continuity, $U^*(\pi(A) \otimes \mathcal{B}_0(L^2(\varphi))) \subseteq \pi(A) \otimes \mathcal{B}_0(L^2(\varphi))$.

Now consider

$$U(1 \otimes \theta_{a\xi_0,\xi_1})(\xi \otimes \xi_2) = (\xi_2|\xi_1)U(\xi \otimes a\xi_0).$$

For $\epsilon > 0$ we can find $(a_i), (b_i) \subseteq A$ with $\| \sum_i \Delta(a_i)(b_i \otimes 1) - 1 \otimes a\| < \epsilon$. Then

$$\Big\| \sum_i (\pi(b_i)\xi \otimes a_i\xi_0) - U(\xi \otimes a\xi_0) \Big\| = \Big\| \sum_i U^*(\pi(b_i)\xi \otimes a_i\xi_0) - \xi \otimes a\xi_0 \Big\|$$
$$= \Big\| \sum_i (\pi \otimes \iota)(\Delta(a_i)(b_i \otimes 1))(\xi \otimes \xi_0) - \xi \otimes a\xi_0 \Big\| < \epsilon \|\xi \otimes \xi_0\|.$$

Thus we can approximate $U(1 \otimes \theta_{a\xi_0,\xi_1})$ by $\sum_i \pi(b_i) \otimes \theta_{a_i\xi_0,\xi_1}$. We conclude that $U(\pi(A) \otimes \mathcal{B}_0(L^2(\varphi))) \subseteq \pi(A) \otimes \mathcal{B}_0(L^2(\varphi))$. Hence $U \in M(\pi(A) \otimes \mathcal{B}_0(L^2(\varphi)))$. □

em:dense **Lemma A.4.** *We have that for $a, b \in A$,*

$$(\iota \otimes \omega_{a\xi_0,b\xi_0})(U) = \pi(\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a)), \quad (\iota \otimes \omega_{a\xi_0,b\xi_0})(U^*) = \pi(\iota \otimes \varphi)((1 \otimes b^*)\Delta(a)).$$

Consequently, the collections $\{(\iota \otimes \omega)(U) : \omega \in \mathcal{B}(L^2(\varphi))_*\}$ and $\{(\iota \otimes \omega)(U^*) : \omega \in \mathcal{B}(L^2(\varphi))_*\}$ are dense in $\pi(A)$.

Proof. For $a, b \in A$ and $\xi_1, \xi_2 \in K$, we have that

$$\big((\iota \otimes \omega_{a\xi_0,b\xi_0})(U)\xi_1 \big| \xi_2\big) = \big(\xi_1 \otimes a\xi_0 \big| U^*(\xi_2 \otimes b\xi_0)\big)$$
$$= \big((\pi \otimes \iota)\Delta(b^*)\xi_1 \otimes a\xi_0 \big| \xi_2 \otimes \xi_0\big) = \big(\pi(\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a))\xi_1 \big| \xi_2\big),$$

which gives the first result. Similarly,

$$\big((\iota \otimes \omega_{a\xi_0,b\xi_0})(U^*)\xi_1 \big| \xi_2\big) = \big((\pi \otimes \iota)((1 \otimes b^*)\Delta(a))(\xi_1 \otimes \xi_0) \big| \xi_2 \otimes \xi_0\big),$$

which gives the second result. As $\Delta(A)(1 \otimes A)$ is linearly dense in $A \otimes A$, the density result follows. □

Suppose now that π is faithful, so we can identify A with $\pi(A)$, and so U is a member of $M(A \otimes \mathcal{B}_0(L^2(\varphi)))$.

egcorep **Proposition A.5.** *Suppose there is a $*$-homomorphism $\Phi : \pi(A) \to \mathcal{B}(K \otimes K)$ with $\Phi\pi = (\pi \otimes \pi)\Delta$. Then $U_{13}U_{23} = (\Phi \otimes \iota)U$. In particular, when π is faithful, U is a unitary corepresentation.*

Proof. We shall instead equivalently show that $(\Phi \otimes \iota)(U^*) = U_{23}^* U_{13}^*$. For $a, b \in A$ and $\xi_1, \xi_2 \in K$, we have that

$$U_{13}^*(\pi(a)\xi_1 \otimes \xi_2 \otimes b\xi_0) = \big((\pi \otimes \iota)((\Delta(b)(a \otimes 1))\big)_{13}(\xi_1 \otimes \xi_2 \otimes \xi_0).$$

Similarly,

$$U_{23}^*(\pi(a_1)\xi_1 \otimes \xi_2 \otimes a_2\xi_0) = \pi(a_1)\xi_1(\pi \otimes \iota)\Delta(a_2)(\xi_2 \otimes \xi_0) = (\pi \otimes \pi \otimes \iota)((\iota \otimes \Delta)(a_1 \otimes a_2))(\xi_1 \otimes \xi_2 \otimes \xi_0).$$

As $\Delta(b)(a \otimes 1) \in A \otimes A$, it follows by continuity that

$$U_{23}^* U_{13}^*(\pi(a)\xi_1 \otimes \xi_2 \otimes b\xi_0) = (\pi \otimes \pi \otimes \iota)((\iota \otimes \Delta)(\Delta(b)(a \otimes 1))(\xi_1 \otimes \xi_2 \otimes \xi_0)$$
$$= (\pi \otimes \pi \otimes \iota)(\Delta^2(b))(\pi(a)\xi_1 \otimes \xi_2 \otimes \xi_0).$$

By hypothesis, this is equal to

$$(\Phi\pi \otimes \iota)\Delta(b)(\pi(a)\xi_1 \otimes \xi_2 \otimes \xi_0).$$

It hence follows that for $a, b \in A$,

$$(\iota \otimes \iota \otimes \omega_{a\xi_0,b\xi_0})(U_{23}^* U_{13}^*) = \Phi\pi\big((\iota \otimes \varphi)(1 \otimes b^*)\Delta(a)\big).$$

By the previous lemma, this is equal to

$$\Phi((\iota \otimes \omega_{a\xi_0,b\xi_0})(U^*)),$$

and the result follows. □

A.2 Irreducible representations

Definition A.6. Let $U \in M(A \otimes \mathcal{B}_0(H))$ be a corepresentation of (A, Δ). A closed subspace H_1 of H is *invariant* for U if $(1 \otimes e)U(1 \otimes e) = U(1 \otimes e)$ where e is the orthogonal projection onto H_1.

U is said to be *irreducible* if the only invariant subspaces are $\{0\}$ and H.

Lemma A.7. *Let H_1 be an invariant subspace for a corepresentation U. Let e be the orthogonal projection onto H_1, and let $U_e = (1 \otimes e)U(1 \otimes e)$. Then U_e is a corepresentation on H_1, unitary if U is.*

Proof. We have that

$$(\Delta \otimes \iota)(U_e) = (1 \otimes 1 \otimes e)U_{13}U_{23}(1 \otimes 1 \otimes e) = (1 \otimes 1 \otimes e)U_{13}(1 \otimes 1 \otimes e)U_{23}(1 \otimes 1 \otimes e) = (U_e)_{13}(U_e)_{23}.$$

Thus U_e is a corepresentation. If U is unitary then

$$U_e^* U_e = (1 \otimes e)U^*(1 \otimes e)U(1 \otimes e) = (1 \otimes e)U^*U(1 \otimes e) = 1 \otimes e.$$

So U_e is unitary, as a member of $M(A \otimes \mathcal{B}_0(H_1))$. $\square$

Definition A.8. A corepresentation of the form U_e is a *sub-corepresentation* of U.

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Proposition A.9. *Let U be a unitary corepresentation of (A, Δ). Let B be the norm closure of $\{(\varphi \otimes \iota)(U(a \otimes 1)) : a \in A\}$. Then B is a non-degenerate C*-subalgebra of $\mathcal{B}(H)$, and $U \in M(A \otimes B)$.*

Proof. Let $a \in A$ and set $x = (\varphi \otimes \iota)(U(a \otimes 1)) \in \mathcal{B}(H)$. Then

$$\begin{aligned} U(\iota \otimes \varphi \otimes \iota)\big(U_{23}(\Delta(a) \otimes 1)\big) &= (\iota \otimes \varphi \otimes \iota)\big(U_{13}U_{23}(\Delta(a) \otimes 1)\big) \\ &= (\iota \otimes \varphi \otimes \iota)\big((\Delta \otimes \iota)(U(a \otimes 1))\big) \\ &= 1 \otimes (\varphi \otimes \iota)(U(a \otimes 1)) = 1 \otimes x. \end{aligned}$$

Thus $U^*(1 \otimes x) = (\iota \otimes \varphi \otimes \iota)(U_{23}(\Delta(a) \otimes 1))$.

So if also $y = (\varphi \otimes \iota)(U(b \otimes 1))$ for some $b \in A$, then

$$\begin{aligned} y^*x = (\varphi \otimes \iota)\big((b^* \otimes 1)U^*(1 \otimes x)\big) &= (\varphi \otimes \iota)\big((b^* \otimes 1)U^*(1 \otimes x)\big) \\ &= (\varphi \otimes \varphi \otimes \iota)((b^* \otimes U)(\Delta(a) \otimes 1)) \\ &= (\varphi \otimes \iota)(U(c \otimes 1)), \end{aligned}$$

where $c = (\varphi \otimes \iota)((b^* \otimes 1)\Delta(a)) \in A$. So we have shown that $B^*B \subseteq B$. As $(A \otimes 1)\Delta(A)$ is dense in $A \otimes A$, as a and b carry, c varies over a dense subset of A. Thus B^*B is dense in B. In particular, B is self-adjoint. Thus also $BB \subseteq B$, and we conclude that B is a C*-algebra.

Now let $\theta \in \mathcal{B}_0(H)$ and $a \in A$, so that $(\varphi \otimes \iota)(U(a \otimes \theta)) \in B\mathcal{B}_0(H)$. As U is a unitary multiplier of $M(A \otimes \mathcal{B}_0(H))$, the set $\{U(a \otimes \theta) : a \in A, \theta \in \mathcal{B}_0(H)\}$ is linearly dense in $A \otimes \mathcal{B}_0(H)$. It follows that $B\mathcal{B}_0(H)$ is linearly dense in $\mathcal{B}_0(H)$, which is enough to show that B acts non-degenerately on H.

Finally, we show that $U \in M(A \otimes B)$. For $b \in A$ and x as above,

$$U^*(b \otimes x) = (\iota \otimes \varphi \otimes \iota)(U_{23}(\Delta(a)(b \otimes 1) \otimes 1)).$$

As $\Delta(a)(b \otimes 1) \in A \otimes A$, we see immediately that $U^*(b \otimes x) \in A \otimes B$. Moreover, as $\Delta(A)(A \otimes 1)$ is dense in $A \otimes A$, we set $\{U^*(b \otimes x) : b \in A, x \in B\}$ is linearly dense in $A \otimes B$. So also $U(A \otimes B) \subseteq A \otimes B$, and $U \in M(A \otimes B)$ as required. $\square$

Proposition A.10. *Let U be a unitary corepresentation of (A, Δ), and let H_1 be an invariant subspace of H for U. Then $H_1^\perp$ is also invariant.*

Proof. Let e be the orthogonal projection of H onto H_1. Let $x = (\varphi \otimes \iota)(U(a \otimes 1)) \in B$, so as $U(1 \otimes e) = (1 \otimes e)U(1 \otimes e)$, it follows that

$$xe = (\varphi \otimes \iota)(U(a \otimes e)) = (\varphi \otimes \iota)((1 \otimes e)U(a \otimes e)) = exe.$$

As $B = B^*$, also $ex = (x^*e)^* = (ex^*e)^* = exe$, and so $ex = xe$. Thus H_1 is an invariant subspace for B, and as B acts non-degenerately on H, it follows that $ex = xe$ for all $x \in M(B)$.[1] As $U \in M(A \otimes B)$, it follows that $(1 \otimes e)U = U(1 \otimes e)$, and then a short calculation shows that

$$(1 \otimes e^\perp)U(1 \otimes e^\perp) = U(1 \otimes e^\perp),$$

where $e^\perp = 1 - e$, as required. □

Definition A.11. Let U_1 and U_2 be unitary corepresentations of (A, Δ) on H_1 and H_2 respectively. The *direct sum* of U_1 and U_2 is $U_1 \oplus U_2 \in M(A \otimes \mathcal{B}_0(H_1 \oplus H_2))$ is

$$U_1 \oplus U_2 = \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix},$$

where here we make the identification

$$\mathcal{B}_0(H_1 \oplus H_2) = \begin{pmatrix} \mathcal{B}_0(H_1) & \mathcal{B}_0(H_2, H_1) \\ \mathcal{B}_0(H_1, H_2) & \mathcal{B}_0(H_2) \end{pmatrix}.$$

The *tensor product* of U_1 and U_2 is $U_1 \textcircled{\scriptsize T} U_2 = (U_1)_{12}(U_2)_{13} \in M(A \otimes \mathcal{B}_0(H_1 \otimes H_2)) \cong M(A \otimes \mathcal{B}_0(H_1) \otimes \mathcal{B}_0(H_2))$.

An *intertwiner* between U_1 and U_2 is a bounded operator $T : H_1 \to H_2$ with $(1 \otimes T)U_1 = U_2(1 \otimes T)$. We denote the collection of intertwiners by $\mathrm{Mor}(U_1, U_2)$. Two corepresentations are *equivalent* if there is an invertible intertwiner, and unitarily equivalent if there is a unitary intertwiner.

lem:one **Lemma A.12.** *Let U and V be corepresentations of (A, Δ) on H_1 and H_2 respectively. Let $x \in \mathcal{B}(H_1, H_2)$, and set*

$$y = (\varphi \otimes \iota)(V^*(1 \otimes x)U).$$

Then $y \in \mathcal{B}(H_1, H_2)$, and $V^*(1 \otimes y)U = 1 \otimes y$.

Proof. We identify $\mathcal{B}(H_1, H_2)$ with a "corner" of $\mathcal{B}(H_1 \oplus H_2)$ in the obvious way. Then U and V are both (on diagonal) corners of $M(A \otimes \mathcal{B}_0(H_1 \oplus H_2))$; thus $V^*(1 \otimes x)U \in M(A \otimes \mathcal{B}_0(H_1 \oplus H_2))$ and so y makes sense as a member of $M(\mathcal{B}_0(H_1 \oplus H_2)) = \mathcal{B}(H_1 \oplus H_2)$. A simple calculation shows that y only has non-zero component in the $\mathcal{B}(H_1, H_2)$ corner; thus y is well-defined.

Notice that

$$(\Delta \otimes \iota)(V^*(1 \otimes x)U) = V_{23}^* V_{13}^*(1 \otimes 1 \otimes x)U_{13}U_{23}.$$

Then observe that

$$(\varphi \otimes \iota \otimes \iota)(\Delta \otimes \iota)(V^*(1 \otimes x)U) = 1 \otimes (\varphi \otimes \iota)(V^*(1 \otimes x)U) = 1 \otimes y,$$

while

$$(\varphi \otimes \iota \otimes \iota)V_{23}^* V_{13}^*(1 \otimes 1 \otimes x)U_{13}U_{23} = V^*(1 \otimes y)U,$$

and the result follows. □

[1]Indeed, let $x \in M(B)$ so for $y \in B, \xi \in H$, we have that $xe(y\xi) = (xy)e\xi = e(xy)\xi = ex(y\xi)$. By non-degeneracy, it follows that $xe = ex$.

The obvious use of this lemma is that if V is unitary, then $(1 \otimes y)U = V(1 \otimes y)$, and so $y \in \mathrm{Mor}(U, V)$. Notice that an obvious modification of the proof shows that if x is compact, then also y will be compact.

nvsimuni **Proposition A.13.** *Let U be an invertible[2] corepresentation of (A, Δ). Then U is equivalent to a unitary corepresentation.*

Proof. Let U act on H, and set

$$y = (\varphi \otimes \iota)(U^*U).$$

By the previous lemma, $U^*(1 \otimes y)U = 1 \otimes y$. Clearly $y \geq 0$ and as U is invertible, $U^*U \geq \epsilon 1$ for some $\epsilon > 0$; thus also $y \geq \epsilon 1$, so y is invertible. Now set

$$V = (1 \otimes y^{1/2})U(1 \otimes y^{-1/2}).$$

Then $(\Delta \otimes \iota)V = (1 \otimes 1 \otimes y^{1/2})U_{13}U_{23}(1 \otimes 1 \otimes y^{-1/2}) = V_{13}V_{23}$ and so V is a corepresentation. Then

$$V^*V = (1 \otimes y^{-1/2})U^*(1 \otimes y)U(1 \otimes y^{-1/2}) = (1 \otimes y^{-1/2})(1 \otimes y)(1 \otimes y^{-1/2}) = 1,$$

and as V is clearly invertible, it follows that V is unitary. By definition, $y^{1/2}$ intertwines U and V, and so U is equivalent to a unitary corepresentation, as required. □

corepdec **Theorem A.14.** *Let U be a unitary corepresentation of (A, Δ) on a Hilbert space H. Then there is a family of mutually orthogonal, finite-dimensional projections $\{e_\alpha : \alpha \in I\}$ with sum 1, with $U(1 \otimes e_\alpha) = (1 \otimes e_\alpha)U$ for each α, and with $U(1 \otimes e_\alpha)$, considered as an element of $A \otimes \mathcal{B}(e_\alpha H)$, being a finite-dimensional unitary corepresentation.*

Proof. Let B be the collection of operators $x \in \mathcal{B}(H)$ with $(1 \otimes x)U = U(1 \otimes x)$. Then B is clearly a norm-closed subalgebra, and as U is unitary, it is easy to see that B is self-adjoint. So B is a C*-algebra.

By Lemma A.12 (lem:one), if $x \in \mathcal{B}_0(H)$ then $y = (\varphi \otimes \iota)(U^*(1 \otimes x)U)$ will be in B, and will be compact. Let (x_i) be an increasing net in $\mathcal{B}_0(H)$ with supremum 1. Then the associated family (y_i) is an increasing net in B with supremum 1. As each y_i is compact, we see that B will contain sufficiently many finite-rank projections to form the required family (e_α). □

The following is then a quantum Schur's Lemma.

op:schur **Proposition A.15.** *Let U, V be corepresentations of (A, Δ). For each $T \in \mathrm{Mor}(U, V)$, the space $\ker T$ is invariant for U, and the closure of the image of T is invariant for V. Suppose that one of the following conditions holds:*

1. *U and V are irreducible;*

2. *U or V are finite-dimensional of the same dimension, and one of U or V is irreducible.*

If U and V are not equivalent, then $\mathrm{Mor}(U, V) = \{0\}$; otherwise $\mathrm{Mor}(U, V) = \mathbb{C}x$ for some invertible $x \in \mathcal{B}(H_U, H_V)$. Furthermore, if U and V are unitary, then x can be chosen to be unitary.

[2] This simply means that there is some operator $U^{-1} \in M(A \otimes \mathcal{B}_0(H))$ with $U^{-1}U = UU^{-1} = 1$.

Proof. Let U act on H_U, and V act on H_V. Let $T \in \mathrm{Mor}(U,V)$. We first show that $\ker T$ and $\overline{T(H_U)}$ are invariant for U and V respectively. Let e be the orthogonal projection onto $\ker T$. Then $0 = V(1\otimes Te) = (1\otimes T)U(1\otimes e)$, and it follows that $(1\otimes e)U(1\otimes e) = U(1\otimes e)$. Similarly, if e is the orthogonal projection onto $T(H_U)$, then we wish to show that $(1\otimes e)V(1\otimes e) = V(1\otimes e)$. Equivalently, as $e(H_V) = T(H_U)$, we wish to show that $(1\otimes e)V(1\otimes T) = V(1\otimes T)$. However,

$$(1\otimes e)V(1\otimes T) = (1\otimes e)(1\otimes T)U = (1\otimes T)U = V(1\otimes T),$$

as required.

Then, if U and V are both irreducible, we immediately see that any $T \in \mathrm{Mor}(U,V)$ is an isomorphism, or is 0. If U is both finite-dimensional and irreducible, then any $T \in \mathrm{Mor}(U,V)$ is 0 or injective, but as $\dim(H_U) = \dim(H_V) < \infty$, then T injective means that T is an isomorphism. Similarly, if V is irreducible then T is either 0 or surjective (and so an isomorphism).

So in either case, if U and V are not equivalent, then $\mathrm{Mor}(U,V) = \{0\}$. If $T \in \mathrm{Mor}(U,V)$ is non-zero, then U and V are equivalent. If now $S \in \mathrm{Mor}(U,V)$ is also non-zero, then for any $\lambda \in \mathbb{C}$, the operator $\lambda T - S$ is in $\mathrm{Mor}(U,V)$ and so is an isomorphism $H_U \to H_V$, or is 0. So choosing λ with $\det(\lambda T - S) = 0$, we see that actually $\lambda T = S$ as required.

Finally, suppose that U and V are unitary, so as $U = (1\otimes T^{-1})V(1\otimes T)$,

$$1 = U^*U = (1\otimes T^*)V^*(1\otimes (TT^*)^{-1})V(1\otimes T), \quad 1 = UU^* = (1\otimes T^{-1})V^*(1\otimes TT^*)V(1\otimes (T^*)^{-1}).$$

Thus

$$1\otimes TT^* = V^*(1\otimes TT^*)V,$$

so as V is unitary, we see that $TT^* \in \mathrm{Mor}(V,V)$. Thus the previous work shows that TT^* is a (necessarily positive) scalar multiple of the identity. We may suppose then that $TT^* = I$, so as T is invertible, T is unitary, as required. □

Now let $\pi : A \to \mathcal{B}(K)$ be a faithful, non-degenerate $*$-homomorphism and form the regular corepresentation U as in Proposition A.5.

Theorem A.16. *Let U be the regular corepresentation, acting on the GNS space H. Let V be an irreducible unitary corepresentation, acting on H_V say. Then V is equivalent to a subcorepresentation of (that is,* contained in*) U.*

Proof. Let $x \in \mathcal{B}_0(H, H_V)$ and set $y = (\varphi\otimes\iota)(V^*(1\otimes x)U) \in \mathcal{B}_0(H,H_V)$ so that $(1\otimes y)U = V(1\otimes y)$, by Lemma A.12. By Proposition A.15, if y is non-zero, then y is surjective. As U, V are unitary,

$$V^*(1\otimes y) = (1\otimes y)U^* \implies (1\otimes y^*)V = U(1\otimes y^*)$$

so $y^* : H_V \to H$ is an intertwiner, and hence y^* is injective, and the image of y^* is invariant for U. So, if y is non-zero, y^* implements the required equivalence between V and a subcorepresentation of U.

Alternatively, $y = 0$ for all choices of x. So for any $\xi \in H$ and $\eta \in H_V$, if $x(\gamma) = (\gamma|\xi)\eta$, then

$$0 = \big((\varphi\otimes\iota)(V^*(1\otimes x)U)\xi_1\big|\eta_1\big) = \langle \varphi, (\iota\otimes\omega_{\eta,\eta_1})(V^*)(\iota\otimes\omega_{\xi_1,\xi})(U)\rangle \qquad (\xi_1 \in H, \eta_1 \in H_V).$$

By Lemma A.4, this means that

$$\langle \varphi, (\iota\otimes\omega)(V^*)a\rangle = 0 \qquad (a \in A, \omega \in \mathcal{B}(H_V)_*).$$

This implies that $(\varphi\otimes\iota)(V^*(a\otimes 1)) = 0$ for all $a \in A$, and so also $(\varphi\otimes\iota)(V^*(a\otimes x)) = 0$ for all $a \in A$ and $x \in \mathcal{B}(H_V)$. As V is irreducible, H_V is finite-dimensional, and so $V \in A\otimes\mathcal{B}(H_V)$. Thus $(\varphi\otimes\iota)(V^*V) = 0$, which contradicts that V is unitary. □

A.3 Contragradient representations

Let U be a finite-dimensional corepresentation of (A, Δ) on H. Given an orthonormal basis $(e_i)_{i=1}^n$ for H we can let (e_{ij}) be the matrix units of $\mathbb{M}_n \cong \mathcal{B}(H)$. Then we can write

$$U = \sum_{i,j=1}^{n} u_{ij} \otimes e_{ij}$$

for some $u_{ij} \in A$. Recall from before that U being a corepresentation is equivalent to $\Delta(u_{ij}) = \sum_k u_{ik} \otimes u_{kj}$.

Let K be another finite-dimensional Hilbert space with orthonormal basis $(f_j)_{j=1}^m$. Then $S \in \mathcal{B}(H, K)$ can be represented by a matrix in $\mathbb{M}_{m,n}$, say (s_{ij}). Then

$$(1 \otimes S)U = \sum_{i,j,p,q} u_{ij} \otimes s_{pq} e_{pq} e_{ij} = \sum_{i,j,p} s_{pi} u_{ij} \otimes e_{pj}.$$

Similarly, if $V = \sum_{i,j=1}^m v_{ij} \otimes e_{ij}$ is a corepresentation on K, then

$$V(1 \otimes S) = \sum_{i,j,p,q} v_{ij} \otimes e_{ij} s_{pq} e_{pq} = \sum_{i,j,q} v_{ij} s_{jq} \otimes e_{iq}.$$

Thus $S \in \mathrm{Mor}(U, V)$ if and only if, using matrix multiplication, $(s_{ij})(u_{pq}) = (v_{pq})(s_{ij})$.

Definition A.17. Given U and (e_n) as above, the *contragradient* corepresentation is $\overline{U} = \sum_{i,j} u_{ij}^* \otimes e_{ij}$.

The definition of $\overline{U}$ does depend upon (e_n). Indeed, picking a new orthonormal basis for H is equivalent to finding a unitary matrix S and setting $V = (1 \otimes S^*)U(1 \otimes S)$. So V is a corepresentation (unitarily) equivalent to U. Then $\overline{V} = (1 \otimes \overline{S}^*)\overline{U}(1 \otimes \overline{S})$, and so $\overline{V}$ is equivalent to $\overline{U}$, but the equivalence is given by the matrix $\overline{S}$, which in general is not equal to S.

ra_irrep **Lemma A.18.** *Let U be a corepresentation. Then $\overline{U}$ is also a corepresentation. If U is irreducible, then so is $\overline{U}$.*

Proof. We see that as Δ is a $*$-homomorphism,

$$\Delta(u_{ij}^*) = \Delta(u_{ij})^* = \Big(\sum_k u_{ik} \otimes u_{kj} \Big)^* = \sum_k u_{ik}^* \otimes u_{kj}^*.$$

So $\overline{U}$ is a corepresentation.

Let $\gamma : \mathbb{M}_n \to \mathbb{M}_n$ be the transpose map, which is an anti-homomorphism. Notice that $\overline{U} = (\iota \otimes \gamma)(U^*)$. Suppose that e is an orthogonal projection on H with $\overline{U}(1 \otimes e) = (1 \otimes e)\overline{U}(1 \otimes e)$. Then applying γ gives that

$$(1 \otimes \gamma(e))U^* = (1 \otimes \gamma(e))U^*(1 \otimes \gamma(e)) \implies U(1 \otimes e') = (1 \otimes e')U(1 \otimes e'),$$

where $e' = \gamma(e)^*$ is still an orthogonal projection. As U is irreducible, $e' = 0$ or 1, and hence also $e = 0$ or 1, showing that $\overline{U}$ is irreducible. □

Notice that $\iota \otimes \gamma$ is not an anti-homomorphism on all of $\mathbb{M}_n(A)$, unless A is commutative. Thus we have to work hard(er) to prove the next result.

junitary **Proposition A.19.** *Let V be a finite-dimensional irreducible unitary corepresentation. Then $\overline{V}$ is equivalent to a unitary corepresentation.*

Proof. We again use Lemma A.12, with U being the left regular representation, acting on the GNS space H. Let V act on the finite-dimensional Hilbert space H_V. Pick $x \in \mathcal{B}_0(H, H_V)$ and set

$$y = (\varphi \otimes \iota)(\overline{V}^*(1 \otimes x)U).$$

So $y \in \mathcal{B}_0(H, H_V)$ with $\overline{V}^*(1 \otimes y)U = 1 \otimes y$. Then $U^*(1 \otimes y^*)\overline{V} = 1 \otimes y^*$ and thus $(1 \otimes y^*)\overline{V} = U(1 \otimes y^*)$. So $y^* \in \mathrm{Mor}(\overline{V}, U)$. By Proposition A.15, as $\overline{V}$ is irreducible, y^* has zero kernel, or $y^* = 0$. As in the proof of Theorem A.16, the image of y^* is an invariant subspace of U, and so either $y = 0$, or y^* implements an isomorphism between $\overline{V}$ and a sub-co-representation of U.

Thus, towards a contradiction, suppose that $y = 0$ for any choice of x. Again, this implies that

$$\langle \varphi, (\iota \otimes \omega)(\overline{V}^*)a \rangle = 0 \qquad (a \in A, \omega \in \mathcal{B}(H_V)_*).$$

Let $\omega \in \mathcal{B}(H_V)_*$ be the functional which sends e_{ij} to 1, and e_{pq} to 0 for all other (p, q). Thus $(\iota \otimes \omega)(\overline{V}^*) = v_{ji}$. We hence see that

$$\langle \varphi, (\iota \otimes \omega)(V)a \rangle = 0 \qquad (a \in A, \omega \in \mathcal{B}(H_V)_*).$$

Thus $(\varphi \otimes \iota)(V(a \otimes x)) = 0$ for $a \in A, x \in \mathcal{B}(H_V)$. This again implies that $(\varphi \otimes \iota)(VV^*) = 0$, contradicting V being unitary. □

In particular, if V is merely an invertible corepresentation, then V is equivalent to the direct sum of finite-dimensional unitary corepresentations; the same is then true of $\overline{V}$, and thus in particular $\overline{V}$ is invertible.

A.4 The Hopf $*$-algebra of matrix elements

Let A_0 be the linear span of the matrix elements[3] of unitary irreducible corepresentations. By the previous work, A_0 is also the linear span of the matrix elements of finite-dimensional invertible corepresentations.

Proposition A.20. *The space A_0 is a dense unital $*$-subalgebra of A.*

Proof. Let U and V be corepresentations, and let $\omega_U \in \mathcal{B}(H_U)_*$ and $\omega_V \in \mathcal{B}(H_V)_*$. Then

$$(\iota \otimes \omega_U \otimes \omega_V)(U \textcircled{\scriptsize T} V) = (\iota \otimes \omega_U \otimes \omega_V)(U_{12}V_{13}) = (\iota \otimes \omega_U)(U)(\iota \otimes \omega_V)(V).$$

If U and V are finite-dimensional and unitary, then $U \textcircled{\scriptsize T} V$ is also finite-dimensional and unitary. We conclude that A_0 is an algebra. Notice that $1 \in A = M(A \otimes \mathbb{C})$ is a unitary corepresentation; thus $1 \in A_0$.

Similarly, as $\overline{U}$ is equivalent to a unitary corepresentation whenever U is finite-dimensional and unitary, it follows easily that A_0 is closed under the $*$ operation.

It remains to show that A_0 is dense in A. Choose a faithful, non-degenerate $*$-representation $\pi : A \to \mathcal{B}(K)$ and form the left regular representation U as in Proposition A.5. By Theorem A.14, U decomposes as a direct sum of finite-dimensional, irreducible unitary corepresentations. By Theorem A.16 every finite-dimensional, irreducible unitary corepresentation is equivalent to a sub-corepresentation of U. Hence A_0 is the span of the matrix elements of finite-dimensional sub-corepresentations of U.

By Lemma A.4, the space $\{(\iota \otimes \omega)(U) : \omega \in \mathcal{B}(H)_*\}$ is dense in A. Given $\xi, \eta \in H$, we claim that we can approximate $(\iota \otimes \omega_{\xi,\eta})(U)$ by elements of A_0; this will show that A_0 is dense in A. Let (e_α) be a family of mutually orthogonal projections with sum 1, as given by

[3] That is, elements of the form $(\iota \otimes \omega)(V)$ where V is a unitary corepresentation, and $\omega \in \mathcal{B}(H_V)_*$.

Theorem A.14 when applied to U. Let $U_\alpha = U(1\otimes e_\alpha) = (1\otimes e_\alpha)U$, a finite-dimensional unitary corepresentation. Then

$$\begin{aligned}(\iota\otimes\omega_{\xi,\eta})(U) &= \sum_\alpha(\iota\otimes\omega_{e_\alpha(\xi),\eta})(U) = \sum_\alpha(\iota\otimes\omega_{e_\alpha(\xi),\eta})(U(1\otimes e_\alpha)) \\ &= \sum_\alpha(\iota\otimes\omega_{e_\alpha(\xi),e_\alpha(\eta)})((1\otimes e_\alpha)U(1\otimes e_\alpha)) = \sum_\alpha(\iota\otimes\omega_{e_\alpha(\xi),e_\alpha(\eta)})(U_\alpha).\end{aligned}$$

Thus $(\iota\otimes\omega_{\xi,\eta})(U)$ is in the closure of A_0, as required. □

Let $\{u_\alpha : \alpha\in I\}$ be a maximal family of non-equivalent unitary corepresentations. For each α, let $u_\alpha\in A\otimes\mathbb{M}_{n_\alpha}$ with $u_\alpha = \sum_{i,j=1}^{n_\alpha} u^\alpha_{ij}\otimes e_{ij}$. We shall prove that $\{u^\alpha_{ij} : \alpha\in I, 1\le i,j\le n_\alpha\}$ is a (linear) basis for A_0.

We first take a small diversion. Let $\sigma : A\otimes A\to A\otimes A$ be the swap map, which is a $*$-homomorphism. It is easy to see that $\sigma\Delta$ is co-associative if and only if Δ is, and so $(A,\sigma\Delta)$ is a C*-bialgebra (called the "opposite" or, less commonly but more accurately, the "co-opposite" quantum group). We see that (A,Δ) satisfies the density conditions to be a compact quantum group if and only if $(A,\sigma\Delta)$ does. In this case, φ remains the Haar measure for $(A,\sigma\Delta)$. Notice however that U is a corepresentation for (A,Δ) if and only if U^* is a corepresentation for $(A,\sigma\Delta)$.

Proposition A.21. *For each $\alpha\in I$, there is a positive invertible matrix F^α such that*

$$\langle\varphi,(u^\beta_{ip})^*u^\alpha_{jq}\rangle = \delta_{\alpha,\beta}\delta_{p,q}F^\alpha_{j,i} \qquad (\beta\in I, 1\le i,p\le n_\beta, 1\le j,q\le n_\alpha).$$

The trace of each matrix F^α is 1.

Proof. Consider the operator $\theta_{e_i,e_j}\in\mathcal{B}_0(\ell^2_{n_\alpha},\ell^2_{n_\beta})$. Then by Lemma A.12,

$$y = (\varphi\otimes\iota)(u_\beta^*(1\otimes x)u_\alpha) = \sum_{p,b,c,q}\langle\varphi,(u^\beta_{bp})^*u^\alpha_{cq}\rangle e_{pb}xe_{cq} = \sum_{p,q}\langle\varphi,(u^\beta_{ip})^*u^\alpha_{jq}\rangle e_{pq}$$

is an operator in $\mathcal{B}_0(\ell^2_{n_\alpha},\ell^2_{n_\beta})$ with $(1\otimes y)u_\alpha = u_\beta(1\otimes y)$. As u_α and u_β are irreducible, by Proposition A.15, we see that $y=0$ if $\alpha\neq\beta$.

When $\alpha=\beta$, by Proposition A.15, we see that y must be a scalar multiple of the identity. Thus we obtain numbers $F^\alpha_{j,i}$ with $\langle\varphi,(u^\beta_{ip})^*u^\alpha_{jq}\rangle = \delta_{\alpha,\beta}\delta_{p,q}F^\alpha_{j,i}$. That u^α is unitary means that

$$\sum_k (u^\alpha_{k,i})^*u^\alpha_{k,j} = \delta_{i,j}1 \implies \delta_{i,j} = \sum_k\langle\varphi,(u^\alpha_{k,i})^*u^\alpha_{k,j}\rangle = \delta_{i,j}\sum_k F^\alpha_{k,k}.$$

Now consider $y = (\varphi\otimes\iota)(\overline{u^\alpha}(\overline{u^\alpha})^*)$. By Lemma A.12, applied to $(A,\sigma\Delta)$, we have that $1\otimes y = \overline{u^\alpha}(1\otimes y)(\overline{u^\alpha})^*$. Now,

$$y = \sum_{i,j,k}\langle\varphi,(\overline{u^\alpha})_{ik}((\overline{u^\alpha})^*)_{kj}\rangle e_{ij} = \sum_{i,j,k}\langle\varphi,(u^\alpha_{ik})^*u^\alpha_{jk}\rangle e_{ij} = n_\alpha\sum_{i,j}F^\alpha_{j,i}e_{ij}.$$

Thus $y = n_\alpha(F^\alpha)^t$. However, clearly y is a positive matrix, and so F^α is positive. Now, as $\overline{u^\alpha}$ is equivalent to a unitary corepresentation, and is hence invertible, we see that y intertwines the corepresentations $(\overline{u^\alpha})^*$ and $(\overline{u^\alpha})^{-1}$, again working with $(A,\sigma\Delta)$. Taking adjoints shows that $\overline{u^\alpha}(1\otimes y^*) = (1\otimes y^*)(\overline{u^\alpha})^{*-1}$. As $\overline{u^\alpha}$ is irreducible and has the same dimension as $(\overline{u^\alpha})^{*-1}$, Proposition A.15 shows that $y^*=0$ or y^* is an isomorphism. As the trace of y is n_α, we conclude that y, and hence also F^α are invertible. □

Proposition A.22. *The collection $\{u^\alpha_{i,j}\}$ is linearly independent, and hence forms a basis for A_0.*

Proof. Suppose that the finite linear combination $\sum_{\alpha,i,j} \lambda^\alpha_{i,j} u^\alpha_{i,j}$ is zero. Then, for any β, p, q

$$0 = \sum_{\alpha,i,j} \lambda^\alpha_{i,j} \langle \varphi, (u^\beta_{p,q})^* u^\alpha_{i,j} \rangle = \sum_i F^\beta_{i,p} \lambda^\beta_{i,q}.$$

As F^β is invertible, $\lambda^\beta = 0$, for any β, as required. □

Let $m : A_0 \odot A_0 \to A_0$ be the multiplication map on the algebraic tensor product $A_0 \odot A_0$. We define linear maps $\kappa : A_0 \to A_0$ and $\epsilon : A_0 \to \mathbb{C}$ by

$$\epsilon(u^\alpha_{i,j}) = \delta_{i,j}, \qquad \kappa(u^\alpha_{i,j}) = (u^\alpha_{j,i})^* \qquad (\alpha \in I, 1 \leq i, j \leq n_\alpha).$$

Notice that then, for any finite-dimensional unitary corepresentation U, we have that

$$(\kappa \otimes \iota)(U) = U^*, \qquad (\epsilon \otimes \iota)(U) = 1.$$

In particular, κ and ϵ are well-defined, independent of our choice of maximal family $\{u^\alpha\}$. If $a_i = (\iota \otimes \omega_i)(U_i)$ for $i = 1, 2$ then

$$\epsilon(a_1 a_2) = (\epsilon \otimes \omega_1 \otimes \omega_2)(U_1 \textcircled{\scriptsize T} U_2) = (\omega_1 \otimes \omega_2)(1) = \langle 1, \omega_1 \rangle \langle 1, \omega_2 \rangle = \epsilon(a_1)\epsilon(a_2),$$

and so ϵ is a character.

m:ishopf **Theorem A.23.** *The maps κ and ϵ turn (A_0, Δ) into a Hopf $*$-algebra. To be more precise,*

$$(\epsilon \otimes \iota)\Delta(a) = (\iota \otimes \epsilon)\Delta(a) = a, \qquad m(\kappa \otimes \iota)\Delta(a) = m(\iota \otimes \kappa)\Delta(a) = \epsilon(a)1 \qquad (a \in A_0).$$

Automatically, κ is an anti-homomorphism, and $\Delta\kappa = \sigma(\kappa \otimes \kappa)\Delta$. Furthermore, $\kappa * \kappa * = \iota$.

Proof. As $\Delta(u^\alpha_{i,j}) = \sum_k u^\alpha_{i,k} \otimes u^\alpha_{k,j}$, it follows that Δ restricts to a map $A_0 \to A_0 \odot A_0$. Then

$$(\epsilon \otimes \iota)\Delta(u^\alpha_{i,j}) = \sum_k \epsilon(u^\alpha_{i,k}) u^\alpha_{k,j} = u^\alpha_{i,j},$$

showing that $(\epsilon \otimes \iota)\Delta = \iota$ on A_0; similarly $(\iota \otimes \epsilon)\Delta = \iota$.

Also

$$m(\kappa \otimes \iota)\Delta(u^\alpha_{i,j}) = \sum_k m((u^\alpha_{k,i})^* \otimes u^\alpha_{k,j}) = \sum_k (u^\alpha_{k,i})^* u^\alpha_{k,j} = \delta_{i,j}1 = \epsilon(u^\alpha_{i,j})1,$$

using that u^α is unitary. Similarly $m(\iota \otimes \kappa)\Delta = \epsilon(\cdot)1$.

That κ is an anti-homomorphism and an anti-co-homomorphism follows from the theory of Hopf algebras, see [4, Section 1.3.3] for example. That $*\kappa * \kappa = \iota$ follows from our definition of κ. □

_in_hopf **Proposition A.24.** *Let $\sum_{i,j} a_{ij} \otimes e_{ij}$ be a finite-dimensional corepresentation of (A, Δ) with $a_{ij} \in A_0$ for all i, j. Then the following are equivalent:*

hopf:one 1. *The matrix (a_{ij}) is invertible;*

hopf:two 2. *If $(\xi_j)_{j=1}^n \subseteq \mathbb{C}$ satisfies that $\sum_j a_{ij}\xi_j = 0$ for all i, then $\xi = 0$.*

opf:twoa 3. *If $(\xi_j)_{j=1}^n \subseteq \mathbb{C}$ satisfies that $\sum_j a_{ij}\xi_i = 0$ for all j, then $\xi = 0$.*

pf:three 4. *$\epsilon(a_{ij}) = \delta_{i,j}$ for all i, j;*

opf:four 5. *The matrix (a_{ij}) is invertible with inverse $(\kappa(a_{ij}))$.*

Proof. Clearly (1) $\implies$ (2) and (1) $\implies$ (3), and (5) $\implies$ (1). If (2) then consider the map $\pi : A_0' \to \mathbb{M}_n; \mu \mapsto (\langle \mu, a_{ij}\rangle)$; here we write A_0' for the vector space of linear (not necessarily bounded) functionals $A_0 \to \mathbb{C}$. This is a homomorphism, and so $\pi(\epsilon)$ is a (not necessarily orthogonal) projection. If $\pi(\epsilon)\xi = 0$ then $\pi(\mu)\xi = \pi(\mu\epsilon)\xi = 0$ for all μ. So $\sum_j \langle \mu, a_{ij}\rangle \xi_j = 0$ for all i and μ, that is, $\sum_j a_{ij}\xi_j = 0$. As (2) holds, $\xi = 0$ and so $\pi(\epsilon) = I$ which shows (4).

Similar, if (3) then, if for all $\mu \in A_0'$ and $(\eta_j)_{j=1}^n \subseteq \mathbb{C}^n$ we have that $\sum_{i,j} \eta_j \xi_i \langle \mu, a_{ij}\rangle = 0$, then $\xi = 0$. Hence the linear span of

$$\Big\{ \sum_j \langle \mu, a_{ij}\rangle \eta_j : \mu \in A_0', \eta \in \mathbb{C}^n \Big\}$$

is all of $\mathbb{C}^n$. Again, this implies that $\pi(\epsilon) = I$, showing (4).

By the previous theorem, if (4) holds then

$$\sum_k \kappa(a_{ik}) a_{kj} = m(\kappa \otimes \iota)\Delta(a_{ij}) = \epsilon(a_{ij})1 = \delta_{i,j}1.$$

Similarly, $\sum_k a_{ik}\kappa(a_{kj}) = \delta_{i,j}1$ and so (5) holds. $\square$

Notice that the proof shows that condition (3) is equivalent to the homomorphism $A_0' \to \mathbb{M}_n$ being non-degenerate. Equivalent conditions are that the induced homomorphisms $A^* \to \mathbb{M}_n$ or $L^1(A) \to \mathbb{M}_n$ are non-degenerate. Theorem A.32 below shows that if the Haar state is faithful on A, then any non-degenerate homomorphism $L^1(A) \to \mathbb{M}_n$ arises from an invertible U in this way (that is, the hypothesis that each $a_{ij} \in A_0$ can be removed).

A.5 Automorphisms

We now study the "F-matrices" F^α more closely.

Proposition A.25. *For $\alpha, \beta \in I$, we have that*

$$\langle \varphi, u^\alpha_{ip}(u^\beta_{jq})^* \rangle = \delta_{\alpha,\beta}\delta_{i,j} \frac{(F^\alpha)^{-1}_{q,p}}{\mathrm{Tr}((F^\alpha)^{-1})}.$$

Proof. Consider the compact quantum group $(A, \sigma\Delta)$. Then $\{(u^\alpha)^* : \alpha \in I\}$ forms a complete set of unitary corepresentations for $(A, \sigma\Delta)$. Thus we can apply Proposition A.21 to find positive, invertible, trace 1 matrices G^α with

$$\langle \varphi, ((u^\alpha)^*_{pi})^* (u^\beta)^*_{qj} \rangle = \langle \varphi, u^\alpha_{ip}(u^\beta_{jq})^* \rangle = \delta_{\alpha,\beta}\delta_{i,j} G^\alpha_{q,p}.$$

The proof of Proposition A.21 shows that $1 \otimes (F^\alpha)^t = \overline{u^\alpha}(1 \otimes (F^\alpha)^t)(\overline{u^\alpha})^*$ and thus also that $1 \otimes (G^\alpha)^t = (\overline{u^\alpha})^*(1 \otimes (G^\alpha)^t)\overline{u^\alpha}$. Thus both $(F^\alpha)^t$ and $((G^\alpha)^{-1})^t$ intertwine $\overline{u^\alpha}$ (which is irreducible) and $((\overline{u^\alpha})^*)^{-1}$ (which is of the same dimension). Thus Proposition A.15 shows that $G^\alpha = \lambda (F^\alpha)^{-1}$ for some $\lambda \in \mathbb{C}$, which may be determined by the condition that G^α has trace 1. $\square$

Lemma A.26. *Let $T \in \mathbb{M}_n$ be such that $(1 \otimes T^{-1})\overline{u^\alpha}(1 \otimes T)$ is unitary. Then F^α is a scalar multiple of $\overline{T}T^t$, and $(F^\alpha)^{-1}$ intertwines u^α and the corepresentation $(\kappa^2(u^\alpha_{ij}))$.*

Proof. By Proposition A.19 there is an invertible $T \in \mathbb{M}_n$ with $v = (1 \otimes T^{-1})\overline{u^\alpha}(1 \otimes T)$ unitary. Thus

$$1 = vv^* = (1 \otimes T^{-1})\overline{u^\alpha}(1 \otimes T)(1 \otimes T^*)\overline{u^\alpha}^*(1 \otimes (T^{-1})^*),$$

and so $(1 \otimes TT^*) = \overline{u^\alpha}(1 \otimes TT^*)\overline{u^\alpha}^*$. Hence by the proof of Proposition A.21, $(F^\alpha)^t$ is a scalar multiple of TT^*, or equivalently, F^α is a scalar multiple of $\overline{TT^*} = \overline{T}T^t$.

Now, as v is unitary, $v^* = \kappa(v)$, where $\kappa(v)$ is the matrix $(\kappa(v_{ij}))_{i,j=1}^n$. So

$$\kappa(v)^t = \overline{v} = (1 \otimes \overline{T}^{-1}) u^\alpha (1 \otimes \overline{T}).$$

However, also

$$\kappa(v)^t = \big((1 \otimes T^{-1})\kappa(\overline{u^\alpha})(1 \otimes T)\big)^t = (1 \otimes T^t)\kappa^2(u^\alpha)(1 \otimes (T^{-1})^t),$$

here using that $(\kappa(\overline{u^\alpha})^t)_{i,j} = \kappa((u^\alpha_{ji})^*) = \kappa^2(u^\alpha_{ij})$. Thus

$$(1 \otimes (T^{-1})^t \overline{T}^{-1}) u^\alpha = \kappa^2(u^\alpha)(1 \otimes (T^{-1})^t \overline{T}^{-1}).$$

So conclude that $(F^\alpha)^{-1}$ intertwines u^α and $\kappa^2(u^\alpha)$. □

Notice that a corollary of this result is that T is determined up to a unitary matrix, and a scalar. Indeed, by rescaling, we may assume that $TT^* = \overline{F}^\alpha$. As $\overline{F}^\alpha$ is positive and invertible, there is a unique unitary[4] matrix U with $T = (\overline{F}^\alpha)^{1/2} U$.

rixkappa **Corollary A.27.** *The matrix* $\overline{(F^\alpha)^{-1}}$ *intertwines the corepresentations* $\overline{u^\alpha}$ *and* $((u^\alpha)^t)^{-1}$*, where of course* $(u^\alpha)^t$ *has matrix* $(u^\alpha_{j,i})$*.*

Proof. Using the properties of κ established in Theorem thm:ishopf A.23 we see that as u^α is unitary, for any i, j

$$\sum_k u^\alpha_{i,k}(u^\alpha_{j,k})^* = \delta_{i,j}1 = \sum_k (u^\alpha_{k,i})^* u^\alpha_{k,j} \implies \sum_k u^\alpha_{i,k}\kappa(u^\alpha_{k,j}) = \delta_{i,j}1 = \sum_k \kappa(u^\alpha_{i,k})u^\alpha_{k,j}$$
$$\implies \sum_k u^\alpha_{k,j}\kappa^{-1}(u^\alpha_{i,k}) = \delta_{i,j}1 = \sum_k \kappa^{-1}(u^\alpha_{k,j})u^\alpha_{i,k}.$$

This implies that $((u^\alpha)^t)^{-1}$ is the matrix $(\kappa^{-1}(u^\alpha_{j,i})) = (\kappa((u^\alpha_{j,i})^*)^*) = (\kappa^2(u^\alpha_{i,j})^*)$. By the previous result, for all i, j,

$$\sum_k (F^\alpha)^{-1}_{i,k} u^\alpha_{k,j} = \sum_k \kappa^2(u^\alpha_{i,k})(F^\alpha)^{-1}_{k,j} \implies \sum_k \overline{(F^\alpha)^{-1}_{i,k}}(u^\alpha_{k,j})^* = \sum_k \kappa^2(u^\alpha_{i,k})^* \overline{(F^\alpha)^{-1}_{k,j}},$$

that is, $\overline{(F^\alpha)^{-1}}$ intertwines $\overline{u^\alpha}$ and $((u^\alpha)^t)^{-1}$ as required. □

In particular, this result shows that

$$(u^\alpha)^t \overline{(F^\alpha)^{-1}}\,\overline{u^\alpha} = \overline{(F^\alpha)^{-1}}, \qquad \overline{u^\alpha}\,\overline{F^\alpha}(u^\alpha)^t = \overline{F^\alpha}.$$

Let us think about how the "F-matrices" are effected by unitary equivalence. Let v be a unitary corepresentation equivalent to u^α, so by Proposition prop:schur A.15, there is a unitary intertwiner, X say. Thus $v = (1 \otimes X^*)u^\alpha(1 \otimes X)$. Then

$$\langle \varphi, v^*_{ip}v_{jq}\rangle = \sum \langle \varphi, \big(\overline{X_{ai}}u^\alpha_{ab}X_{bp}\big)^* \overline{X_{cj}}u^\alpha_{cd}X_{dq}\rangle = \sum X_{ai}\overline{X_{bp}X_{cj}}X_{dq}\langle\varphi, (u^\alpha_{ab})^*u^\alpha_{cd}\rangle$$
$$= \sum X_{ai}\overline{X_{bp}}X_{bq}\overline{X_{cj}}F^\alpha_{c,a} = \delta_{p,q}\big(X^*F^\alpha X\big)_{j,i}.$$

Thus the "F-matrix" associated with v is $X^*F^\alpha X$.

[4]For any vector x we have that $\|T^*x\|^2 = (TT^*x|x) = (\overline{F}^\alpha x|x) = \|(\overline{F}^\alpha)^{1/2}x\|^2$. So there is a well-defined isometry U with $UT^* = (\overline{F}^\alpha)^{1/2}$. As $(\overline{F}^\alpha)^{1/2}$ is invertible, U is everywhere defined and invertible, so a unitary. Then $TU^* = (\overline{F}^\alpha)^{1/2}$ so $T = (\overline{F}^\alpha)^{1/2}U$ as required.

For each α, set $t_\alpha = \mathrm{Tr}((F^\alpha)^{-1})$. As F^α is a positive invertible matrix, $t_\alpha > 0$. For each $z \in \mathbb{C}$, define a linear map by

$$f_z : A_0 \to \mathbb{C}; \qquad u^\alpha_{i,j} \mapsto ((F^\alpha)^{-z})_{i,j} t_\alpha^{-z/2}.$$

Here we use the functional calculus to define $F^z = \exp(z \log F)$ for a positive matrix F.

Because $(T^*FT)^z = T^*F^zT$ for any positive invertible F, unitary T and $z \in \mathbb{C}$, we see that f_z is well-defined, independent of the choice of irreducibles $\{u^\alpha\}$ (of course, t_α is well-defined).

As is standard, we turn A^* into a Banach algebra, with the product denoted by $*$, by

$$\langle \mu * \lambda, a\rangle = \langle \mu \otimes \lambda, \Delta(a)\rangle \qquad (a \in A, \mu, \lambda \in A^*).$$

Notice that $*$ is also well-defined on the algebraic dual of A_0, because of Theorem thm:ishopf A.23. Define $\sigma : A_0 \to A_0$ by

$$\sigma(a) = f_1 * a * f_1 = (f_1 \otimes \iota \otimes f_1)\Delta^2(a) \qquad (a \in A).$$

tpropsfz **Proposition A.28.** *The maps f_z have the following properties:*

1. *For $a \in A_0$, the map $\mathbb{C} \to \mathbb{C}; z \mapsto f_z(a)$ is entire and of exponential growth in the right half-plane (meaning that there are $C > 0$ and $d \in \mathbb{R}$ with $|f_z(a)| \leq Ce^{d\,\mathrm{Re}(z)}$ when $\mathrm{Re}(z) > 0$);*
2. *$f_0 = \epsilon$ the counit, and $f_z * f_w = f_{z+w}$ for all $z, w \in \mathbb{C}$;*
3. *for $a, b \in A_0$, we have that $\langle \varphi, ab\rangle = \langle \varphi, b\sigma(a)\rangle$.*

Proof. (1) follows almost immediately. To see this easily, suppose that F^α is diagonal (as we may, as F^α is positive, so diagonalisable). Then, if $t > 0$, the function $z \mapsto t^{-z} = e^{-z\log t}$ is of exponential growth in the right half-plane, as $|e^{-zs}| = e^{-s\,\mathrm{Re}(z)}$ for $s \in \mathbb{R}$. As any $a \in A_0$ is a finite linear combination of elements of the form $u^\alpha_{i,j}$ the result follows.

For (ii), first notice that $F^0 = \exp(0) = I$ for any positive matrix F, and so $f_0(u^\alpha_{ij}) = \delta_{i,j}$ as required to show that $f_0 = \epsilon$. Now notice that

$$\begin{aligned}\langle f_z * f_w, u^\alpha_{ij}\rangle &= \sum_k \langle f_z, u^\alpha_{ik}\rangle\langle f_w, u^\alpha_{kj}\rangle = \sum_k (F^\alpha)^{-z}_{ik}(F^\alpha)^{-w}_{kj} t_\alpha^{-z/2} t_\alpha^{-w/2} \\ &= \big((F^\alpha)^{-z}(F^\alpha)^{-w}\big)_{ij} t_\alpha^{-(z+w)/2} = (t_\alpha^{1/2}F^\alpha)^{-(z+w)}_{ij} = \langle f_{z+w}, u^\alpha_{ij}\rangle.\end{aligned}$$

For (iii), notice that

$$\sigma(u^\alpha_{ij}) = \sum_{k,l} \langle f_1, u^\alpha_{il}\rangle u^\alpha_{lk} \langle f_1, u^\alpha_{kj}\rangle = t_\alpha^{-1} \sum_{k,l} (F^\alpha)^{-1}_{il}(F^\alpha)^{-1}_{kj} u^\alpha_{lk}.$$

Thus, if $a = u^\alpha_{ip}$ and $b = (u^\beta_{jq})^*$, then

$$\begin{aligned}\langle \varphi, b\sigma(a)\rangle &= t_\alpha^{-1} \sum_{k,l} (F^\alpha)^{-1}_{il}(F^\alpha)^{-1}_{kp} \langle \varphi, (u^\beta_{jq})^* u^\alpha_{lk}\rangle = t_\alpha^{-1}\delta_{\alpha,\beta} \sum_l (F^\alpha)^{-1}_{il}(F^\alpha)^{-1}_{qp} F^\alpha_{lj} \\ &= t_\alpha^{-1}\delta_{\alpha,\beta}\delta_{i,j}(F^\alpha)^{-1}_{qp} = \langle \varphi, ab\rangle,\end{aligned}$$

where the final equality uses Proposition prop:haarotherway A.25. Then (iii) follows by linearity. □

Theorem A.29. *Each f_z is a character on A_0. Furthermore:*

1. *$f_z(1) = 1$, $f_z(\kappa(a)) = f_{-z}(a)$ and $f_z(a^*) = \overline{f_{-\overline{z}}(a)}$ for all $a \in A, z \in \mathbb{C}$;*
2. *$\kappa^2(a) = (f_1 \otimes \iota \otimes f_{-1})\Delta^2(a)$ for each $a \in A$.*

The characters f_z are uniquely determined by the properties shown in the previous proposition.

Proof. We first claim that σ is a character. For $a,b,c\in A_0$,

$$\langle\varphi,abc\rangle=\langle\varphi,c\sigma(ab)\rangle=\langle\varphi,bc\sigma(a)\rangle=\langle\varphi,c\sigma(a)\sigma(b)\rangle.$$

As this holds for all c, we conclude that $\sigma(ab)=\sigma(a)\sigma(b)$ as required. Then, for $a\in A_0$,

$$\langle f_2,a\rangle=\langle f_1*f_0*f_1,a\rangle=\langle\epsilon,\sigma(a)\rangle,$$

and so $f_2=\epsilon\circ\sigma$ is a character. Then $f_4=f_2*f_2=(f_2\otimes f_2)\circ\Delta$ is a character, as Δ is a homomorphism. Similarly, f_{2k} is a character for all $k\in\mathbb{N}$. Thus, for $a,b\in A_0$, the functions

$$z\mapsto f_z(ab),\quad\text{and}\quad z\mapsto f_z(a)f_z(b)$$

are both entire and of exponential growth in the right-half plane, and are equal on $\{2k:k\in\mathbb{N}\}$. Thus they agree everywhere (see woro1 [5]Page 228). So f_z is a character for all z.

In this argument, we have only used the properties of the family (f_z) established by the previous proposition. Then σ is uniquely determined by condition (3) (of the previous proposition), and so $f_2=\epsilon\circ\sigma$ is uniquely determined. Thus also f_{2k} is uniquely determined, given condition (2). But then (f_z) is uniquely determined by the same complex analysis argument.

Clearly $f_z(1)=1$ for all z. Then

$$f_z\kappa=(f_z\kappa\otimes\epsilon)\Delta=(f_z\kappa\otimes f_0)\Delta=(f_z\kappa\otimes f_z\otimes f_{-z})\Delta^2=(f_z\otimes f_z\otimes f_{-z})\big((\kappa\otimes\iota)\Delta\otimes\iota\big)\Delta.$$

That f_z is a character means that $f_zm=f_z\otimes f_z$, and so

$$f_z\kappa=(f_z\otimes f_{-z})\big(m(\kappa\otimes\iota)\Delta\otimes\iota\big)\Delta=f_z(1)(\epsilon\otimes f_{-z})\Delta=f_{-z},$$

as required. Notice now that if $t>0$ then $\overline{t^{\overline{z}}}=\overline{\exp(\overline{z}\log t)}=\exp(z\log t)=t^z$. Being careful, this shows that $(F^{\overline{z}})^*=F^z$ for a positive invertible matrix F. Thus

$$f_z((u^\alpha_{ij})^*)=f_z(\kappa(u^\alpha_{ji}))=f_{-z}(u^\alpha_{ji})=(F^\alpha)^z_{j,i}t_\alpha^{z/2}=\overline{(F^\alpha)^{\overline{z}}_{i,j}t_\alpha^{\overline{z}/2}}=\overline{f_{\overline{z}}(u^\alpha_{ij})},$$

which completes showing (1).

By Lemma lem:fmatrixkappa A.26, $(1\otimes(F^\alpha)^{-1})u^\alpha(1\otimes F^\alpha)=\kappa^2(u^\alpha)$, and so

$$\kappa^2(u^\alpha_{ij})=\sum_{k,l}(F^\alpha)^{-1}_{i,k}u^\alpha_{k,l}F^\alpha_{l,j}t_\alpha^{-1/2}t_\alpha^{1/2}=(f_1\otimes\iota\otimes f_{-1})\Delta^2(u^\alpha_{ij}),$$

which shows (2). □

Proposition A.30. *For $z,z'\in\mathbb{C}$, define a map $\rho_{z,z'}:A_0\to A_0$ by*

$$\rho_{z,z'}=(f_{z'}\otimes\iota\otimes f_z)\Delta^2.$$

Then $\rho_{z,z'}$ is an algebra homomorphism, and for any $w,w'\in\mathbb{C}$,

$$\begin{aligned}\rho_{0,0}&=\iota, & \rho_{z,z'}\circ\rho_{w,w'}&=\rho_{z+w,z'+w'},\\ \varphi\circ\rho_{z,z'}&=\varphi, & \rho_{z,z'}\circ *&=*\circ\rho_{-\overline{z},-\overline{z}'}\\ \rho_{z,z'}\circ\kappa&=\kappa\circ\rho_{-z',-z}, & \Delta\circ\rho_{z,z'}&=(\rho_{w,z'}\otimes\rho_{z,-w})\circ\Delta,\\ \kappa^{-1}&=\rho_{1,-1}\circ\kappa.\end{aligned}$$

Proof. These are all immediate from the previous proposition. □

In particular, define two one-parameter families of $*$-homomorphisms of A_0 by

$$\sigma_t=\rho_{it,it},\qquad\tau_t=\rho_{-it,it}\qquad(t\in\mathbb{R}).$$

These have analytic extensions to all of $\mathbb{C}$, and we see that $\sigma=\rho_{1,1}=\sigma_{-i}$ while $\kappa^2=\rho_{-1,1}=\tau_{-i}$. Also $\Delta\tau_t=(\tau_t\otimes\tau_t)\Delta$ and $\Delta\sigma_t=(\tau_t\otimes\sigma_t)\Delta$. It follows that (σ_t) is the modular automorphism group of φ, while (τ_t) is the scaling group of (A,Δ). Notice that $\rho_{z,z'}=\sigma_{-i(z+z')/2}\tau_{-i(z'-z)/2}$.

A.6 Slicing against coreps

We take a slight diversion, and follow [6, Section 4].

Proposition A.31. *Let $U \in M(A \otimes \mathcal{B}_0(H))$ be a unitary corepresentation, and let $\omega \in \mathcal{B}_0(H)^*$. Then:*

1. *Set $a = (\iota \otimes \omega)(U) \in A$. If $\varphi(aa^*) = 0$ then $a = 0$.*
2. *$(\iota \otimes \omega)(U) = 0$ if and only if $(\iota \otimes \omega)(U^*) = 0$.*

For any $a, b \in A$ fixed, we have that $(\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a)) = 0$ if and only if $(\iota \otimes \varphi)((1 \otimes b^*)\Delta(a)) = 0$.

Proof. By Proposition A.9, if B is the norm closure of $\{(c\varphi \otimes \iota)(U) : c \in A\}$, then B is a non-degenerate C*-algebra acting on H, and $U \in M(A \otimes B)$. In particular, we can find $b_0 \in B, \omega_0 \in \mathcal{B}_0(H)^*$ with $\omega = b_0\omega_0$.

For (1), for any $c \in A$, we have by Cauchy-Schwarz that $|\varphi(ac)|^2 \leq \varphi(aa^*)\varphi(c^*b) = 0$, and so $\langle (c\varphi \otimes \iota)(U), \omega \rangle = \langle c\varphi, a \rangle = 0$. Thus $\langle b, \omega \rangle = 0$ for all $b \in B$. As $U \in M(A \otimes B)$ we can find a bounded net (u_i) in $A \otimes B$ with $u_i \to U$ strictly. Then

$$a = (\iota \otimes \omega)(U) = (\iota \otimes \omega_0)(U(1 \otimes b_0)) = \lim_i (\iota \otimes \omega_0)(u_i(1 \otimes b_0)) = \lim_i (\iota \otimes \omega)(u_i) = 0,$$

as $u_i \in A \otimes B$.

For (2), suppose that $(\iota \otimes \omega)(U) = 0$. As just argued, this certainly implies that $(\iota \otimes \omega)(V) = 0$ for any $V \in M(A \otimes B)$. In particular, $(\iota \otimes \omega)(U^*) = 0$. Conversely, if $(\iota \otimes \omega)(U^*) = 0$ then $(\iota \otimes \omega)(U^*)^* = (\iota \otimes \omega^*)(U) = 0$, and so $0 = (\iota \otimes \omega^*)(U^*) = (\iota \otimes \omega)(U)^*$ as required.

Finally, follow Section A.1, as applied to some faithful representation of A, to form the left regular corepresentation U. Then Lemma A.4 combined with (2) gives immediately the final claim. □

Theorem A.32. *Suppose that φ is faithful. If $a \in A$ with $\Delta(a)$ in the algebraic tensor product of A with itself, then $a \in A_0$.*

Proof. Let $\Delta(a) = \sum_{i=1}^n a_i \otimes b_i$. For $b \in A$, notice that

$$(\iota \otimes \varphi)((1 \otimes b^*)\Delta(a)) = \sum_{i=1}^n \varphi(b^* b_i) a_i = \sum_{i=1}^n \langle b_i\varphi, b^* \rangle a_i.$$

Thus $(\iota \otimes \varphi)((1 \otimes b^*)\Delta(a)) = 0$ if and only if $b^* \in \ker(b_1\varphi) \cap \cdots \cap \ker(b_n\varphi)$. By the previous proposition, this is equivalent to $(\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a)) = 0$. In particular, we conclude that $\{(\iota \otimes \varphi)(\Delta(b^*)(1 \otimes a)) : b \in A\}$ is a finite-dimensional subspace of A.

Now let $b = u^\alpha_{i,j}$ to see that

$$\Big\{ \sum_k (u^\alpha_{i,k})^* \varphi((u^\alpha_{k,j})^* a) : \alpha \in I, 1 \leq i, j \leq n_\alpha \Big\}$$

is also a finite-dimensional subspace of A (actually, of A_0). As the set $\{u^\alpha_{i,j}\}$ is a basis for A_0, it follows that there is a finite subset $F \subseteq I$ such that

$$\varphi((u^\alpha_{k,j})^* a) = 0 \qquad (\alpha \not\in F, 1 \leq j, k \leq n_\alpha).$$

Using Proposition A.21, if we set $H_\alpha = \operatorname{lin}\{u^\alpha_{i,j}\xi_0 : 1 \leq i, j \leq n_\alpha\} \subseteq L^2(\varphi)$, then $L^2(\varphi)$ is the orthogonal direct sum of the finite-dimensional subspaces $\{H_\alpha : \alpha \in I\}$. We have just shown that $a\xi_0 \in \operatorname{lin}\{H_\alpha : \alpha \in F\}$. As φ is faithful, the GNS map $A \to L^2(\varphi); b \mapsto b\xi_0$ is injective, and so $a \in \operatorname{lin}\{u^\alpha_{i,j} : \alpha \in F\} \subseteq A_0$ as required. □

An example given in [2] shows that this result may fail if φ is not faithful.

A.7 Faithfulness of the Haar state

ith_haar

aithhopf **Proposition A.33.** *The restriction of φ to A_0 is a faithful state.*

Proof. Let $a \in A_0$ with $\langle \varphi, a^*a \rangle = 0$. By Cauchy-Schwarz, $\langle \varphi, a^*b \rangle = 0$ for all $b \in A_0$. Thus, if $a = \sum_{\alpha,i,j} \lambda^\alpha_{i,j} u^\alpha_{i,j}$ a finite linear combination, then taking $b = u^\beta_{p,q}$ shows that

$$0 = \sum_{i,j} \overline{\lambda^\beta_{i,j}} \delta_{j,q} F^\beta_{p,i} = \sum_i \overline{\lambda^\beta_{i,q}} F^\beta_{p,i}.$$

Again, as F^β is invertible, this shows that $\lambda^\beta = 0$ for all β, as required. □

Proposition A.34. *For any $a \in A$, we have that $\langle \varphi, a^*a \rangle = 0$ if and only if $\langle \varphi, aa^* \rangle = 0$. In particular:*

1. *$N_\varphi = \{a \in A : \langle \varphi, a^*a \rangle = 0\}$ is a two-sided closed ideal of A;*
2. *Let $(L^2(\varphi), \pi, \Lambda)$ be the GNS construction for φ. Then $\ker \Lambda = \ker \pi = N_\varphi$.*

Proof. Suppose $\langle \varphi, a^*a \rangle = 0$. By Cauchy-Schwarz, $\langle \varphi, a^*b \rangle = 0$ for all $b \in A$, in particular, for all $b \in A_0$. As A_0 is dense in A, we can find a sequence (a_n) in A_0 with $a_n^* \to a^*$ in norm. So

$$0 = \langle \varphi, a^*\sigma(b) \rangle = \lim_n \langle \varphi, a_n^*\sigma(b) \rangle = \lim_n \langle \varphi, ba_n^* \rangle = \langle \varphi, ba^* \rangle$$

where here we use Proposition prop:firstpropsfz A.28. As this holds for all $b \in A_0$, again by density, we conclude that $\langle \varphi, aa^* \rangle = 0$, as required.

That N_φ is a left ideal follows from the inequality $a^*x^*xa \leq \|x\|^2 a^*a$; clearly N_φ is closed. However, we have just shown that N_φ is self-adjoint, and hence is a right ideal as well, showing (1).

By definition, $\ker \Lambda = N_\varphi$. Suppose that $\pi(a) = 0$, so $0 = \pi(a)\Lambda(1) = \Lambda(a)$, so $a \in N_\varphi$. Conversely, if $a \in N_\varphi$ then for $b \in A$, as $abb^*a^* \leq \|b\|^2 aa^*$ and $\langle \varphi, aa^* \rangle = 0$, also $\langle \varphi, abb^*a^* \rangle = 0$, so also $\langle \varphi, b^*a^*ab \rangle = 0$, showing that $\pi(a)\Lambda(b) = 0$. As b was arbitrary, $\pi(a)\xi = 0$ for all $\xi \in H$, showing that $\pi(a) = 0$. Thus (2) holds. □

So we can form the quotient algebra $A_r = A/N_\varphi$, and let φ_r be the functional induced by φ on A_r; it follows that φ_r is a faithful state on A_r. Let $(L^2(\varphi), \pi, \Lambda)$ be the GNS construction for φ on A, and let (H_r, π_r, Λ_r) be the GNS construction for φ_r on A_r. Let $q : A \to A_r$ be the quotient map. By Proposition prop:haarfaithhopf A.33, we see that q restricts to an injection on A_0, and hence we can identify A_0 as a dense subalgebra of A_r.

Theorem A.35. *The map $\Lambda(a) \mapsto \Lambda_r(q(a))$ extends to an isometric isomorphism θ from $L^2(\varphi)$ to H_r. Then $\pi_r(q(a))\theta = \theta\pi(a)$ for all $a \in A$, and so $\pi(A), \pi(A_r)$ and A_r are all isometrically isomorphic.*

There is a unital $*$-homomorphism $\Delta_r : A_r \to A_r \otimes A_r$ with $(q \otimes q)\Delta = \Delta_r q$, and such that (A_r, Δ_r) becomes a compact quantum group. Δ_r restricts to Δ on A_0. The corepresentation theory of (A_r, Δ_r) agrees with that of (A, Δ).

Proof. As $\ker q = N_\varphi = \ker \Lambda$, the map θ is well-defined on $\Lambda(A)$. Then $\|\theta\Lambda(a)\|^2 = \langle \varphi_r, q(a^*a) \rangle = \langle \varphi, a^*a \rangle = \|\Lambda(a)\|^2$, and so θ is an isometry with dense range, and hence extends to an isometric isomorphism. Clearly θ intertwines $\pi_r q$ and π, and so we can identify $\pi(A)$ with $\pi_r(A_r) \cong A_r$.

We now use Proposition prop:corepgivescomult A.3. Use $\pi : A \to \mathcal{B}(L^2(\varphi))$ to form U, a unitary in $M(\pi(A) \otimes \mathcal{B}_0(L^2(\varphi))) \subseteq \mathcal{B}(L^2(\varphi) \otimes L^2(\varphi))$ with $(\pi \otimes \pi)\Delta(a) = U^*(1 \otimes \pi(a))U$ for $a \in A$. Using the isomorphism with H_r, we obtain a unitary $W \in \mathcal{M}(A_r \otimes \mathcal{B}_0(H_r))$ with $W^*(1 \otimes q(a))W = (\pi_r q \otimes \pi_r q)\Delta(a)$ for $a \in A$. Thus, if $a \in \ker q$, then $(q \otimes q)\Delta(a) = 0$ (as $\pi_r \otimes \pi_r$ injects on $A_r \otimes A_r$).

Then we can set $\Delta_r(a) = W^*(1 \otimes a)W$ for $a \in A_r$, and we see that $\Delta_r(q(a)) = (q \otimes q)\Delta(a)$, as required.

It is clear that Δ_r agrees with Δ on A_0. The statement about corepresentations follows as we can phrase everything in terms of A_0.[5] □

As an aside, from LCQG theorem, we define

$$W^*(\Lambda(a) \otimes \Lambda(b)) = (\Lambda \otimes \Lambda)(\Delta(b)(a \otimes 1)).$$

This is the same definition as given by Proposition A.3.

We can now also construct the von Neumann algebraic version of A_r, as $M = A_r''$ in $\mathcal{B}(L^2(\varphi))$. It is easy to see that we can extend Δ to a M by defining $\Delta(x) = W^*(1 \otimes x)W$ for $x \in M$ (σ-weak continuity shows that Δ does map into $M \overline{\otimes} M$, and that Δ is coassociative). We extend φ to M by identifying it with normal state $\omega_{\Lambda(1)}$.

Lemma A.36. *The extension of φ to M is a faithful normal state on M.*

Proof. We argue above. If $x \in M$ with $\varphi(x^*x) = 0$, then $x\Lambda(1) = 0$. We can find a net (a_i) in A_0 which converges strongly on x (by Kaplansky Density). Then, for $b, c \in A_0$,

$$\begin{aligned}\big(x\Lambda(\sigma(b))\big|\Lambda(c)\big) &= \lim_n \varphi(c^* a_n \sigma(b)) = \lim_n \varphi(bc^* a_n) = \lim_n \big(a_n\Lambda(1)\big|\Lambda(cb^*)\big) \\ &= \big(x\Lambda(1)\big|\Lambda(cb^*)\big) = 0.\end{aligned}$$

By density, $(x\xi|\eta) = 0$ for all $\xi, \eta \in L^2(\varphi)$, so $x = 0$. □

Theorem A.37. *Let $x \in M$ with $\Delta(x)$ in the algebraic tensor product of M with itself. Then $x \in A_0$.*

Proof. We copy the proof of Theorem A.32. To do so, we need to use a version of Proposition A.31, where $a \in M$ in the final claim. In turn, this follows from a version of Lemma A.4, which in turn follows from the construction of $W \in \mathcal{B}(L^2(\varphi) \otimes L^2(\varphi))$ as $W^*(\xi \otimes \Lambda(a)) = \Delta(a)(\xi \otimes \Lambda(1))$ for $a \in A, \xi \in L^2(\varphi)$. For $x \in M$, if (a_n) is a net in A converging strongly to x, then $\Delta(x)$ will be the strong limit of $\Delta(a_n)$, and $\Delta(x) = x\Delta(1) = \lim_n a_n \Delta(1) = \lim_n \Delta(a_n)$ in norm. Thus $W^*(\xi \otimes \Lambda(x)) = \Delta(x)(\xi \otimes \Lambda(1))$ for all $x \in M$, and the proof is complete. □

B Character theory

Much of this theory comes from [7, Section 5].

Definition B.1. Let $U = (U_{ij}) \in A \otimes M_n$ be a (finite-dimensional, unitary) corepresentation. Then the *character* of U is the element $\chi(U) = \chi_U = \sum_{i=1}^n U_{ii} \in A$.

If Tr denotes the (non-normalised) trace, then $\chi_U = (\iota \otimes \mathrm{Tr})U$, showing χ_U to be coordinate independent.

Lemma B.2. *Let U, V be corepresentations of A. Then $\chi(U \oplus V) = \chi(U) + \chi(V)$, $\chi(U \mathbin{\top\!\!\!\!\bigcirc} V) = \chi(U)\chi(V)$, $\chi(\overline{U}) = \chi(U)^* = \kappa(\chi(U))$. If U and V are equivalent of dimension n, then $\chi(U) = \chi(V)$ and $\epsilon(\chi(U)) = n$.*

[5] Should probably be more precise here– a target would be to prove: For $V \in M(A \otimes \mathcal{B}_0(H))$ a unitary corepresentation of A, clearly $(q \otimes \iota)V$ is a unitary corepresentation of A_r; we claim that this establishes a bijection between unitary corepresentations of A and of A_r.

Proof. We only prove the non-obvious claims. We may suppose that U is unitary, so then $\chi(\overline{U}) = \sum_i U_{ii}^* = \sum_i \kappa(U_{ii})$ and so $\chi(\overline{U}) = \chi(U)^* = \kappa(\chi(U))$. Similarly, $\epsilon(\chi(U)) = \sum_i \epsilon(U_{ii}) = n$. □

r_are_on **Proposition B.3.** *If U, V are irreducible (unitary) corepresentations, then $\varphi(\chi_U^* \chi_V) = \varphi(\chi_U \chi_V^*) = 1$ if U is equivalent to V, and equals 0 otherwise.*

Proof. This follows immediately from Proposition prop:fmatrices A.21 and Proposition prop:haarotherway A.25. □

Then, as for classical compact groups, knowing χ_U allows us to find how U is decomposed as irreducibles. To be precise, if we set $n_\alpha = \varphi(\chi_{u_\alpha}^* \chi_U)$, then

$$U \cong \bigoplus_\alpha (u^\alpha)^{\oplus n_\alpha}, \quad \chi_U = \sum_\alpha n_\alpha \chi(u^\alpha).$$

Furthermore, the space of intertwiners between U and itself has dimension $\sum_\alpha n_\alpha^2 = \varphi(\chi_U^* \chi_U)$.

Lemma B.4. *Assume diagonalised F-matrices.*[6] *Then $f_1(\chi_U) = f_{-1}(\chi_U) = \Lambda_\alpha$.*

Proof. Simply note that $f_z(\chi_U) = \sum_i (\lambda_i^\alpha)^z$ and so $f_1(\chi_U) = f_{-1}(\chi_U) = \Lambda_\alpha$. □

Notice that

$$\Delta(\chi_U) = \sum_i \Delta(U_{ii}) = \sum_{i,j} U_{ij} \otimes U_{ji},$$

and so $\Delta(\chi_U) = \sigma\Delta(\chi_U)$. Woronowicz says that this corresponds to the classical situation where characters are always invariant under inner-automorphisms.[7]

B.1 Woronowicz's question

Let $A_{\text{cen}} = \{a \in A : \Delta(a) = \sigma\Delta(a)\}$ and $A_{\text{cen}}^0 = A_0 \cap A_{\text{cen}}$.

Lemma B.5. *Let $a \in A_{\text{cen}}^0$. Then a is a finite linear combination of characters.*

Proof. As $a \in A_0$, we can write $a = \sum a_{\alpha,i,j} u_{ij}^\alpha$. Then

$$\Delta(a) = \sum a_{\alpha,i,j} u_{ik}^\alpha \otimes u_{kj}^\alpha = \sigma\Delta(a) = \sum a_{\alpha,i,j} u_{kj}^\alpha \otimes u_{ik}^\alpha.$$

Then for all β, p, q,

$$\sum_i a_{\beta,i,q} u_{ip}^\beta = \sum_j a_{\beta,p,j} u_{qj}^\beta.$$

But then looking at the $u_{r,s}^\gamma$ component shows that for all γ, p, q, r, s we have that

$$a_{\gamma,r,q}\delta_{s,p} = a_{\gamma,p,s}\delta_{r,q}.$$

So if $s \neq p$ then $a_{\gamma,p,s} = 0$, while taking $r = q$ and $s = p$ shows that $a_{\gamma,r,r} = a_{\gamma,s,s}$ for all r, s. So there are scalars b_α such that $a_{\alpha,i,j} = \delta_{i,j} b_\alpha$. Hence

$$a = \sum_\alpha b_\alpha \sum_i u_{ii}^\alpha = \sum_\alpha b_\alpha \chi(u^\alpha),$$

as required. □

Woronowicz asked:

- Is A_{cen}^0 dense in A_{cen}?
- Equivalently, is the span of characters dense in A_{cen}.

Again, if we believe that when $A = C(G)$ then A_{cen} is the space of functions invariant under inner-automorphisms (i.e. the space of "class functions") then this is true in the classical group case.

[6] Maybe we don't need to do this– but then we need to define the "quantum-dimension" somewhere!
[7] Can we expand?

C Diagonalisation

sec:diag

Recall (from Proposition prop:fmatrices A.21) that the F-matrices satisfy

$$\langle \varphi, (u^\beta_{ip})^* u^\alpha_{jq}\rangle = \delta_{\alpha,\beta}\delta_{p,q} F^\alpha_{j,i} \qquad (\alpha,\beta\in I, 1\le i,p\le n_\beta, 1\le j,q\le n_\alpha),$$

where F^α is a positive invertible matrix with trace 1.

Then we can find a unitary matrix X^α such that $(X^\alpha)^* F^\alpha X^\alpha$ is diagonal, say with diagonal entries $(\mu^{(\alpha)}_i) \subseteq (0,1]$, with $\sum_i \mu^{(\alpha)}_i = 1$.

Set $v^\alpha = (X^\alpha)^* u^\alpha X^\alpha$, a unitary corepresentation (unitarily) equivalent to u^α. Then

$$\begin{aligned}
\langle \varphi, (v^\beta_{i,p})^* v^\alpha_{j,q}\rangle &= \sum_{a,b,c,d} \langle \varphi, ((X^\beta)^*_{i,a} u^\beta_{a,b} X^\beta_{b,p})^* (X^\alpha_{j,c})^* u^\alpha_{c,d} X^\alpha_{d,q}\rangle \\
&= \delta_{\alpha,\beta} \sum_{a,b,c,d} X^\alpha_{a,i} \overline{X^\alpha_{b,p} X^\alpha_{c,j}} X^\alpha_{d,q} \langle \varphi, (u^\alpha_{a,b})^* u^\alpha_{c,d}\rangle = \delta_{\alpha,\beta} \sum_{a,b,c,d} X^\alpha_{a,i} \overline{X^\alpha_{b,p} X^\alpha_{c,j}} X^\alpha_{d,q} \delta_{b,d} F^\alpha_{c,a} \\
&= \delta_{\alpha,\beta} \sum_{a,c} X^\alpha_{a,i} \overline{X^\alpha_{c,j}} ((X^\alpha)^* X^\alpha)_{p,q} F^\alpha_{c,a} = \delta_{\alpha,\beta}\delta_{p,q} ((X^\alpha)^* F^\alpha X^\alpha)_{j,i} \\
&= \delta_{\alpha,\beta}\delta_{p,q}\delta_{i,j} \mu^{(\alpha)}_i .
\end{aligned}$$

We now use Proposition prop:haarotherway A.25. First note that $(X^\alpha)^*(F^\alpha)^{-1} X^\alpha$ is diagonal with entries (μ_i^{-1}). As before, set $t_\alpha = \mathrm{Tr}((F^\alpha)^{-1}) = \sum_i \mu_i^{-1}$. So we see that

$$\begin{aligned}
\langle \varphi, v^\beta_{i,p} (v^\alpha_{j,q})^*\rangle &= \sum_{a,b,c,d} \langle \varphi, (X^\beta)^*_{i,a} u^\beta_{a,b} X^\beta_{b,p} ((X^\alpha_{j,c})^* u^\alpha_{c,d} X^\alpha_{d,q})^*\rangle \\
&= \delta_{\alpha,\beta} \sum_{a,b,c,d} (X^\alpha)^*_{i,a} X^\alpha_{b,p} X^\alpha_{c,j} \overline{X^\alpha_{d,q}} \delta_{a,c} \frac{(F^\alpha)^{-1}_{d,b}}{t_\alpha} \\
&= \delta_{\alpha,\beta}\delta_{i,j} \sum_{b,d} X^\alpha_{b,p} (X^\alpha)^*_{q,d} \frac{(F^\alpha)^{-1}_{d,b}}{t_\alpha} \\
&= \delta_{\alpha,\beta}\delta_{i,j} \frac{((X^\alpha)^*(F^\alpha)^{-1} X^\alpha)_{q,p}}{t_\alpha} = \delta_{\alpha,\beta}\delta_{i,j}\delta_{p,q} (\mu^\alpha_p)^{-1} t_\alpha^{-1}.
\end{aligned}$$

Let $\lambda^\alpha_i = (\mu^\alpha_i)^{-1} t_\alpha^{-1/2}$, so that

$$\sum_i (\lambda^\alpha_i)^{-1} = (t^\alpha)^{1/2}, \quad \sum_i \lambda^\alpha_i = (t^\alpha)^{-1/2} t_\alpha = (t^\alpha)^{1/2}.$$

So with $\Lambda_\alpha = (t^\alpha)^{1/2}$, we see that

$$\langle \varphi, (v^\beta_{i,p})^* v^\alpha_{j,q}\rangle = \delta_{\alpha,\beta}\delta_{p,q}\delta_{i,j} \frac{1}{\lambda^\alpha_i \Lambda_\alpha}, \qquad \langle \varphi, v^\beta_{i,p} (v^\alpha_{j,q})^*\rangle = \delta_{\alpha,\beta}\delta_{i,j}\delta_{p,q} \frac{\lambda^\alpha_p}{\Lambda_\alpha}.$$

Thus, to recap, for the new family of unitary corepresentations (v^α), the associated "F-matrices" are diagonal, with entries $(\mu^{(\alpha)}_i)$ or equivalently, with entries $((\lambda^\alpha_i)^{-1}\Lambda_\alpha^{-1})$.

Thus this does agree with my PAMS paper.

Notice that Lemma lem:fmatrixkappa A.26 shows that

$$\frac{\delta_{i,j}}{\lambda^\alpha_i \Lambda_\alpha} = \sum_{k,l} (\overline{v^\alpha})_{i,k} \frac{\delta_{k,l}}{\lambda^\alpha_k \Lambda_\alpha} (\overline{v^\alpha})^*_{l,j} = \sum_k (v^\alpha_{i,k})^* v^\alpha_{j,k} \frac{1}{\lambda^\alpha_k \Lambda_\alpha}.$$

C.1 Decomposing the left-regular corepresentation

[8]

Form the left-regular corepresentation U as in Proposition prop:regcorep A.5, so that $U \in M(A \otimes \mathcal{B}_0(L^2(\varphi))$. Recall that $L^2(\varphi)$ is the GNS space for φ, with cyclic vector ξ_0. As at the start, $L^2(\varphi)$ decomposes as the orthogonal direct sum $L^2(\varphi) = \bigoplus_\alpha H_\alpha$ where H_α is the span of the vectors $(v^\alpha_{ij})^*\xi_0$. There is then a unitary

$$U_\alpha : H_\alpha \to \ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}; \quad (v^\alpha_{ij})^*\xi_0 \mapsto \sqrt{\frac{\lambda^\alpha_j}{\Lambda_\alpha}}\delta_i \otimes \delta_j.$$

Let $X = \bigoplus_\alpha U_\alpha : L^2(\varphi) \to \bigoplus_\alpha \ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$. Then as before,

$$(1\otimes X)U^*(1\otimes X^*)\big(\xi\otimes\delta^\alpha_i\otimes\delta^\alpha_j\big) = \sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}(1\otimes X)U^*(\xi\otimes(v^\alpha_{ij})^*\xi_0)$$

$$= \sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}\sum_k(1\otimes X)\big((v^\alpha_{ik})^*\xi\otimes(v^\alpha_{kj})^*\xi_0\big) = \sum_k(v^\alpha_{ik})^*\xi\otimes\delta^\alpha_k\otimes\delta^\alpha_j.$$

It follows that

$$(1\otimes X)U^*(1\otimes X^*) = \sum_{\alpha,i,k}(v^\alpha_{ik})^*\otimes e^\alpha_{ki}\otimes 1,$$

and so

$$(1\otimes X)U(1\otimes X^*) = \sum_{\alpha,i,k} v^\alpha_{ik}\otimes e^\alpha_{ik}\otimes 1.$$

Hence $(1\otimes X)U(1\otimes X^*)$ decomposes as (v^α) where each $v^\alpha \in M_n(A) = A\otimes M_n$ acts on the first component of $\ell^2_{n_\alpha}\otimes\ell^2_{n_\alpha}$.

C.2 The right regular representation

Again, let $(A,\sigma\Delta)$ be the opposite quantum group. Then φ remains the Haar weight for $(A,\sigma\Delta)$, and so we can form the regular representation U^{op} for $(A,\sigma\Delta)$, acting on $L^2(\varphi)$. It is easy to see that Y is a (unitary) corepresentation of (A,Δ) if and only if Y^* is a (unitary) corepresentation of $(A,\sigma\Delta)$. Set $V = (U^{\mathrm{op}})^*$, the *right regular representation* of (A,Δ). By definition,

$$V(\xi\otimes a\xi_0) = \sigma\Delta(a)(\xi\otimes\xi_0).$$

Thus we find that

$$(1\otimes X)V(1\otimes X^*)(\xi\otimes\delta^\alpha_i\otimes\delta^\alpha_j) = \sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}(1\otimes X)V(\xi\otimes(v^\alpha_{ij})^*\xi_0)$$

$$= \sqrt{\frac{\Lambda_\alpha}{\lambda^\alpha_j}}(1\otimes X)\sum_k(v^\alpha_{kj})^*\xi\otimes(v^\alpha_{ik})^*\xi_0 = \sum_k(v^\alpha_{kj})^*\xi\otimes\delta^\alpha_i\otimes\sqrt{\frac{\lambda^\alpha_k}{\lambda^\alpha_j}}\delta^\alpha_k.$$

Hence we see that

$$(1\otimes X)V(1\otimes X^*) = \sum_{\alpha,j,k}(v^\alpha_{kj})^*\otimes 1\otimes\sqrt{\frac{\lambda^\alpha_k}{\lambda^\alpha_j}}e^\alpha_{kj} = \sum_{\alpha,j,k}(\tau_{-i/2}(v^\alpha_{kj}))^*\otimes 1\otimes e^\alpha_{kj}$$

$$= \sum_{\alpha,j,k}R(v^\alpha_{jk})\otimes 1\otimes e^\alpha_{kj}.$$

[8] This is just a variant of the construction at the start, but where now we don't work with the *reduced* version of A.

C.3 Products of compact quantum groups

Let (A, Δ_A) and (B, Δ_B) be compact quantum groups, with Haar states φ_A and φ_B. We form a coproduct Δ on $A \otimes B$ by $\Delta = (1 \otimes \sigma \otimes 1)(\Delta_A \otimes \Delta_B)$. This is clearly a map $A \otimes B \to (A \otimes B) \otimes (A \otimes B)$. A tedious but easy calculation shows that this is cocommutative. We call $(A \otimes B, \Delta)$ the *product* of A and B.

Let U be a corepresentation of A, and V be a corepresentation of B, both acting on the same space H. We shall say that U and V *commute* if $U_{13}V_{23} = V_{23}U_{13}$. Under this assumption, if we set $X = U_{13}V_{23} = U \times V \in M(A \otimes B \otimes \mathcal{B}_0(H))$, then

$$\begin{aligned}(\Delta \otimes \iota)X &= (\iota \otimes \sigma \otimes \iota \otimes \iota)\big((\Delta_A \otimes \iota)(U)_{125}(\Delta_B \otimes \iota)(V)_{345}\big) = (\iota \otimes \sigma \otimes \iota \otimes \iota)\big(U_{15}U_{25}V_{35}V_{45}\big) \\ &= U_{15}U_{35}V_{25}V_{45} = U_{15}V_{25}U_{35}V_{45} = X_{13}X_{23}.\end{aligned}$$

Hence X is a corepresentation of $A \otimes B$.

In particular, set $B = A$ and let U, V be the left (respectively, right) regular representations. Thanks to the previous calculations, we see that U and V commute. Furthermore, by taking suitable $\mu \in A^* \odot A^* \subseteq (A \otimes A)^*$, we have

$$(\mu \otimes \iota)(U_{13}V_{23}) = e^{\alpha}_{ik} \otimes e^{\alpha}_{jl}$$

for any α, i, j, k, l. Hence $U_{13}V_{23}$ is irreducible. This is in some sense the analogue of the classical Peter-Weyl theorem.

- Can we show that every irrep of $A \times A$ occurs in this way?

C.4 "Central" elements

In a similar manner, we can show that UV (or VU) is a unitary corepresentation of (A, Δ); indeed

$$(\Delta \otimes \iota)(UV) = U_{13}U_{23}V_{13}V_{23} = U_{13}V_{13}U_{23}V_{23} = (UV)_{13}(UV)_{23}.$$

We shall say that $\eta \in L^2(\varphi)$ is *central* or *invariant* if $(UV)(\xi \otimes \eta) = \xi \otimes \eta$ for all ξ. It is easy to see that this is equivalent to

$$(\mu \otimes \iota)(UV)\eta = \mu(1)\eta \qquad (\mu \in A^*),$$

which also shows that the original definition is independent of the chosen faithful representation of A.

Lemma C.1. *The operator $p = (\varphi \otimes \iota)(UV)$ is a projection, and $\eta \in L^2(\varphi)$ is central if and only if $p\eta = \eta$.*

Proof. Let X be any (unitary) corepresentation of A, and for now, let $p = (\varphi \otimes \iota)X$. Applying $\varphi \otimes \iota \otimes \iota$ to the relation $(\Delta \otimes \iota)(X) = X_{13}X_{23}$ shows that $(1 \otimes p)X = 1 \otimes p$. Similarly, applying $\iota \otimes \varphi \otimes \iota$ yields that $X(1 \otimes p) = 1 \otimes p$. Then, applying $\varphi \otimes \iota$ gives that $p^2 = p$. Finally, as φ is a state and $\|X\| \leq 1$, it follows that $\|p\| \leq 1$, and so p must be an orthogonal projection.

Now say that η is invariant for X if $(\mu \otimes \iota)(X)\eta = \mu(1)\eta$ for all $\mu \in A^*$. It follows immediately that if η is invariant, then $p\eta = \eta$. Conversely, if $p\eta = \eta$ then

$$\xi \otimes \eta = (1 \otimes p)(\xi \otimes \eta) = X(1 \otimes p)(\xi \otimes \eta) = X(\xi \otimes \eta),$$

and so η is invariant.

The lemma now follows from the special case $X = UV$. □

- What happens if we instead use VU?

Dropping now the isomorphism X, we see that

$$p = (\varphi \otimes \iota)(UV) = \sum \varphi(v^\alpha_{ik}(v^\alpha_{lj})^*) e^\alpha_{ik} \otimes e^\alpha_{lj} \sqrt{\frac{\lambda^\alpha_l}{\lambda^\alpha_j}} = \sum_{\alpha,i,j} \frac{\sqrt{\lambda^\alpha_i \lambda^\alpha_j}}{\Lambda_\alpha} e^\alpha_{ij} \otimes e^\alpha_{ij}$$

$$= \sum_\alpha \sum_{i,j} \sqrt{\frac{\lambda^\alpha_i}{\Lambda_\alpha}} \sqrt{\frac{\lambda^\alpha_j}{\Lambda_\alpha}} \theta_{\delta^\alpha_i \otimes \delta^\alpha_i, \delta^\alpha_j \otimes \delta^\alpha_j} = \sum_\alpha \theta_{e_\alpha, e_\alpha},$$

say, where $e_\alpha = \sum_i \sqrt{\frac{\lambda^\alpha_i}{\Lambda_\alpha}} \delta^\alpha_i \otimes \delta^\alpha_i$. Here we use the obvious isomorphism $\mathbb{M}_{n_\alpha} \otimes \mathbb{M}_{n_\alpha} \cong \mathbb{M}_{n_\alpha \times n_\alpha}$. Notice that actually $e_\alpha = X(\chi^*_\alpha \xi_0)$ where χ_α is the character of v^α. It immediately follows that $p(e_\alpha) = e_\alpha$ for each α. Less obvious in this picture is that $X(\chi_\alpha \xi_0)$ is also invariant. We can prove this by observing that

$$V(\xi \otimes \chi_\alpha \xi_0) = \sum_i \sigma \Delta(v^\alpha_{ii})(\xi \otimes \xi_0) = \sum_{ij} v^\alpha_{ji} \xi \otimes v^\alpha_{ij} \xi_0 = \sum_j \Delta(v^\alpha_{jj})(\xi \otimes \xi_0) = U^*(\xi \otimes \chi_\alpha \xi_0).$$

Hence $UV(\xi \otimes \xi_0) = \xi \otimes \xi_0$, which is true for any ξ, showing that ξ_0 is invariant.

Corollary C.2. *The family (e_α) is an orthonormal basis for the subspace of central vectors in $L^2(\varphi)$.*

Proof. From Proposition prop:char_are_on B.3 we know that $\varphi(\chi_\alpha \chi^*_\beta) = \delta_{\alpha,\beta}$, showing that $(\chi^*_\alpha \xi_0) = (e_\alpha)$ is an orthonormal set. The result now follows given the form of p established above. □

C.4.1 Actions

In the commutative case, we can consider the action of G on itself given by $s \cdot t = sts^{-1}$. This gives a coaction $\alpha : C(G) \to C(G \times G)$ given by $\alpha(f)(s,t) = f(sts^{-1})$. This is a left coaction, as

$$(\iota \otimes \alpha)\alpha(f)(s,t,r) = \alpha(f)(s, trt^{-1}) = f(strt^{-1}s^{-1}) = (\Delta \otimes \iota)\alpha(f)(s,t,r).$$

First observe that $V\xi(s,t) = \xi(s,ts)$ for $\xi \in L^2(G \times G)$. Hence

$$V^*U^*(1 \otimes f)UV\xi(s,t) = V^*\Delta(f)V\xi(s,t) = \Delta(f)V\xi(s,ts^{-1}) = f(sts^{-1})V\xi(s,ts^{-1}) = \alpha(f)(s,t)\xi(s,t),$$

and so $V^*U^*(1 \otimes f)UV = \alpha(f)$.

However, in the compact quantum group case, this doesn't work, because in general $V^*\Delta(v^\alpha_{ij})V \in M(A \otimes \mathcal{B}_0(L^2(\varphi)))$ is not in $A \otimes A$. **How to show this? Is it true in the Kac case?**

C.5 Convolution product

We identify a dense subspace of $L^1(A)$ with a (dense) subspace of A by saying that $\omega \in L^1(A)$ corresponds to $a \in A$ when $\hat{\Lambda}(\lambda(\omega)) = a\xi_0$ in $L^2(\varphi)$. This is equivalent to

$$\big(a\xi_0 \big| b\xi_0\big) = \varphi(b^*a) = \langle \varphi, b^*a \rangle = \langle b^*, a\varphi \rangle = \big(\hat{\Lambda}(\lambda(\omega)) \big| b\xi_0\big) = \langle b^*, \omega \rangle \qquad (b \in A).$$

That is, if and only if $a\varphi = \omega$. Then, given $a, b \in A$ we define the *convolution product* $a * b$ to be (if it exists) the element c of A which corresponds to $(a\varphi) * (b\varphi) \in L^1(A)$, that is, $c\varphi = (a\varphi) * (b\varphi)$.

Let $a = v^\alpha_{ij}$ and $b = v^\beta_{kl}$. Then to find c, it is enough that

$$\langle (v^\gamma_{st})^*, c\varphi \rangle = \langle (v^\gamma_{st})^*, (a\varphi) * (b\varphi) \rangle$$

for all γ, s, t. However,

$$\langle (v^\gamma_{st})^*, (a\varphi) * (b\varphi)\rangle = \sum_r \varphi((v^\gamma_{sr})^* v^\alpha_{ij})\varphi((v^\gamma_{rt})^* v^\beta_{kl}) = \delta_{\alpha,\beta}\delta_{\alpha,\gamma}\delta_{s,i}\delta_{t,l}\delta_{j,k}\frac{1}{\Lambda_\alpha^2\lambda^\alpha_i\lambda^\alpha_j}$$
$$= \delta_{\alpha,\beta}\delta_{j,k}\frac{1}{\Lambda_\alpha\lambda^\alpha_j}\varphi((v^\gamma_{st})^* v^\alpha_{il}),$$

from which it follows that

$$v^\alpha_{ij} * v^\beta_{kl} = \delta_{\alpha,\beta}\delta_{j,k}\frac{1}{\Lambda_\alpha\lambda^\alpha_j}v^\alpha_{il}.$$

In particular,

$$\chi_\alpha * \chi_\beta = \delta_{\alpha,\beta}\sum_i \frac{1}{\Lambda_\alpha\lambda^\alpha_i}v^\alpha_{i,i}.$$

We could instead consider "twisted" convolution:

$$a \star \omega = \hat\lambda\big(\hat\omega[a\xi_0, \hat\Lambda(\lambda(\omega)^*)]\big).$$

Note quite sure where this goes– to copy the Dixmier idea, we'd need to find a "central bai" of such ω, and it's not clear when we can do this– at the very best, we'd need $\mathbb{G}$ coamenable!

(So, maybe, spend some time thinking about what happens when for $A(G)$ with G discrete??)

C.6 Todo

- We do know that WV (and/or VW) is a corep of $\mathbb{G}$, and so can talk about "central" $L^2(\mathbb{G})$ vectors. However, should show that this does not (in non-Kac case?) give a coaction of A (unfortunately).

- Then think about Dixmier's proof:

  - Does "convolution" of central elements of $L^2(\mathbb{G})$ make sense?
  - I think want something like central η such that there is a bounded operator T with $\Lambda(\hat\lambda(\hat\omega_{\xi,\eta})) = (\xi * \eta^*) = T(\xi)$? Then want these to give a bai...

D Commutative case

Suppose now that (A, Δ) is a compact quantum group with A commutative. We shall show that $A = C(G)$ for some compact group G, and that Δ is the canonical comultiplication.

As A is commutative, $A = C(G)$ for some compact Hausdorff space G. Then $\Delta : C(G) \to C(G \times G)$ is a $*$-homomorphism, and so corresponds to some map $G \times G \to G$. That Δ is coassociative means that G becomes a compact semigroup. At this stage, we remark that it is possible to use some compact semigroup theory to show directly that the cancellation conditions imply that G must be a compact group. Instead, we shall use some general theory.

Let $U \in M_n(C(G))$ be a finite-dimensional corepresentation, and let $\pi : G \to M_n$ be the associated continuous map, given by the isomorphism $M_n(C(G)) = C(G; M_n)$. Then U being a corepresentation corresponds to π being a homomorphism. We now adapt an argument from woro3 [7]. For any finite-dimensional unitary representation $\pi : G \to U(n)$ (where $U(n)$ is the n-dimensional unitary group) we note that $\pi(G)$ is a compact sub-semigroup of $U(n)$. If $A \in \pi(G)$ then by compactness, we can find a sequence $n(i)$ of naturals with $n(i+1) > n(i)+1$, and with $A^{n(i)} \to B$ as $i \to \infty$. Notice that $B \in \pi(G)$. Then set $m(i) = n(i+1) - (n(i)+1) > 0$, so that

$$A^{m(i)} = A^{n(i+1)}(A^{-1})^{n(i)+1} \to BB^{-1}A^{-1}.$$

Hence $A^{-1} \in \pi(G)$, and so $\pi(G)$ is a compact subgroup of $U(n)$.

By following the general theory, we find a dense Hopf $*$-algebra $P(G)$ inside $C(G)$; we see that $P(G)$ is precisely the collection of coefficients of finite-dimensional unitary representations of G.

Proposition D.1. *We have that G is a compact group, and the counit ϵ and antipode κ extend to $C(G)$ with the usual definitions coming from the group structure of G.*

Proof. That $P(G)$ is dense in $C(G)$ means that $P(G)$ separates the points of G; that is, for $s, t \in G$ distinct, there is a unitary representation $\pi : G \to U(n)$ with $\pi(s) \neq \pi(t)$.

Consider the collection $\mathcal{N}$ of all subsets $N_\pi = \{s \in G : \pi(s) = 1\}$ where π is a finite-dimensional unitary representation. Then each N_π is a non-empty compact set, as $\pi(G)$ is a compact group. Then $\mathcal{N}$ has the finite-intersection property, and $N_{\pi_1} \cap \cdots \cap N_{\pi_n} = N_\pi$ where $\pi = \pi_1 \oplus \cdots \oplus \pi_n$. So $\bigcap \mathcal{N}$ is non-empty, and thus there is some $e_G \in G$ with $e_G \in N_\pi$ for all π. As such π separate points, e_G is unique. Then, for any π and $t \in G$, we find that $\pi(te_G) = \pi(t) = \pi(e_G t)$, so by the separation of points property, e_G is the identity of G.

Now fix $t_0 \in G$. For each π there is at least one $t \in G$ with $\pi(t) = \pi(t_0)^{-1}$ so that $\pi(tt_0) = \pi(t_0 t) = \pi(e_G)$. Again by a finite-intersection property argument, we can show that there is at least one such t that works for all π. Then separation of points shows that t is unique, and that $t = t_0^{-1}$. So G is a group.

The defining properties of ϵ and κ now easily show that, for $f \in P(G)$, we have $\epsilon(f) = f(e_G)$, and $\kappa(f)(s) = f(s^{-1})$ for $s \in G$. These maps obviously extend by continuity to $C(G)$. ☐

The Haar state φ corresponds to a Borel probability measure, ds, on G. That φ is left and right invariant means that

$$\int_G f(st)\ ds = \int_G f(ts)\ ds = \int_G f(s)\ ds \qquad (t \in G, f \in C(G)).$$

Then by uniqueness, ds must be the Haar measure on G. We quickly remind the reader why ds has full support (equivalently, why φ is faithful). Towards a contradiction, suppose that $\varphi(f) = 0$ for some non-zero positive $f \in C(G)$. Then there is a non-empty open set U with $|U| = 0$. Then all (left) translates of U have zero measure; but as G is a group, these cover G, so by compactness, there is a finite subcover, and hence $|G| = 0$, contradiction. So φ is faithful. Hence A is already reduced, and we can identify $L^2(G)$ with the GNS space for φ.

Let U be a (unitary) corepresentation, and consider the contragradient corepresentation $\overline{U}$, corresponding to $\overline{\pi}$. Then

$$\overline{\pi}(s) = \sum_{i,j=1}^{n} u_{ij}^*(s) e_{ij} = \sum_{i,j=1}^{n} \overline{u_{ij}(s)} e_{ij} = \overline{\pi(s)},$$

where for $x \in M_n$, we again denote by $\overline{x} = (x^*)^t = (x^t)^*$ the matrix obtained by pointwise conjugation of complex numbers. As A is commutative, it is clear that U unitary (respectively, invertible) implies also that $\overline{U}$ is unitary (respectively, invertible), and so Proposition A.19 prop:conjunitary becomes a triviality in this case.

From Lemma A.26 lem:fmatrixkappa we see that each "F-matrix" is a scalar multiple of the identity, and so in particular diagonal. Taking the normalisation that $\mathrm{Tr}(F^\alpha) = \mathrm{Tr}((F^\alpha)^{-1})$, we must have that $F^\alpha = I_{n_\alpha}$, and so $\Lambda_\alpha = n_\alpha$, for all α. Then each character f_z is equal to the counit, and the scaling group (and of course the modular group) is trivial. Hence $\kappa = R$ the unitary antipode.

D.1 Some formulae

The GNS construction for φ has the concrete form that $H = L^2(G)$, the map $\Lambda : C(G) \to L^2(G)$ is formal identification of functions, and $\pi : C(G) \to \mathcal{B}(L^2(G))$ is such that $\pi(f)$ is the operator given by multiplication by f. Then $Jf(s) = \overline{f(s)}$ for $s \in G, f \in L^2(G)$ and $\hat{J}(f)(s) = \overline{f(s^{-1})}$. Also

$$W \in \mathcal{B}(L^2(G \times G)); \quad W\xi(s,t) = \xi(s, s^{-1}t) \qquad (\xi \in L^2(G \times G), s, t \in G).$$

Let (v^α) be a complete family of pairwise non-equivalent irreducible unitary corepresentations, with associated unitary representations (π_α). Then we identify $\ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$ with a subspace of $L^2(G)$ via

$$\delta_i^\alpha \otimes \delta_j^\alpha \mapsto \sqrt{n_\alpha}\overline{v_{ij}^\alpha}.$$

Then identifying $L^2(G)$ with $\bigoplus \ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$ we again find that

$$W = \big(w_\alpha\big) = \Big(\sum_{i,j} v_{ij}^\alpha \otimes e_{ij} \otimes 1\Big) \in \mathcal{B}\Big(L^2(G) \otimes \bigoplus \ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}\Big).$$

The left-regular representation is $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$ given by

$$\lambda(\omega) = (\omega \otimes \iota)(W); \quad \lambda(\omega)(f) = \omega * f \qquad (\omega \in L^1(G), f \in L^2(G)),$$

that is, $\lambda(\omega)$ is the operator of left convolution by ω. In the above picture,

$$\lambda(\omega) = \big((\omega \otimes \iota)w_\alpha\big) \in \bigoplus_\alpha \mathbb{M}_{n_\alpha} \otimes \mathbb{M}_{n_\alpha},$$

where for each α,

$$(\omega \otimes \iota)w_\alpha = \sum_{ij} \langle v_{ij}^\alpha, \omega\rangle e_{ij} \otimes 1 = \int_G \omega(s)\pi_\alpha(s) ds \otimes 1.$$

So as usual, as a C*-algebra, $C_r^*(G)$ is isomorphic to $\bigoplus_n \mathbb{M}_{n_\alpha}$, but when concretely acting on $L^2(G)$, we have to remember that each factor $\mathbb{M}_{n_\alpha}$ acts with multiplicity n_α; here I have chosen to write this as $e_{ij} \otimes 1$, whereas classical sources usually add an "n_α" term to indicate multiplicity.

Let's just check this:

$$\begin{aligned}
\lambda(\omega)(\delta_i^\alpha \otimes \delta_j^\alpha) \leftrightarrow n_\alpha^{1/2}\lambda(\omega)(\overline{v_{ij}^\alpha}) &= n_\alpha^{1/2} \int_G \omega(s)\overline{v_{ij}^\alpha}(s^{-1}t)\ ds \\
&= n_\alpha^{1/2} \int_G \omega(s)\overline{\sum_k v_{ik}^\alpha(s^{-1})v_{kj}^\alpha(t)}\ ds = n_\alpha^{1/2} \int_G \omega(s) \sum_k v_{ki}^\alpha(s)\overline{v_{kj}^\alpha(t)}\ ds \\
&\leftrightarrow \sum_k \int_G \omega(s)v_{ki}^\alpha(s)\ ds\ \delta_k^\alpha \otimes \delta_j^\alpha = \int_G \omega(s)\pi_\alpha(s)\delta_i^\alpha \otimes \delta_j^\alpha.
\end{aligned}$$

From general LCQG theory, it's easy[9] to see that $\hat{\Lambda}(\lambda(\omega)) = \omega$ for $\omega \in L^1(G) \cap L^2(G)$. From above, we find that the weight on $C_r^*(G) \cong \bigoplus_\alpha \mathbb{M}_{n_\alpha}$ is

$$\hat{\varphi}\big((x_\alpha)\big) = \sum_\alpha n_\alpha \operatorname{Tr}(x_\alpha),$$

[9] We have $(\hat{\Lambda}(\lambda(\omega))|\Lambda(a)) = \langle a^*, \omega\rangle = \int_G \omega(s)\overline{a(s)}\ ds$ and as $\Lambda(a) = a$ under formal identification of functions $C(G) \subseteq L^2(G)$ the result follows.

where $\mathrm{Tr} : \mathbb{M}_{n_\alpha} \to \mathbb{C}$ is the usual trace $\mathrm{Tr}(x) = \sum_{i=1}^{n_\alpha} x_{ii}$. Then

$$\begin{aligned}\big(\hat\Lambda((x_\alpha))\big|\hat\Lambda((y_\alpha))\big) &= \sum_\alpha n_\alpha \,\mathrm{Tr}(y_\alpha^* x_\alpha) = \sum_\alpha n_\alpha \sum_{ij} \overline{y_{ij}^\alpha} x_{ij}^\alpha \\ &= \sum_\alpha \Big(\sum_{ij} \sqrt{n_\alpha} x_{ij}^\alpha \delta_i^\alpha \otimes \delta_j^\alpha \Big| \sum_{kl} \sqrt{n_\alpha} x_{kl}^\alpha \delta_k^\alpha \otimes \delta_l^\alpha \Big).\end{aligned}$$

Hence there is an isomorphism $H_{\hat\varphi} \to \bigoplus_\alpha \ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$,

$$\hat\Lambda((x_\alpha)) \mapsto \Big(\sum_{ij} \sqrt{n_\alpha} x_{ij}^\alpha \delta_i^\alpha \otimes \delta_j^\alpha \Big).$$

Under this, for $\omega \in L^1(G) \cap L^2(G)$,

$$\omega = \hat\Lambda(\lambda(\omega)) \mapsto \Big(\sum_{ij} \sqrt{n_\alpha} \langle v_{ij}^\alpha, \omega\rangle \delta_i^\alpha \otimes \delta_j^\alpha \Big).$$

If we identify $\ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha}$ with the space of Hilbert-Schmidt operators on $\ell^2_{n_\alpha}$, then the α-component of $\hat\Lambda(\lambda(\omega))$ is precisely $\sqrt{n_\alpha} \int_G \omega(s)\pi_\alpha(s)\, ds$. We need to be a little careful: here

$$\ell^2_{n_\alpha} \otimes \ell^2_{n_\alpha} \ni \delta_i \otimes \delta_j \mapsto e_{ij} \in \mathcal{HS}(\ell^2_{n_\alpha}),$$

where $e_{ij} : \delta_k \mapsto \delta_{j,k}\delta_i$ and so $e_{ij} = \theta_{\delta_i,\delta_j}$.

D.2 The Fourier algebra

As usual, $L^1(\hat A)$ is the Fourier algebra $A(G)$. Let $\xi, \eta \in L^2(G)$ and let $\hat\omega_{\xi,\eta} \in A(G)$ be the functional

$$VN(G) \mapsto \mathbb{C}; \quad x \mapsto (x\xi|\eta).$$

Now, as $VN(G) \cong \prod \mathbb{M}_{n_\alpha}$ it follows that $A(G) \cong \ell^1 - \bigoplus \mathbb{T}_{n_\alpha}$, an ℓ^1-direct sum of trace-class spaces.

Let us introduce some notation. For a Hilbert space H, let $\omega_{\xi,\eta} \in \mathcal{B}(H)_*$ be $x \mapsto (x\xi|\eta)$. Then let $\theta_{\xi,\eta}$ be the (rank-one) operator $\gamma \mapsto (\gamma|\eta)\xi$. Then the map $\omega_{\xi,\eta} \mapsto \theta_{\xi,\eta}$ extends to the identification of $\mathcal{B}(H)_*$ with the trace-class operators $\mathcal{T}(H)$. For $x \in \mathcal{B}(H)$ we have $\mathrm{Tr}(x\theta_{\xi,\eta}) = \mathrm{Tr}(\theta_{\xi,\eta}x) = (x\xi|\eta) = \langle x, \omega_{\xi,\eta}\rangle$.

When $H = \ell^2_n$, as usual we have $e_{ij} = \theta_{\delta_i,\delta_j}$ and so $\omega_{ij} = \omega_{\delta_i,\delta_j} = e_{ij}$ as a trace-class operator. Then

$$\langle e_{ij}, \omega_{kl}\rangle = \mathrm{Tr}(e_{ij}e_{kl}) = \delta_{jk}\delta_{il}.$$

So ω_{kl} sends e_{lk} to 1, and all the other matrix units to 0. (This is sometimes called "trace-duality" to distinguish it from "parallel-duality").

Suppose $\xi, \eta \in L^1(G) \cap L^2(G)$ so that $\xi = \hat\Lambda(\lambda(\xi))$ and the same for η. Then

$$\begin{aligned}\langle e_{ij}^\alpha, \hat\omega_{\xi,\eta}\rangle &= \big(e_{ij}^\alpha \hat\Lambda(\lambda(\xi))\big|\hat\Lambda(\lambda(\eta))\big) = \hat\varphi\big(\lambda(\eta)^* e_{ij}^\alpha \lambda(\xi)\big) \\ &= n_\alpha \,\mathrm{Tr}\,\big(\pi_\alpha(\eta)^* e_{ij}^\alpha \pi_\alpha(\xi)\big) = n_\alpha \,\mathrm{Tr}\,\big(e_{ij}^\alpha \pi_\alpha(\xi)\pi_\alpha(\eta)^*\big).\end{aligned}$$

Hence, using that the integrated form of π_α is a $*$-homomorphism $L^1(G) \to \mathbb{M}_{n_\alpha}$,

$$\hat\omega_{\xi,\eta} = (\omega_\alpha) \in \ell^1 - \bigoplus_\alpha \mathbb{T}_{n_\alpha} \qquad \omega_\alpha = n_\alpha \int_G (\xi * \eta^*)(s)\pi_\alpha(s)\, ds.$$

Here $\eta^*(s) = \overline{\eta(s^{-1})}$ and $\xi * \eta^*$ is the convolution product (again, this reflects the use of the Takesaki–Tatsumma, aka quantum-group, embedding of $A(G)$ into $C_0(G)$, not the Eymard embedding).

D.3 Contragradient representations

For each α consider the contragradient $\overline{v_\alpha}$. We have that

$$(\overline{v_\alpha})_{ij}(s) = \overline{v^\alpha_{ij}(s)} = \overline{\pi_\alpha(s)_{ij}} = \pi_\alpha(s^{-1})_{ji}.$$

Let $\overline{\pi}_\alpha$ be the induced representation, which in this (commutative) situation is unitary. We can have two situations: either $\overline{\pi}_\alpha$ is equivalent to π_α, or it is not.

Example D.2. If $G = SU(2)$ then it's well-known that for each n there is exactly one equivalence class of irreducible representations of dimension n. Hence here $\overline{\pi}_\alpha$ is always equivalent to π_α.

Example D.3. If G is abelian, then every irreducible representation is one-dimensional, and so is a continuous character $\alpha : G \to \mathbb{T}$. Then $\overline{\alpha}$ is just $\overline{\alpha}(s) = \overline{\alpha(s)}$. Then observe equivalence of one-dimensional representations corresponds exactly to genuine equality of functions $G \to \mathbb{T}$. Then $\alpha = \overline{\alpha}$ if and only if $\alpha(s) \in \{1, -1\}$ for all s.

D.4 Todo

Maybe try to write-down the coproduct (and/or product on $A(G)$) using the "Fusion-rules"?? Try to write down the antipode on $VN(G)$??

E Completions of the Hopf algebra

It somewhat folklore that the Hopf $*$-algebra $\mathcal{A}$ can be completed to give back $C(\mathbb{G})$ or $C^u(\mathbb{G})$. We justified this (at the reduced level) in Section sec:faith_haar A.7.

However, there are some subtle points here, going back to Woronowicz and especially highlighted by Dijkhuizen and Koornwinder. The issues is that in general a (unital) $*$-algebra $\mathcal{A}$ need not have any interesting C*-algebra completion. Let $\mathcal{A}^+$ be the positive cone generated by elements of the form $\{a^*a : a \in \mathcal{A}\}$. Then a linear map $\phi : \mathcal{A} \to \mathbb{C}$ is positive if $\phi(\mathcal{A}^+) \subseteq [0, \infty)$ and is a state if additionally $\phi(1) = 1$. If ϕ is a state on $\mathcal{A}$ then we can form the pre-GNS space (H, ξ_0). Indeed, the Cauchy-Schwarz inequality is enough to show that $N_\phi = \{a \in \mathcal{A} : \phi(a^*a) = 0\}$ is a left ideal in $\mathcal{A}$ (compare tak1 [3, Chapter I, Lemma 9.6] for example), and so we define $H = \mathcal{A}/N_\phi$, let ξ_0 be the equivalence class of 1, so that we can identify the equivalence class of a with $a\xi_0$, and then equip H with the inner-product $(a\xi_0|b\xi_0) = \phi(b^*a)$. Note that we have not completed H and so H is only a pre-Hilbert space.

Then for $a \in \mathcal{A}$ define $\pi(a) : H \to H$ by $\pi(a)(b\xi_0) = (ab)\xi_0$. That N_ϕ is a left ideal shows that $\pi(a)$ is well-defined; clearly $\pi(a)$ is linear and adjointable, in the sense that

$$\big(\pi(a)b\xi_0\big|c\xi_0)\big) = \big(b\xi_0\big|\pi(a^*)c\xi_0)\big) \qquad (a, b, c \in \mathcal{A}).$$

So the only missing piece of the usual GNS construction is whether $\pi(a)$ is bounded, and hence extends to the completion of H. For a C*-algebra this is a subtle point going back to the early days of the axiomatisation of the subject.

The following can be found in Dijkhuizen and Koornwinder.

Proposition E.1. *Let $\mathcal{A}$ be the Hopf $*$-algebra associated to a CQG (A, Δ). Then if $\pi : \mathcal{A} \to \mathcal{L}(H_0)$ is a $*$-map into the adjointable linear maps on an inner-product space H_0, then π is bounded, and so extends to a $*$-homomorphism $A \to \mathcal{B}(H)$ where H is the completion of H_0.*

Proof. Let (u_{ij}) be a finite-dimensional unitary corepresentation of A, so each $u_{ij} \in \mathcal{A}$. As $\sum_k u_{ki}^* u_{kj} = \delta_{ij} 1$, for $\xi \in H_0$, and any i, j,

$$\|\xi\|^2 = (\xi|\xi) = \sum_k (\pi(u_{ki}^* u_{ki})\xi|\xi) = \sum_k (\pi(u_{ki})\xi|\pi(u_{ki})\xi) \geq \|\pi(u_{ji})\xi\|^2.$$

It follows that $\|\pi(u_{ij})\| \leq 1$ for all i, j. As $\mathcal{A}$ is spanned by such elements, we have shown that $\pi(a)$ is bounded for all $a \in \mathcal{A}$. □

As such, for any state ϕ on $\mathcal{A}$ we can find a Hilbert space H, a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ and $\xi \in H$ such that $\phi(a) = (\pi(a)\xi|\xi)$ for all $a \in \mathcal{A}$. So states on $\mathcal{A}$ biject with states on the universal C*-algebra completion of $\mathcal{A}$, namely $C^u(\mathbb{G})$.

F Do we need to be so careful?

In the section on von Neumann algebras, we seemingly used the Hopf $*$-algebra quite a bit– this is equivalent to using that the Haar state is KMS. Here we present some examples to show that *some sort of condition* is needed.

F.1 Counter-example

We find a C*-algebra A which admits a faithful state, but such that in the GNS representation, the state is not faithful on A''.

The following was suggested to us by Narutaka Ozawa[10]

Let $A = C([0,1], \mathbb{M}_2)$. Let $C \subseteq [0,1]$ be a closed set with empty interior but positive (Lebesgue) measure. For example, let (ϵ_n) be a sequence in $(0,1)$ with $\sum_n \epsilon_n < 1/2$, let (q_n) be an enumeration of the rationals in $[0,1]$, and let $C = [0,1] \setminus \bigcup_n (q_n - \epsilon_n, q_n + \epsilon_n)$.

Define a state ϕ on A by

$$\phi(a) = \int_C a(x)_{11} \ dx + \frac{1}{2} \int_{[0,1]\setminus C} a(x)_{11} + a(x)_{22} \ dx.$$

Here a is a continuous function $[0,1] \to \mathbb{M}_2$, and $a(x)_{ij}$ is the (i,j)th entry of the matrix $a(x)$.

Now, $a \geq 0$ if and only if $a(x) \geq 0$ for all x, which implies that $a(x)_{11}, a(x)_{22} \geq 0$. So ϕ is positive, and faithful because $[0,1] \setminus C$ is dense and open. Clearly $\phi(1) = 1$, so ϕ is a state.

For $a, b \in A$ the pre-inner-product induced by ϕ is

$$(a|b) = \phi(b^* a) = \int_C a(x)_{11}\overline{b(x)_{11}} + a(x)_{21}\overline{b(x)_{21}} \ dx + \frac{1}{2} \int_{[0,1]\setminus C} \sum_{i,j} a(x)_{ij}\overline{b(x)_{ij}} \ dx.$$

Let μ_1 be the measure of $[0,1]$ given by

$$\int f \ d\mu_1 = \int_C f + \frac{1}{2} \int_{[0,1]\setminus C} f.$$

Let μ_2 be $1/2$ of Lebesgue measure, restricted to $[0,1] \setminus C$. As Lebesgue measure dominates both μ_1 and μ_2, it's easy to see that μ_1, μ_2 are regular measures. Then the GNS space for ϕ can thus be identified with

$$\mathbb{M}_{2,1}(L^2(\mu_1)) \oplus \mathbb{M}_{2,1}(L^2(\mu_2)),$$

thought of as column vectors, with A acting by matrix multiplication, and then $C([0,1])$ acting by pointwise multiplication, in the obvious way. The cyclic vector is $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \oplus \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. To ease notation, let the GNS space be $H_1 \oplus H_2$, and let $\pi_i : A \to \mathcal{B}(H_i)$ be the resulting representations.

[10] See `http://mathoverflow.net/questions/93295/separating-vectors-for-c-algebras/93383♯93383`

Lemma F.1. *Let A be a C^*-algebra, $\pi_1 : A \to \mathcal{B}(H_1)$ a non-degenerate representation, let $H_2 \subseteq H_1$ be an invariant subspace, and let $\pi_2 : A \to \mathcal{B}(H_2)$ be the restriction of π_1. Let $\pi : A \to \mathcal{B}(H_1 \oplus H_2)$ be the direct sum of π_1 with π_2. Then $\pi(A)'' = \{(T,S) : T \in \pi_1(A)'', S = T|_{H_2}\}$ acting diagonally on $H_1 \oplus H_2$, a von Neumann algebra which is isomorphic to $\pi_1(A)''$.*

Proof. As π_1 is non-degenerate, so is π_2, and hence so is π. So we need to compute the σ-weak closure of $\pi(A)$. On bounded sets this agrees with the strong closure, and from this is it obvious that $\pi(A)''$ has the stated form. □

Notice that in our case $L^2(\mu_2)$ is a subspace of $L^2(\mu_1)$ if we identify $\xi \in L^2(\mu_2)$ with $\xi\chi_{[0,1]\setminus C} \in L^2(\mu_1)$.

Let $\mathfrak{A}$ be the commutant of $C([0,1])$ in $\mathcal{B}(L^2(\mu_1))$. Then $\pi_1(A)'$ consists of matrices $\begin{pmatrix} T & 0 \\ 0 & T \end{pmatrix}$ with $T \in \mathfrak{A}$. Thus $\pi_1(A)'' = \mathbb{M}_2(\mathfrak{A}')$. So we need to compute the bicommutant of $C([0,1])$ in $\mathcal{B}(L^2(\mu_1))$. By duality arguments, and (for example) Lusin's theorem, this is $L^\infty(\mu_1) \cong L^\infty([0,1])$.

Thus $\pi(A)'' \cong L^\infty([0,1])$. However, the cyclic vector for the GNS construction yields the state

$$\tilde{\phi}(a) = \int a_1 1 \; d\mu_1 + \int a_2 2 \; d\mu_2,$$

which is not faithful (there are measurable, non-continuous functions supported on C which are not zero almost everywhere).

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