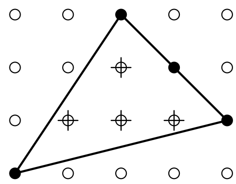
Pick's Theorem

We consider a grid (or "lattice") of points. A **lattice polygon** is a polygon all of whose corners (or "vertices") are at grid points. We will assume our polygons are **simple** so that edges cannot intersect each other, and there can be no "holes" in a polygon.

Let A be the area of a lattice polygon, let I be the number of grid points which are inside (in the "interior") of the polygon, and let B be the number of grid points which exactly hit an edge or corner (are on the "boundary") of the polygon.



In the example, $B = 4$ and $I = 4$ and $A = 5$.

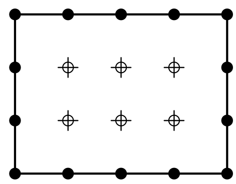
Pick's Theorem then says that $A = I + \frac{1}{2}B - 1$. Our aim is to prove this ("prove" means to give a convincing mathematical argument which works for *any* polygon– not just checking a few examples). We will do this by building the result up from simple, easy cases.

Rectangles. We'll first work with rectangles (where the sides are parallel to the grid). Let the rectangle have width w and height h, so $A = hw$.

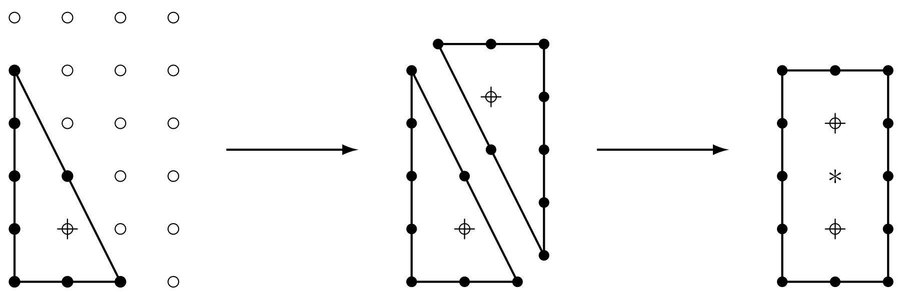
Carefully counting the interior points, we see that we have a grid $(w-1)$ wide and $(h-1)$ height, and so $I = (w-1)(h-1) = hw - h - w + 1$.

Carefully counting the boundary points (say starting in one corner and proceeding clockwise) we get $B = h + h + w + w$.

Hence $I + \frac{1}{2}B - 1 = (hw - h - w + 1) + (h + w) - 1 = hw = A$ which verifies Pick's Theorem.



Right-angled triangles. We now look at right-angled triangles where two sides are parallel to the grid. Counting the interior and boundary points here would be really hard (but not impossible!) so instead we use a "trick": if you copy the triangle, rotate it, and "glue" it to the original triangle along the hypotenuse, then we get a rectangle, and we can use the above result.



We now have some careful counting to do! Let A, I, B be for the triangle. We want to count B carefully, so write

$$B = B_s + B_h, \quad \text{where} \quad \begin{aligned} B_s &= \text{boundary points on the sides,} \\ B_h &= \text{boundary points on the hypotenuse.} \end{aligned}$$

In the example, $B_h = 1, B_s = 7$ and $B = 8$.

Let I_r, B_r, A_r be for the rectangle, so from above,

$$A_r = I_r + \frac{1}{2}B_r - 1.$$

If we count the boundary points of the rectangle, then we count the boundary points on the sides of both rectangles, but doing this, we *double count* the corners on the hypotenuse, so

$$B_r = 2B_s - 2.$$

We now count the interior points of the rectangle– doing this, we count the interior points of both triangles, and we count (but only once!) the points on the hypotenuse. So

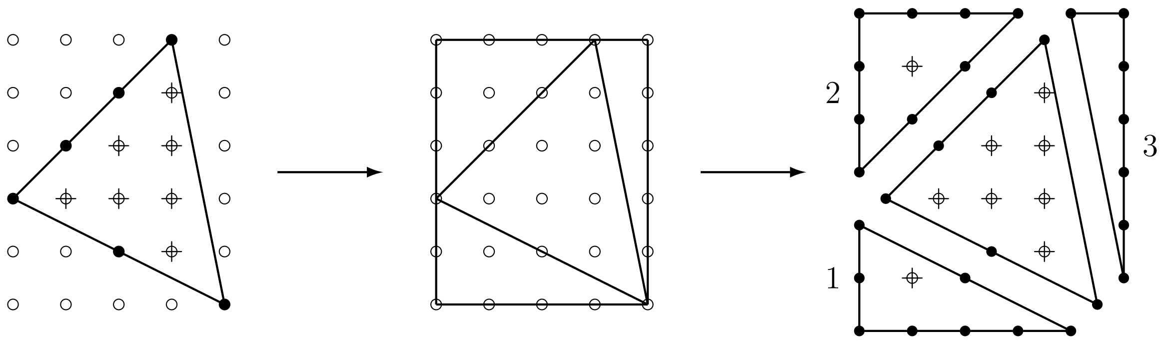
$$I_r = 2I + B_h.$$

Finally, the area of the rectangle is of course twice the area of the triangle. So

$$\begin{aligned} A = \frac{1}{2}A_r &= \frac{1}{2}\Big(I_r + \frac{1}{2}B_r - 1\Big) \\ &= \frac{1}{2}\Big(2I + B_h + \frac{1}{2}(2B_s - 2) - 1\Big) \\ &= \frac{1}{2}\Big(2I + B - 1 - 1\Big) \\ &= I + \frac{1}{2}B - 1, \end{aligned}$$

which is exactly what we want.

General triangles. We now consider general triangles (but of course with the corners on the grid points).



Again, let A, I, B be for the triangle. For $i = 1, 2, 3$ let A_i, B_i, I_i be the numbers for the added triangle numbered i. Finally, let A_r, B_r, I_r be the for big rectangle surrounding everything. So in our example above, $B_1 = 8, I_3 = 0, I = 7, I_r = 12$.

Let us could the boundary points:

$$B_1 + B_2 + B_3 - 3 = B_r + B - 3.$$

We get this because on the right-hand side of the equation, we are counting the boundaries of each added triangle. If we do this then we count the boundary points of the rectangle, and the boundary points of the original triangle. We end up counting the corner points of the original triangle twice, which we show by the "-3" on each side– but these conveniently cancel out.

We now count the interior points of the big rectangle:

$$I_1 + I_2 + I_3 + (B - 3) + I = I_r.$$

That is, we count the interior points of all the added triangles, then we add the boundary points of the original triangle (but we don't count the corner points– they aren't in the interior of the rectangle) and finally we count the interior points of the original triangle.

From what we learnt about, we know that

$$A_r = I_r + \frac{1}{2}B_r - 1, \qquad A_i = I_i + \frac{1}{2}B_i - 1.$$

We then put these together:

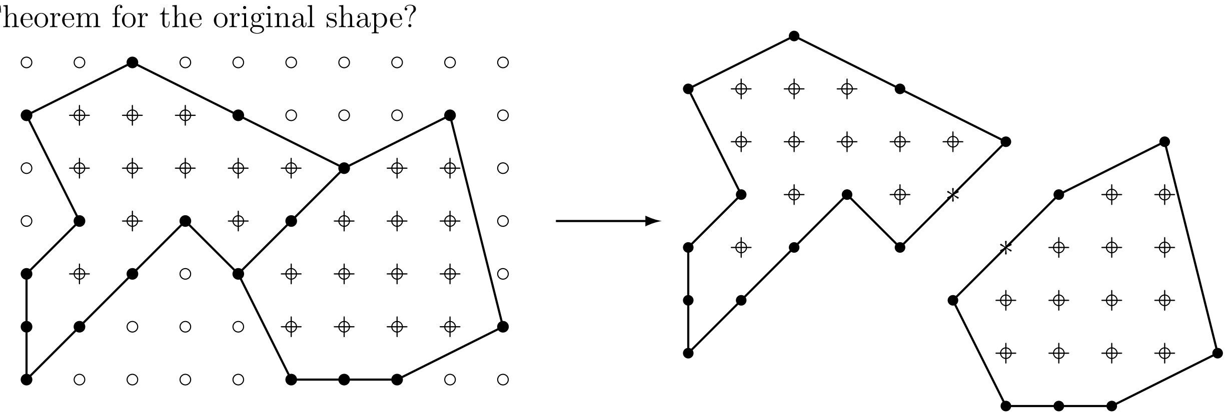
$$\begin{aligned}
A &= A_r - A_1 - A_2 - A_3 \\
&= \big(I_r + \frac{1}{2}B_r - 1\big) - \big(I_1 + \frac{1}{2}B_1 - 1\big) - \big(I_2 + \frac{1}{2}B_2 - 1\big) - \big(I_3 + \frac{1}{2}B_3 - 1\big) \\
&= 2 + \big(I_r - I_1 - I_2 - I_3\big) + \frac{1}{2}\big(B_r - B_1 - B_2 - B - 3\big) \\
&= 2 + \big(B - 3 + I\big) + \frac{1}{2}(-B) \\
&= I + \frac{1}{2}B - 1,
\end{aligned}$$

exactly as we want.

The general case. Finally we treat a general lattice polygon. The idea is, roughly, as follows:

- We know the result is true for any triangle.
- **Suppose** we have some way of combining two shapes.
- Then given any quadrilateral, we add a new edge and split it into two triangles. "Adding" the result for the triangles gives the result for the quadrilaterals.
- So now the result is true for any quadrilateral.
- Then given a pentagon, add an edge and split it into a quadrilateral and a triangle. "Adding" the results for the bits gives the result for pentagons.
- Now do hexagons, heptagon, octagons, etc. etc.

However, we better justify the 2nd step: having split a polygon into two pieces, and assuming that Pick's Theorem holds for each bit, how do we reassemble that information to prove Pick's Theorem for the original shape?



The strategy should be clear by now. Let A, I, B be for the big polygon, let A_1, I_1, B_1 be for the first smaller polygon, and A_2, I_2, B_2 for the second. Let X be the number of points on the new edge (marked with a $*$ in our example). Thus, if we count interior points, we get

$$I = I_1 + I_2 + X.$$

Counting boundary points is a bit trickier: if we count the boundary points of the two parts, then we will double count both the points counted by x, and we will double count the corner points for the next edge. Hence

$$B_1 + B_2 = B + 2x + 2.$$

If we assume that Pick's Theorem holds for the smaller polygons, then

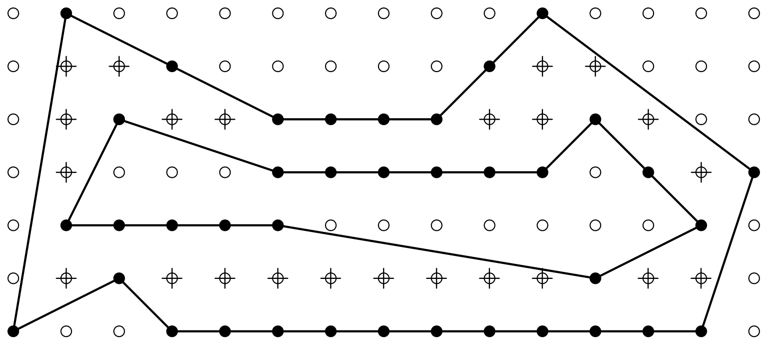
$$\begin{aligned}A &= A_1 + A_2 = I_1 + \frac{1}{2}B_1 - 1 + I_2 + \frac{1}{2}B_2 - 1 \\ &= I_1 + I_2 + X + \frac{1}{2}(B_1 + B_2 - 2X) - 2 \quad \text{so we add and then subtract } X\text{, cancelling out.} \\ &= I + \frac{1}{2}(B + 2) - 2 = I + \frac{1}{2}B - 1,\end{aligned}$$

hence showing Pick's Theorem for the largest polygon.

A note on induction. What we did above was first prove the result for triangles. Then we used this result to show the result for 4 sided polygons (by splitting into two triangles). Then we showed the result for 5 sided polygons (by splitting into a triangle and a 4 sided polygon). And then for 6 sided polygons, and so on.

This method of proof— first splitting the problem into special cases (cases for 3, then cases for 4, then cases for 5 etc.) and then proving the lowest case (triangles), then using this to attack the next problem, and then using that to attack the next problem, and so on, is called "Mathematical Induction". It's an incredibly powerful proof technique, because you sort of "boot-strap" the result to prove more and more complicated claims.

Polygons with holes. Suppose we start with a polygon, and then "chop a hole in it" by removing some other polygon from the inside. What is the remaining area?



Well, let A_1, B_1, I_1 for the starting polygon, and A_2, B_2, I_2 for the removed polygon. Let A, B, I be the remaining polygon. Counting boundary points (both "outside" and "inside" boundary) we see that $B = B_1 + B_2$. Of course, $A = A_1 - A_2$. Counting interior points, we see that all the points (both interior and boundary) of the removed polygon were interior to the original polygon, so $I_1 = I_2 + B_2 + I$. Putting this together, and using Pick's Theorem,

$$\begin{aligned}A &= A_1 - A_2 = \big(I_1 + \frac{1}{2}B_1 - 1\big) - \big(I_2 + \frac{1}{2}B_2 - 1\big) = I_2 + B_2 + I + \frac{1}{2}B_1 - 1 - I_2 - \frac{1}{2}B_2 + 1 \\ &= I + \frac{1}{2}B_1 + \frac{1}{2}B_2 = I + \frac{1}{2}B.\end{aligned}$$

This is like before, but the "-1" terms has been removed.

In fact, if we chop h holes then Pick's Theorem becomes

$$A = I + \frac{1}{2}B + h - 1.$$

This works when $h = 1$, and also for the original problem, where there was no hole, so $h = 0$.