The group algebra derivation problem

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Abstract

We present a short, self-contained (and somewhat idiosyncratic) presentation of the recent simplification to the proof of the $L^1(G)$ derivation problem: every derivation $d : L^1(G) \to L^1(G)$ is given by an inner-derivation at a point of $M(G)$.

1 Introduction

Let $\mathcal{A}$ be a (Banach) algebra and E be an $\mathcal{A}$-bimodule. A *derivation* $d : \mathcal{A} \to E$ is a linear map such that $d(ab) = a \cdot d(b) + d(a) \cdot b$ for all $a, b \in \mathcal{A}$. For $x \in E$, we define an *inner derivation* by $\delta_x : \mathcal{A} \to E; a \mapsto a \cdot x - x \cdot a$.

Let G be a locally compact group, and consider the group algebra $L^1(G)$. Recall that $L^1(G)$ is an ideal in $M(G)$, the measure algebra of G. A problem going back to Williamson, asked around 1965 (see the introduction to [8]), asks whether every derivation $d : L^1(G) \to L^1(G)$ is of the form $d = \delta_\mu$ for some $\mu \in M(G)$. We remark that the ideas of [7] shows that any derivation $L^1(G) \to L^1(G)$ is automatically bounded. Indeed, if $d : L^1(G) \to M(G)$ is a derivation, then by Cohen factorisation, $L^1(G) = \{ab : a, b \in L^1(G)\}$ and so $d(L^1(G)) \subseteq L^1(G) \cdot M(G) + M(G) \cdot L^1(G) = L^1(G)$, so we can also consider derivations $L^1(G) \to M(G)$.

Work of Johnson, Sinclair and Ringrose showed that this conjecture is true for many classes of groups; in particular, discrete, SIN, amenable and connected groups. Further results were obtained by Ghahramani, Runde and Willis. Finally, in 2008, Losert solved the general case. Recently, in 2010, Bader, Gelander and Monod provided a remarkably simple proof of the main results of Losert's paper. This note is to provide a short, self-contained proof of the derivation problem, based upon this recent simplification.

2 Reformulation using G-spaces

We follow [5]. A G-space is a locally compact space X on which G acts by homeomorphisms. That is, we have a continuous map $G \times X \to X; (s, x) \mapsto s \cdot x$, with $s \cdot (t \cdot x) = st \cdot x$ and $e \cdot x = x$, where $e \in G$ is the unit. We have an associated action on $M(X)$ given by

$$\langle s \cdot \mu, f \rangle = \int_X f(s \cdot x) \, d\mu(x) \qquad (s \in G, f \in C_0(X), \mu \in M(X)).$$

A *crossed homomorphism* is a map $\Phi : G \to M(X)$ which is continuous when $M(X)$ has the weak*-topology, and with $\Phi(st) = \Phi(s) + s \cdot \Phi(t)$ for $s, t \in G$. Notice that then $\Phi(e) = \Phi(t) - e \cdot \Phi(t) = 0$. We call Φ *bounded* if $\sup_s \|\Phi(s)\| < \infty$. Finally, Φ is *principal* if there exists $\mu \in M(X)$ with $\Phi(s) = s \cdot \mu - \mu$ for each $s \in G$.

Consider the special case when $X = G$ and G acts by $s \cdot x = sxs^{-1}$ for $s, x \in G$. Given a derivation $d : L^1(G) \to L^1(G)$, there is a derivation $D : M(G) \to M(G)$ which extends d, and which may be defined by setting

$$\langle D(\mu), a \cdot f\rangle = \langle d(\mu a), f\rangle - \langle d(a), f \cdot \mu\rangle \qquad (\mu \in M(G), a \in L^1(G), f \in C_0(G)).$$

See [6, Section 1.d] or [3, Theorem 5.6.34]. Here $M(G)$ (and hence $L^1(G)$) acts on $C_0(G)$ by convolution. Then we define $\Phi : G \to M(G)$ by $\Phi(s) = D(\delta_s)\delta_{s^{-1}}$. When D is bounded, so is Φ. Then, for $s, t \in G$,

$$\Phi(st) = D(\delta_s)\delta_{tt^{-1}s^{-1}} + \delta_s D(\delta_t)\delta_{t^{-1}s^{-1}} = \Phi(s) + s \cdot \Phi(t),$$

so that Φ is a crossed homomorphism. If Φ is principal, then there exists $\mu \in M(G)$ such that, for $s \in G$,

$$D(\delta_s)\delta_{s^{-1}} = \Phi(s) = \delta_s\mu\delta_{s^{-1}} - \mu \implies D(\delta_s) = \delta_s\mu - \mu\delta_s.$$

Hence D is inner, and so also d is inner.

3 Crossed homomorphisms are principal

We now follow [1], where crossed homomorphisms are called *cocycles*. We continue to use our terminology.

Let (E, d) be a metric space, and let $A \subseteq E$ be a bounded subset. The *circumradius* of A is the quantity

$$\rho_E(A) = \inf\big\{r \geq 0 : \exists\, x \in E \text{ such that } d(x, a) \leq r \ (a \in A)\big\}.$$

The *Chebyshev centre* of A is

$$C_E(A) = \big\{c \in E : d(c, a) \leq \rho_E(A) \ (a \in A)\big\}.$$

Note that $C_E(A)$ might be empty! However, if $c \in C_E(A)$, then by definition of $C_E(A)$, we see that $\sup_{a\in A} d(c, a) \leq \rho_E(A)$. But then by the definition of $\rho_E(A)$, we must actually have that $\sup_{a\in A} d(c, a) = \rho_E(A)$.

Notice that we can write

$$C_E(A) = \bigcap_{r>\rho_E(A)} C^r_E(A) \quad \text{where} \quad C^r_E(A) = \bigcap_{a\in A}\{x \in E : d(a, x) \leq r\}.$$

So, when E is a normed space and d is given by the norm, we have that each $C^r_E(A)$ is the intersection of closed balls, and so is closed and convex. As A is bounded, so is $C^r_E(A)$. Hence also $C^r_E(A)$ is bounded, closed and convex. If $E = F^*$ is a dual Banach space, then each $C^r_E(A)$ is the intersection of weak*-compact sets, and so is weak*-compact. Thus $C_E(A)$ is also weak*-compact. Furthermore, each $C^r_E(A)$ is non-empty, and as $C^r_E(A) \subseteq C^s_E(A)$ for $r \leq s$, it follows by compactness that $C_E(A)$ is also non-empty.

We say that E is an *L-embedded Banach space* if there is a closed subspace $E_0 \subseteq E^{**}$ such that $E^{**} = E \oplus E_0$, and such that, for $x \in E$ and $c_0 \in E_0$, we have that $\|x + x_0\| = \|x\| + \|x_0\|$. It is classical that any L^1 space is L-embedded; in fact, this is true for the predual of any von Neumann algebra, see [9, Chapter III, Theorem 2.14].

When dealing with fixed points, there is little point considering complex scalars; so we shall restrict to real vector spaces for now. When K is a convex set, an *affine* map $T : K \to K$ satisfies

$$T\big(\lambda x + (1 - \lambda)y\big) = \lambda T(x) + (1 - \lambda)T(y) \qquad (x, y \in K, 0 \leq \lambda \leq 1).$$

When E is a vector space, it is easy to see[1] that an affine map $T : E \to E$ with $T(0) = 0$ is actually linear. Thus a general affine map on E is a combination of a linear map with a translation. The Mazur-Ulam Theorem states that a surjective isometry between normed spaces is automatically affine.

Theorem 3.1. *Let E be an L-embedded Banach space, and let $A \subseteq E$ be a non-empty bounded subset. Let $\mathcal{G}$ be the collection of affine isometries[2] of E which preserve A. There exists $x \in E$ such that $T(x) = x$ for each $T \in \mathcal{G}$. Furthermore, we have that $\sup_{a\in A} \|x-a\| \leq \sup_{a\in A} \|y-a\|$ for any $y \in E$.*

Proof. Consider A as a subset of E^{**}. Then $C_{E^{**}}(A)$ is non-empty, weak*-compact, and convex. Let $c \in C_{E^{**}}(A)$, and write $c = c_E + c_0$ with $c_E \in E$ and $c_0 \in E_0$. For $a \in A$, as $c - a = (c_E - a) + c_0 \in E \oplus E_0$, we see that $\|c - a\| = \|c_E - a\| + \|c_0\|$. Thus

$$\rho_{E^{**}}(A) = \sup_{a\in A} \|c-a\| = \sup_{a\in A} \|c_E - a\| + \|c_0\| \geq \rho_E(A) + \|c_0\|.$$

However, clearly $\rho_{E^{**}}(A) \leq \rho_E(A)$, so $c_0 = 0$ and hence $c \in E$. We also see that $\rho_E(A) = \rho_{E^{**}}(A)$, from which it follows that $C_{E^{**}}(A) = C_E(A)$, and so in particular, $C_E(A)$ is non-empty, weakly compact, and convex.

As the definition of $C_E(A)$ only involves the metric space structure of E, we see that $C_E(A)$ is invariant under $\mathcal{G}$. By the Ryll-Nardzewski Theorem, there exists $x \in C_E(A)$ with $T(x) = x$ for each $T \in \mathcal{G}$. Then by definition, $\sup_{a\in A} \|x - a\| = \rho_E(A)$ which is the minimum (attained!) of $\sup_{a\in A} \|y - a\|$ as y varies over E. □

To be precise, we use the following version of the Ryll-Nardzewski Theorem, see [2, Theorem 10.8]: If E is a locally convex space, Q is a weakly compact convex subset of E and $\mathcal{P}$ is a non-contracting family of weakly continuous affine maps of Q to Q, then $\mathcal{P}$ has a fixed point in Q. That $\mathcal{P}$ is *non-contracting* means that for distinct points $x, y \in Q$, the closure of $\{T(x) - T(y) : T \in \mathcal{P}\}$ does not contain 0.

For us, E is a Banach space with the norm topology, and $\mathcal{P}$ consists of affine isometries of E. Thus, if $x, y \in Q$ are distinct, then $\|T(x) - T(y)\| = \|x - y\|$ for all $T \in \mathcal{P}$, and so 0 is not in the norm closure of $\{T(x) - T(y) : T \in \mathcal{P}\}$.

Theorem 3.2. *Let X be a G-space. Any bounded crossed homomorphism $\Phi : G \to M(X)$ is principal. Indeed, we can find $\mu \in M(X)$ with $\|\mu\| \leq \sup_{s\in G} \|\Phi(s)\|$ such that $\Phi(s) = s \cdot \mu - \mu$ for $s \in G$.*

Proof. We have that $M(X) = C_0(X)^*$ and so $M(X)$ is L-embedded. For $s \in G$ and $\mu \in M(X)$, define $s \circ \mu = s \cdot \mu + \Phi(s)$. Here, as before, for $f \in C_0(X)$, we have $\langle s \cdot \mu, f\rangle = \int_X f(s \cdot x)\, d\mu(x)$. Then, for $s, t \in G$,

$$s \circ (t \circ \mu) = s \circ (t \cdot \mu + \Phi(t)) = st \cdot \mu + s \cdot \Phi(t) + \Phi(s) = st \cdot \mu + \Phi(st) = st \circ \mu.$$

So $G \times M(G) \to M(G); (s, \mu) \mapsto s \circ \mu$ is an action (actually, this is not central to the proof!) Then, for $s \in G$, and $\mu, \lambda \in M(X)$,

$$\|s \circ \mu - s \circ \lambda\| = \|s \cdot \mu + \Phi(s) - s \cdot \lambda - \Phi(s)\| = \|s \cdot (\mu - \lambda)\| = \|\mu - \lambda\|.$$

Hence $M(X) \to M(X); \mu \mapsto s \circ \mu$ is an isometry.

[1] If $T(0) = 0$ then $T(\lambda x) = \lambda T(x)$ for $x \in E, \lambda \in [0, 1]$. For $\lambda > 0$, we have that $T(x/\lambda) = T(x)/\lambda \implies \lambda T(y) = T(\lambda y)$ for $y = x/\lambda \in E$. So T is positive homogeneous. Then $T(x + y) = T((2x)/2 + (2y)/2) = T(2x)/2 + T(2y)/2 = T(x) + T(y)$ so T is additive. Then $0 = T(x + (-x)) = T(x) + T(-x)$ so $T(-x) = -T(x)$; thus T is (real) linear.

[2] By Mazur-Ulam, we could also consider general bijective isometries.

Let $A = \{\Phi(s) : s \in G\}$. For $s, t \in G$, we have that $s \circ \Phi(t) = s \cdot \Phi(t) + \Phi(s) = \Phi(st) \in A$. So A is preserved under the action $\circ$, and so by the main theorem, there exists $\mu \in M(X)$ with $s \circ \mu = \mu$ for each s. That is, $s \cdot \mu + \Phi(s) = \mu$, or $\Phi(s) = s \cdot (-\mu) - (-\mu)$. Finally, we have that $\sup_{s\in G} \|\mu - \Phi(s)\| \leq \sup_{s\in G} \|\lambda - \Phi(s)\|$ for any $\lambda \in M(X)$. As $\Phi(e) = 0$, we have that $\|-\mu\| = \|\mu\| \leq \sup_{s\in G} \|\mu - \Phi(s)\| \leq \sup_{s\in G} \|\Phi(s)\|$ as claimed. $\square$

We finally come to our application. Let $d : L^1(G) \to L^1(G)$ be a (bounded) derivation, let $D : M(G) \to M(G)$ be the extension, and let $\Phi : G \to M(G)$ be the associated crossed homomorphism, given by $\Phi(s) = D(\delta_s)\delta_{s^{-1}}$. There exists $\mu \in M(G)$, with $\|\mu\| \leq \sup_s \|\Phi(s)\|$, and such that $\Phi(s) = s \cdot \mu - \mu$ for $s \in G$. As before, it follows that

$$d(a) = a \cdot \mu - \mu \cdot a, \quad (a \in L^1(G)) \qquad \|\mu\| \leq \sup_{s\in G} \|D(\delta_s)\| \leq \|d\|.$$

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