Semisimplicity of $\mathcal{B}(E)''$

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Abstract

We study the semi-simplicity of the second dual of the Banach algebra of operators on a Banach space, $\mathcal{B}(E)''$, endowed with either Arens product. It was previously shown that if E is a Hilbert space, then $\mathcal{B}(E)$ is Arens regular and $\mathcal{B}(E)''$ is semi-simple. We show that for a large class of Banach spaces E, including subspaces of L^p spaces not isomorphic to a Hilbert space, $\mathcal{B}(E)''$ is not semi-simple. This is achieved by deriving a new representation of $\mathcal{B}(l^p)'$, and then constructing a member of the radical of $\mathcal{B}(l^p)''$, for $p \neq 2$.

Key words:
Banach algebra, Banach space, Arens products
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1 Introduction and algebraic background

When E is a Banach space, E'' is its second dual space, and we have a canonical isometry $\kappa : E \to E''$. We can thus view E'' as an "extension" of E. The same is true of a Banach algebra $\mathfrak{A}$: the first and second Arens products, $\Box$ and $\Diamond$, are defined on $\mathfrak{A}''$ extending the algebra product on $\mathfrak{A}$. When these two natural products coincide, we say that $\mathfrak{A}$ is Arens regular.

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In [3], it was shown that $\mathcal{B}(E)$, the Banach algebra of operators on a Banach space, is Arens regular whenever E is super-reflexive. The proof uses an injective homomorphism $\mathcal{B}(E)'' \rightarrow \mathcal{B}(F)$ (for either Arens product) where F is another reflexive Banach space– one can take $F = (l^2(E))_{\mathcal{U}}$ where $(l^2(E))_{\mathcal{U}}$ is an ultrapower. This is a natural approach to take, as ultrapowers are another form of "extension", and one which is closely linked to second duals (see [8, Section 2]).

When E is a Hilbert space, $\mathcal{B}(E)$ is a C^*-algebra, which gives another way to show that $\mathcal{B}(E)$ is Arens regular in this special case, and to show that $\mathcal{B}(E)''$ is semi-simple. It thus seems natural to ask whether $\mathcal{B}(E)''$ is semi-simple for any super-reflexive Banach space. In this paper, we shall show that, for a large class of spaces E, including $E = L^p(\nu)$ for any measure ν and $p \neq 2$, $\mathcal{B}(E)''$ is not semi-simple. Indeed, the only spaces E for which $\mathcal{B}(E)''$ is known to be semi-simple are those spaces which are isomorphic to a Hilbert space.

1.1 Algebraic Background

Throughout, if E is a Banach space, then E' is its dual space, the space of all continuous linear functionals on E. If $x \in E$ and $\lambda \in E'$ then we write $\langle \lambda, x \rangle = \lambda(x)$. We maintain the convention that the left-hand side of $\langle ., . \rangle$ is a member of the dual of the space which contains the right-hand side member of $\langle ., . \rangle$.

For a Banach space E there is a natural map $\kappa_E : E \rightarrow E''$ given by

$$\langle \kappa_E(x), \mu \rangle = \langle \mu, x \rangle \qquad (x \in E, \mu \in E').$$

Then κ_E is an isometry, and we say that E is reflexive if κ_E is an isomorphism.

When E and F are Banach spaces, $\mathcal{B}(E,F)$ is the Banach space of all bounded linear maps from E to F, with the operator norm. By $\mathcal{K}(E,F)$ we denote the ideal of compact operators in $\mathcal{B}(E,F)$; by $\mathcal{F}(E,F)$ the ideal the finite-rank operators. The closure of $\mathcal{F}(E,F)$ in $\mathcal{B}(E,F)$ is the ideal of approximable operators, $\mathcal{A}(E,F)$. We write $\mathcal{B}(E) = \mathcal{B}(E,E)$ for the Banach algebra of operators on a Banach space E, and similarly $\mathcal{K}(E), \mathcal{F}(E)$ and $\mathcal{A}(E)$.

We denote the tensor product of Banach spaces E and F by $E \otimes F$. Then we can give $E \otimes F$ the projective tensor norm, defined for $u \in E \otimes F$ by

$$\|u\|_\pi = \inf \left\{ \sum_{i=1}^n \|e_i\| \|f_i\| : u = \sum_{i=1}^n e_i \otimes f_i \right\}.$$

Then the completion of $E \otimes F$ under $\|\cdot\|_\pi$ is $E\widehat{\otimes}F$, the projective tensor product of E and F. See [10, Chapter 2] for more details.

There is a natural norm-decreasing map from $E\widehat{\otimes}E'$ to $\mathcal{B}(E)$ given by

$$\left(\sum_{i=1}^\infty x_i \otimes \mu_i \right)(x) = \sum_{i=1}^\infty x_i \langle \mu_i, x \rangle \qquad \left(\sum_{i=1}^\infty x_i \otimes \mu_i \in E\widehat{\otimes}E', x \in E \right).$$

We say that E has the approximation property (AP) when this map has trivial kernel. In this case, $\mathcal{A}(E) = \mathcal{K}(E)$. See [10, Chapter 4] for more details.

Finally, we can identify $\mathcal{B}(E,F')$ with $(E\widehat{\otimes}F)'$ by

$$\langle T, e \otimes f \rangle = \langle T(e), f \rangle \qquad (T \in \mathcal{B}(E,F'), e \otimes f \in E\widehat{\otimes}F)$$

and linearity. In particular, if E is reflexive, then $(E\widehat{\otimes}E')' = \mathcal{B}(E)$.

1.2 Arens products

For a Banach algebra $\mathfrak{A}$, $a, b \in \mathfrak{A}$, $\lambda \in \mathfrak{A}'$ and $\Phi \in \mathfrak{A}''$ we define $a.\lambda \in \mathfrak{A}'$, $\lambda.a \in \mathfrak{A}'$, $\lambda.\Phi \in \mathfrak{A}'$ and $\Phi.\lambda \in \mathfrak{A}'$ by

$$a.\lambda : b \mapsto \langle \lambda, ba \rangle \quad , \quad \lambda.a : b \mapsto \langle \lambda, ab \rangle,$$
$$\lambda.\Phi : b \mapsto \langle \Phi, b.\lambda \rangle \quad , \quad \Phi.\lambda : b \mapsto \langle \Phi, \lambda.b \rangle,$$

and then define two products $\Box$ and $\Diamond$ on $\mathfrak{A}''$ by

$$\langle \Phi \Box \Psi, \lambda \rangle = \langle \Phi, \Psi.\lambda \rangle \quad , \quad \langle \Phi \Diamond \Psi, \lambda \rangle = \langle \Psi, \lambda.\Phi \rangle \qquad (\Phi, \Psi \in \mathfrak{A}'', \lambda \in \mathfrak{A}').$$

Then $(\mathfrak{A}'', \Box)$ and $(\mathfrak{A}'', \Diamond)$ become Banach algebras, and both $\Box$ and $\Diamond$ agree with the original algebra product on $\mathfrak{A}$. We call $\Box$ and $\Diamond$ the first and second Arens products respectively. If $\Box$ and $\Diamond$ agree on the whole of $\mathfrak{A}''$, then $\mathfrak{A}$ is said to be *Arens regular*. For further details we refer to reader to [1, Section 2.6] or [2].

In [3] (or see [2] for a different presentation) it is shown that whenever a Banach space E is a super-reflexive, $\mathcal{B}(E)$ is Arens regular.

For a Banach space E, an index set I and an ultrafilter $\mathcal{U}$ define

$$l^\infty(E, I) = \{(x_i)_{i\in I} \subset E : \sup_{i\in I} \|x_i\| < \infty\},$$
$$N_{\mathcal{U}} = \{(x_i) \in l^\infty(E, I) : \lim_{i\in\mathcal{U}} \|x_i\| = 0\}.$$

Then $N_{\mathcal{U}}$ is a closed subspace of $l^\infty(E, I)$, and we define $(E)_{\mathcal{U}}$ to be the quotient space $l^\infty(E, I)/N_{\mathcal{U}}$. It is easy to check that if (x_i) is some representative of an equivalence class in $(E)_{\mathcal{U}}$, then $\|(x_i)\| = \lim_{i\in\mathcal{U}} \|x_i\|$. For more details see [3] and [8].

If F is a reflexive left $\mathcal{B}(E)$-module, then define a map $\phi : F\widehat{\otimes}F' \to \mathcal{B}(E)'$ by

$$\langle \phi(f \otimes \mu), T\rangle = \langle \mu, T.f\rangle \quad (f \otimes \mu \in F\widehat{\otimes}F', T \in \mathcal{B}(E)).$$

In [3] it is shown that $\phi' : \mathcal{B}(E)'' \to \mathcal{B}(F)$ is a homomorphism for either Arens product on $\mathcal{B}(E)''$. In particular, if ϕ is surjective, then ϕ' is an isomorphism onto its range, so that $\mathcal{B}(E)$ is Arens regular.

It would be natural, in the above construction, to consider using $F = (E)_{\mathcal{U}}$ for some ultrapower $\mathcal{U}$, but it seems unlikely that, in general, ϕ even has dense range in this case. However, we can make $l^2(E)$ into a left $\mathcal{B}(E)$-module by letting $\mathcal{B}(E)$ act co-ordinate wise, and then $(l^2(E))_{\mathcal{U}}$ naturally becomes a left $\mathcal{B}(E)$-module as well. As E is super-reflexive, $l^2(E)$ is super-reflexive, so $(l^2(E))_{\mathcal{U}}$ is reflexive. In [3] it was shown that for a suitable ultrafilter $\mathcal{U}$, if we set $F = (l^2(E))_{\mathcal{U}}$, then ϕ is a surjection. In section 3.1 of this paper, we shall show that for a suitable ultrafilter $\mathcal{U}$, if $E = l^p$ for $1 < p < \infty$, then ϕ is a surjection with $F = (E)_{\mathcal{U}}$.

1.3 Semi-simplicity and radicals

We state (see [1]) that for a unital Banach algebra $\mathcal{A}$, with unit e, the radical of $\mathcal{A}$ is

$$\begin{aligned}\operatorname{rad}(\mathcal{A}) &= \{a \in \mathcal{A} : e - ba \text{ is invertible } (b \in \mathcal{A})\} \\ &= \{a \in \mathcal{A} : e - ab \text{ is invertible } (b \in \mathcal{A})\} \\ &= \{a \in \mathcal{A} : \operatorname{Sp}(ab) = \{0\}\ (b \in \mathcal{A})\} \\ &= \{a \in \mathcal{A} : \operatorname{Sp}(ba) = \{0\}\ (b \in \mathcal{A})\} \\ &= \{a \in \mathcal{A} : \lim_{n\to\infty} \|(ab)^n\|^{1/n} = 0\ (b \in \mathcal{A})\} \\ &= \{a \in \mathcal{A} : \lim_{n\to\infty} \|(ba)^n\|^{1/n} = 0\ (b \in \mathcal{A})\},\end{aligned}$$

where $\operatorname{Sp}(c) = \{\lambda \in \mathbb{C} : \lambda e - c \text{ is not invertible}\}$ is the spectrum of c in $\mathcal{A}$.

2 A case when $\mathcal{B}(E)''$ is not semi-simple

For this section, let E be a reflexive Banach space. Let $\kappa : E\widehat{\otimes}E' \to \mathcal{B}(E)'$ be the usual isometry from the Banach space $E\widehat{\otimes}E'$ to its second dual. Then κ' is a linear map from $\mathcal{B}(E)''$ onto $\mathcal{B}(E)$.

Proposition 2.1 *Let E and κ be as above. Then we have the following:*

(1) κ is a $\mathcal{B}(E)$-bimodule homomorphism;

(2) κ' is a $\mathcal{B}(E)$-bimodule homomorphism;

(3) for $\Phi \in \mathcal{B}(E)''$ and $\tau \in E\widehat{\otimes}E'$, we have $\Phi.\kappa(\tau) = \kappa(\kappa'(\Phi).\tau)$ and $\kappa(\tau).\Phi = \kappa(\tau.\kappa'(\Phi))$;

(4) κ' is a homomorphism for both Arens products on $\mathcal{B}(E)''$;

(5) if we identify $\mathcal{B}(E)$ with its image in $\mathcal{B}(E)''$, then κ' is a projection onto $\mathcal{B}(E)$, and so we have $\mathcal{B}(E)'' = \mathcal{B}(E) \oplus \ker \kappa'$.

(6) *Writing* $\mathcal{B}(E)'' = \mathcal{B}(E) \oplus \ker \kappa'$, *we have*

$$(T, \Gamma_1)\Box(S, \Gamma_2) = (TS, T.\Gamma_2 + \Gamma_1.S + \Gamma_1\Box\Gamma_2) \in \mathcal{B}(E) \oplus \ker \kappa',$$

for $(T, \Gamma_1), (S, \Gamma_2) \in \mathcal{B}(E) \oplus \ker \kappa'$, *and similarly for the product* $\Diamond$.

Proof.

(1) For $S, T \in \mathcal{B}(E)$ and $\tau \in E\widehat{\otimes}E'$ we have

$$\langle \kappa(T.\tau), S\rangle = \langle S, T.\tau\rangle = \langle ST, \tau\rangle = \langle \kappa(\tau), ST\rangle = \langle T.\kappa(\tau), S\rangle$$

and similarly $\kappa(\tau.T) = \kappa(\tau).T$.

(2) This is now standard from (1).

(3) For $T \in \mathcal{B}(E)$ we have

$$\begin{aligned}\langle \Phi.\kappa(\tau), T\rangle &= \langle \Phi, \kappa(\tau).T\rangle = \langle \Phi, \kappa(\tau.T)\rangle = \langle \kappa'(\Phi), \tau.T\rangle \\ &= \langle T \circ \kappa'(\Phi), \tau\rangle = \langle T, \kappa'(\Phi).\tau\rangle = \langle \kappa(\kappa'(\Phi).\tau), T\rangle,\end{aligned}$$

and similarly $\kappa(\tau).\Phi = \kappa(\tau.\kappa'(\Phi))$.

(4) For $\Phi, \Psi \in \mathcal{B}(E)''$ and $\tau \in E\widehat{\otimes}E'$ we have

$$\langle \kappa'(\Phi\Box\Psi), \tau\rangle = \langle \Phi, \Psi.\kappa(\tau)\rangle = \langle \Phi, \kappa(\kappa'(\Psi).\tau)\rangle = \langle \kappa'(\Phi) \circ \kappa'(\Psi), \tau\rangle$$

and

$$\langle \kappa'(\Phi\Diamond\Psi), \tau\rangle = \langle \Psi, \kappa(\tau).\Phi\rangle = \langle \Psi, \kappa(\tau.\kappa'(\Phi))\rangle = \langle \kappa'(\Phi) \circ \kappa'(\Psi), \tau\rangle.$$

(5) We wish to show that for $T \in \mathcal{B}(E)$, we have $\kappa'(T) = T$, which follows because $\langle \kappa'(T), \tau\rangle = \langle T, \kappa(\tau)\rangle = \langle T, \tau\rangle$.

(6) We have $\kappa'((T + \Gamma_1)\Box(S + \Gamma_2)) = \kappa'(TS) + \kappa'(\Gamma_1).S + T.\kappa'(\Gamma_2) + \kappa'(\Gamma_1) \circ \kappa'(\Gamma_2) = TS$.

□

Proposition 2.2 *Let* $\Phi \in \mathcal{B}(E)''$ *and suppose that* $\kappa'(\Phi) \neq 0$. *Then* $\Phi \notin \operatorname{rad} \mathcal{B}(E)''$ *for either Arens product.*

Proof. Pick $x \in E$ and $\mu \in E'$ with $\kappa'(\Phi)(x) \neq 0$ and $\langle \mu, \kappa'(\Phi)(x) \rangle = 1$. Then let $T = x \otimes \mu \in \mathcal{B}(E)$, so that $\kappa'(T \Box \Phi)(x) = T(\kappa'(\Phi)(x)) = x$, and hence $\kappa'(\mathrm{Id} - T \Box \Phi)$ has non-trivial kernel and so cannot be invertible. Thus $\mathrm{Id} - T \Box \Phi$ is not invertible in $\mathcal{B}(E)''$, so that $\Phi \not\in \operatorname{rad} \mathcal{B}(E)''$. The same holds for the product $\Diamond$. $\Box$

Note that Proposition 2.1(6) shows that $\ker \kappa'$ is an ideal of $\mathcal{B}(E)''$ for either Arens product. Consequently, by Proposition 2.2, $\operatorname{rad} \mathcal{B}(E)'' = (\operatorname{rad} \mathcal{B}(E)'') \cap \ker \kappa' = \operatorname{rad} \ker \kappa'$. Thus we can concentrate on $\ker \kappa' \subseteq \mathcal{B}(E)''$ when considering the radical of $\mathcal{B}(E)''$.

2.1 An example where $\mathcal{B}(E)''$ is not semi-simple

We look at a Banach space $E = F \oplus G$, where E is reflexive (so that F and G are reflexive), and use the results of the last section. We can regard $\mathcal{B}(E)$ as an algebra of two-by-two matricies with entries from $\mathcal{B}(F)$, $\mathcal{B}(F,G)$ etc. Indeed,

$$\mathcal{B}(E) = \left\{ \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} : \begin{matrix} A_{11} \in \mathcal{B}(F), A_{21} \in \mathcal{B}(G,F), \\ A_{12} \in \mathcal{B}(F,G), A_{22} \in \mathcal{B}(G) \end{matrix} \right\},$$

and so

$$\mathcal{B}(E)'' = \left\{ \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} : \begin{matrix} \Phi_{11} \in \mathcal{B}(F)'', \Phi_{12} \in \mathcal{B}(G,F)'', \\ \Phi_{21} \in \mathcal{B}(F,G)'', \Phi_{22} \in \mathcal{B}(G)'' \end{matrix} \right\}.$$

Lemma 2.3 *Let $\mathcal{A}$ be a unital Banach algebra, and let $p, q \in \mathcal{A}$ be orthogonal idemopotents (that is, $p^2 = p, q^2 = q$ and $pq = qp = 0$) such that $p + q = e_{\mathcal{A}}$. Then*

$$\mathcal{A} = \begin{pmatrix} p\mathcal{A}p & p\mathcal{A}q \\ q\mathcal{A}p & q\mathcal{A}q \end{pmatrix}.$$

Let $\mathfrak{A}$ be a subalgebra of $\mathcal{A}$, and let $\mathfrak{B}$ be an ideal in $\mathfrak{A}$, so that

$$\mathfrak{A} \subseteq \begin{pmatrix} p\mathcal{A}p & 0 \\ q\mathcal{A}p & q\mathcal{A}q \end{pmatrix} \quad , \quad \mathfrak{B} \subseteq \begin{pmatrix} 0 & 0 \\ q\mathcal{A}p & 0 \end{pmatrix}.$$

Then $\mathfrak{B}$ lies in the radical of $\mathfrak{A}$.

Proof. Firstly note that if $a \in \mathfrak{A}$, then $a = e_{\mathfrak{A}} a e_{\mathfrak{A}} = pap + paq + qap + qaq$, so that $\mathfrak{A}$ does have the form of a two-by-two matrix algebra. Pick $b \in \mathfrak{B}$ and $a \in \mathfrak{A}$. Then

$$e_{\mathfrak{A}} + ba = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ qbp & 0 \end{pmatrix} \begin{pmatrix} pap & 0 \\ qap & qaq \end{pmatrix} = \begin{pmatrix} p & 0 \\ qbpap & q \end{pmatrix},$$

which has inverse $\left(\begin{smallmatrix} p & 0 \\ -qbpap & q \end{smallmatrix} \right)$. Thus, as $a \in \mathfrak{A}$ was arbitrary, $b \in \operatorname{rad} \mathfrak{A}$. □

We can certainly apply this lemma to $\mathcal{A} = \mathcal{B}(F \oplus G)'' = \mathcal{B}(E)''$, with either of the Arens products (with p and q being the projections onto F and G respectively). Then, with reference to the comment after Proposition 2.2, we wish to impose conditions on F and G so that $\ker \kappa' = \mathfrak{A}$ (by which we mean that $\ker \kappa'$ has, as a matrix algebra, the correct form to apply the preceding Lemma).

Lemma 2.4 *If every bounded linear map from G to F is compact, then* $\ker \kappa' = \mathfrak{A}$.

Proof. We need to show that, if $\mathcal{B}(G,F) = \mathcal{K}(G,F)$, then if $\Phi \in \mathcal{B}(G,F)''$ with $\kappa' \left(\begin{smallmatrix} 0 & \Phi \\ 0 & 0 \end{smallmatrix} \right) = 0$, then $\Phi = 0$. Now, $\kappa' \left(\begin{smallmatrix} 0 & \Phi \\ 0 & 0 \end{smallmatrix} \right) = 0$ if and only if $\langle \Phi, \lambda \rangle = 0$ for each $\lambda \in G \widehat{\otimes} F'$ (noting that $(G \widehat{\otimes} F')' = \mathcal{B}(G,F)$). Thus it is enough to show that $\kappa_{G \widehat{\otimes} F'} : G \widehat{\otimes} F' \to \mathcal{B}(G,F)'$ is surjective, that is, $G \widehat{\otimes} F'$ is reflexive.

Now, $G \widehat{\otimes} F'$ is reflexive if and only if $\mathcal{B}(G,F)$ is reflexive. By [10, Theorem 4.19], if $\mathcal{B}(G,F) = \mathcal{K}(G,F)$, then $\mathcal{B}(G,F)$ is reflexive, so we are done. □

Finally, we would like $\mathfrak{B}$ to not be the zero space.

Lemma 2.5 *With F, G and κ as above, there is a non-zero $\Psi \in \ker \kappa' \cap \mathcal{B}(F,G)''$ if and only if $\mathcal{B}(F,G)$ is not reflexive. If one of F or G has the approximation property, then $\mathcal{B}(F,G)$ is not reflexive if and only if $\mathcal{B}(F,G) \neq \mathcal{K}(F,G)$.*

Proof. As κ' restricts to a projection of $\mathcal{B}(F,G)''$ onto $\mathcal{B}(F,G)$, the first part is clear.

As $(F\widehat{\otimes}G')' = \mathcal{B}(F,G)$, the space $\mathcal{B}(F,G)$ is reflexive if and only if $F\widehat{\otimes}G'$ is reflexive. The second part of the lemma then follows from [10, Theorem 4.21]. □

Theorem 2.6 *Let F and G be reflexive Banach spaces such that one has the approximation property, $\mathcal{B}(F,G) = \mathcal{K}(F,G)$ and $\mathcal{B}(G,F) \neq \mathcal{K}(G,F)$. Then $\mathcal{B}(F \oplus G)''$, with either Arens product, is not semisimple.*

Proof. This follows directly from the above results. □

Corollary 2.7 *Choose p and q so that $1 < p < q < \infty$. Then $\mathcal{B}(l^p \oplus l^q)''$ is not semi-simple.*

Proof. By [10, Theorem 4.23], $\mathcal{B}(l^q, l^p) = \mathcal{K}(l^q, l^p)$. By considering the formal identity map from l^p to l^q we see that $\mathcal{B}(l^p, l^q) \neq \mathcal{K}(l^p, l^q)$. □

3 The case where $E = l^p$

In this section, we will show that $\mathcal{B}(l^p)''$ is not semi-simple for $1 < p < \infty$, $p \neq 2$.

If $\mathcal{A}$ is a Banach algebra, denote by $\mathcal{A}^{\mathrm{op}}$ the Banach algebra whose underlying Banach space is $\mathcal{A}$ but with reversed product. It is then clear that $\mathcal{A}$ is semi-simple if and only if $\mathcal{A}^{\mathrm{op}}$ is, and that $(\mathcal{A}'')^{\mathrm{op}} = (\mathcal{A}^{\mathrm{op}})''$ when $\mathcal{A}$ is Arens regular. Thus we can restrict ourselves to the case where $1 < p < 2$, the other cases following from the anti-isomorphism $\mathcal{B}(l^p) \to \mathcal{B}(l^q), T \mapsto T'$ (where, as usual,

$p^{-1} + q^{-1} = 1$).

Our approach is to try to adapt the method used in Section 2, but instead of writing $E = F \oplus G$ with $\mathcal{B}(E, F)$ being very small (that is, all compact operators), we shall construct an operator $T \in \mathcal{B}(E)$ which is "in the limit" compact, in the sense that we can find a system of operators (P_A) so that weak*-$\lim_A TP_A$ is in the radical. If $\mathcal{B}(E, F) = \mathcal{K}(E, F)$, then any T would do, with P_A being such that weak*-$\lim_A(\mathrm{Id} - P_A) = \mathrm{Id}$. We have to work somewhat harder for the space $E = l^p$.

3.1 Action of $\mathcal{B}(E)''$ on $(E)_\mathcal{U}$ and $(l^2(E))_\mathcal{U}$

For an ultrafilter $\mathcal{U}$ and a super-reflexive Banach space E, recall that we define $\phi : (E)_\mathcal{U} \widehat{\otimes} (E')_\mathcal{U} \to \mathcal{B}(E)'$ by

$$\langle \phi((x_i) \otimes (\mu_i)), T \rangle = \langle (\mu_i), T.(x_i) \rangle = \lim_{i \in \mathcal{U}} \langle \mu_i, T(x_i) \rangle$$

for $T \in \mathcal{B}(E)$ and $(x_i) \otimes (\mu_i) \in (E)_\mathcal{U} \widehat{\otimes} (E')_\mathcal{U}$. When we need to stress which ultrafilter is being used, we shall write $\phi_\mathcal{U}$. Then we have $\phi' : \mathcal{B}(E)'' \to \mathcal{B}((E)_\mathcal{U})$ given by

$$\langle \mu, \phi'(\Phi)(x) \rangle = \langle \Phi, \phi(x \otimes \mu) \rangle \quad (\Phi \in \mathcal{B}(E)'', x \in (E)_\mathcal{U}, \mu \in (E')_\mathcal{U}).$$

Then ϕ' is a homomorphism for either Arens product (by results in [3]). If $\Phi \in \mathcal{B}(E)''$, then we know that, for some ultrafilter $\mathcal{W}$ and some bounded family (T_α) in $\mathcal{B}(E)$, we have weak*-$\lim_{\alpha \in \mathcal{W}} T_\alpha = \Phi$. Thus we see that, for $x \in (E)_\mathcal{U}$ and $\mu \in (E')_\mathcal{U}$, we have $\langle \mu, \phi'(\Phi)(x) \rangle = \lim_{\alpha \in \mathcal{W}} \langle \mu, T_\alpha(x) \rangle$ and so

$$\phi'(\Phi)(x) = \operatorname*{weak\text{-}lim}_{\alpha \in \mathcal{W}} T_\alpha(x) \quad (x \in (E)_\mathcal{U}),$$

which makes sense because $(E)_\mathcal{U}$ is reflexive.

Lemma 3.1 *For each $\Phi \in \mathcal{B}(E)''$, $x \in (E)_\mathcal{U}$ and $\varepsilon > 0$ we can find $S \in \mathcal{B}(E)$ with $\|S\| \le \|\Phi\|$ and $\|\phi'(\Phi)(x) - S(x)\| < \varepsilon$.*

Proof. Let $X = \{S(x) : S \in \mathcal{B}(E), \|S\| \leq \|\Phi\|\}$ so that, by the above, $\phi'(\Phi)(x)$ is in the weak closure of X. Since X is convex and bounded, $\phi'(\Phi)(x)$ is thus in the norm closure of X, so we are done. $\square$

As stated above, in general, it is not the case that ϕ is surjective. However, define a map $\rho : E \times E' \to (E)_{\mathcal{U}} \widehat{\otimes} (E')_{\mathcal{U}}$ by $\rho(x, \mu) = x \otimes \mu$, where we identify E with its image in $(E)_{\mathcal{U}}$ and E' with its image in $(E')_{\mathcal{U}}$. Then ρ is norm-decreasing and so extends to a norm-decreasing map $\rho : E \widehat{\otimes} E' \to (E)_{\mathcal{U}} \widehat{\otimes} (E')_{\mathcal{U}}$.

Lemma 3.2 *The map ρ is an isometry, and $\phi \circ \rho : E \widehat{\otimes} E' \to \mathcal{B}(E)'$ is the map $\kappa : E \widehat{\otimes} E' \to \mathcal{B}(E)'$.*

Proof. If $T \in \mathcal{B}(E)$ then

$$\langle \phi(\rho(x \otimes \mu)), T \rangle = \langle \mu, T(x) \rangle = \langle \kappa(x \otimes \mu), T \rangle,$$

so, by linearity and continuity, $\phi \circ \rho = \kappa$. As κ is an isometry, and ϕ and ρ are norm-decreasing, ρ must also be an isometry. $\square$

In the rest of this section, we shall prove that, when $E = l^p$ for $1 < p < \infty$, the map ϕ actually is surjective for a suitable ultrafilter $\mathcal{U}$.

Let E be a reflexive Banach space with the approximation property, so that $\mathcal{A}(E)' = E \widehat{\otimes} E'$, with the duality given by

$$\langle x \otimes \mu, T \rangle = \langle \mu, T(x) \rangle \qquad (x \otimes \mu \in E \widehat{\otimes} E', T \in \mathcal{A}(E)).$$

For more details, see [10, Theorem 5.33]. Consequently we shall identify $\mathcal{A}(E)''$ with $\mathcal{B}(E)$, and it is easy to check that the canonical map $\kappa_{\mathcal{A}(E)} : \mathcal{A}(E) \to \mathcal{A}(E)'' = \mathcal{B}(E)$ is just the inclusion map. Thus $E \widehat{\otimes} E'$ is complemented in $\mathcal{B}(E)'$ with projection $\kappa'_{\mathcal{A}(E)} : \mathcal{B}(E)' \to E \widehat{\otimes} E'$ and $\mathcal{B}(E)' = E \widehat{\otimes} E' \oplus \mathcal{A}(E)^\circ$ where

$$\mathcal{A}(E)^\circ = \{\lambda \in \mathcal{B}(E)' : \langle \lambda, T \rangle = 0 \ (T \in \mathcal{A}(E))\}.$$

We can form the quotient algebra $\mathcal{B}(E)/\mathcal{A}(E)$, which in a natural way has dual space $\mathcal{A}(E)^\circ$. For $T \in \mathcal{B}(E)$, write $T + \mathcal{A}(E)$ for the image of T in

$\mathcal{B}(E)/\mathcal{A}(E)$, so that

$$\|T + \mathcal{A}(E)\| = \inf\{\|T + S\| : S \in \mathcal{A}(E)\}.$$

Then in the case where $E = l^p$ (which does have the approximation property), define $P_n \in \mathcal{B}(l^p)$ to be projection onto the first n co-ordinates, and $Q_n = \mathrm{Id} - P_n$, for $n \in \mathbb{N}$. Then we have the following.

Proposition 3.3 *For $T \in \mathcal{B}(l^p)$, we have*

$$\|T + \mathcal{A}(l^p)\| = \lim_{n\to\infty} \|TQ_n\| = \lim_{n\to\infty} \|Q_n T Q_n\|.$$

We may also replace $\lim_{n\to\infty}$ *by* $\inf_n$.

Proof. As $(\|TQ_n\|)_{n=1}^\infty$ and $(\|Q_nTQ_n\|)_{n=1}^\infty$ are decreasing sequences, we can interchange taking limits and taking infima. Then as $TQ_n = T - TP_n$ and $TP_n \in \mathcal{A}(l^p)$, we have $\|T + \mathcal{A}(l^p)\| \leq \|TQ_n\|$ for every n. Assume that we have $S \in \mathcal{A}(l^p)$ with $\|T + S\| < \inf_n \|TQ_n\|$, so that as $S = \lim_n SP_n$, we have $\lim_n \|SQ_n\| = 0$, and so $\lim_n \|TQ_n\| = \lim_n \|(T + S)Q_n\| \leq \|T + S\| < \lim_n \|TQ_n\|$. This contradiction shows that

$$\|T + \mathcal{A}(l^p)\| = \lim_n \|TQ_n\|.$$

For $n \in \mathbb{N}$, we have $Q_nTQ_n = T - TP_n - P_nT + P_nTP_n$, and so $\|T + \mathcal{A}(l^p)\| \leq \|Q_nTQ_n\|$. Hence

$$\|T + \mathcal{A}(l^p)\| \leq \lim_n \|Q_nTQ_n\| \leq \lim_n \|TQ_n\| = \|T + \mathcal{A}(l^p)\|$$

so we must have equality throughout, completing the proof. □

The following is a variant of Helley's Lemma, and is a standard result.

Proposition 3.4 *Let F be a Banach space, $\Phi \in F''$ and $M \subset F'$ be a finite-dimensional subspace. Then for $\varepsilon > 0$ we can find $x \in F$ so that $\langle \mu, x\rangle = \langle \Phi, \mu\rangle$ for each $\mu \in M$, and*

$$\|x\| \leq \varepsilon + \max\{|\langle \Phi, \mu\rangle| : \mu \in M, \|\mu\| = 1\}.$$

Proof. This follows easily from [7, Lemma I.6.2]. □

Let $(e_i)_{i=1}^{\infty}$ be the standard unit basis vectors of l^p. For $x = \sum_{i=1}^{\infty} x_i e_i \in l^p$, define the support of x to be $\mathrm{supp}(x) = \{i \in \mathbb{N} : x_i \neq 0\}$. Then $P_n(x) = x$ if and only if $\mathrm{supp}(x) \subseteq \{1, \ldots, n\}$, and $Q_n(x) = x$ if and only if $\mathrm{supp}(x) \subseteq \{n+1, n+2, \ldots\}$.

Lemma 3.5 *Let $M \subset \mathcal{B}(l^p)$ be a finite-dimensional subspace, $\varepsilon > 0$ and $x \in l^p$. Then there exists an $N_0 \in \mathbb{N}$ so that $\|Q_n(T(x))\| < \varepsilon \|T\|$ for each $T \in M$ and $n \geq N_0$. For each $m \in \mathbb{N}$, there exists $N_1 \in \mathbb{N}$ so that $\|P_m T Q_n\| < \varepsilon \|T\|$ for each $T \in M$ and $n \geq N_1$.*

Proof. Firstly, assume towards a contradiction that for each $n \in \mathbb{N}$, we can find $T_n \in M$ with $\|T_n\| = 1$ and $\|Q_n(T_n(x))\| \geq \varepsilon \|T_n\| = \varepsilon$. Then, as M has compact unit ball, we can find a subsequence (n_i) so that for some $T \in M$, $T_{n_i} \to T$ as $i \to \infty$. Then we have

$$0 = \lim_i \|Q_{n_i}(T(x))\| = \lim_i \|Q_{n_i}(T_{n_i}(x))\| \geq \varepsilon$$

which is the required contradiction.

For the second part, pick $\delta > 0$ and, by the compactness of the unit ball of M, let $(T_i)_{i=1}^{N}$ be in M with $\|T_i\| = 1$ for each i, so that for each $T \in M$ with $\|T\| = 1$, we can find i with $\|T - T_i\| < \delta$. Then we claim that we can find $N_1 \in \mathbb{N}$ so that $n \geq N_1$ implies that $\|P_m T_i Q_n\| < \delta \|T_i\|$ for $1 \leq i \leq N$.

It is enough to show this for each separate i as we have only finitely many to consider. Then, towards a contradiction, if $\lim_n \|P_m T_i Q_n\| \neq 0$, then we can find $\theta > 0$ and $n_1 < n_2 < \cdots$ so that $\|P_m T_i Q_{n_j}\| \geq 2\theta$ for each j. Then we can find $(x_j)_{j=1}^{\infty}$ with $\|x_j\| = 1$ and $Q_{n_j}(x_j) = x_j$ so that $\|P_m T_i(x_j)\| \geq \theta$ for each j. However, we have

$$\begin{aligned} \lim_{j\to\infty} \|P_m T_i(x_j)\| &= \lim_{j\to\infty} \left(\sum_{k=1}^{m} |\langle e_k, T_i(x_j)\rangle|^p \right)^{1/p} \\ &= \left(\sum_{k=1}^{m} \lim_{j\to\infty} |\langle T_i'(e_k), x_j\rangle|^p \right)^{1/p} = 0, \end{aligned}$$

which is the required contradiction.

So if $T \in M$ with $\|T\| = 1$ and $n \geq N_1$, for some i we have $\|T - T_i\| < \delta$ and so

$$\|P_m T Q_n\| \leq \|P_m T_i Q_n\| + \delta < \delta\|T_i\| + \delta = 2\delta.$$

Thus, if $\delta = \varepsilon/2$, we have $\|P_m T Q_n\| < \varepsilon$ as required. $\square$

A block-basis in l^p is a sequence of norm-one vectors $(x_n)_{n=1}^{\infty}$ in l^p such that $\mathrm{supp}(x_n)$ is finite for each n, and such that $\max \mathrm{supp}(x_n) < \min \mathrm{supp}(x_{n+1})$ for each n.

For $A \subseteq \mathbb{N}$, let P_A be the projection on l^p defined by

$$P_A(e_n) = \begin{cases} e_n & (n \in A), \\ 0 & (n \notin A). \end{cases}$$

Proposition 3.6 *Let $\lambda \in \mathcal{A}(l^p)^\circ$ with $\|\lambda\| = 1$, $M \subset \mathcal{B}(l^p)$ be a finite-dimensional subspace with $M \cap \mathcal{A}(l^p) = \{0\}$, $n_1 \in \mathbb{N}$ and (ε_n) be a sequence of positive reals. Then we can find a block-basis (x_n) in l^p and $(A_n)_{n=1}^{\infty}$ a sequence of pairwise-disjoint subsets of $\mathbb{N}$ such that:*

(1) $|\langle \lambda, T\rangle| \leq (1 + \varepsilon_1) \sup_n \|T(x_n)\|$ *for each* $T \in M$*;*

(2) $\|P_{\mathbb{N}\setminus A_n}(T(x_n))\| < \varepsilon_n \|T\|$ *and* $\|P_{A_n}(T(x_m))\| < \varepsilon_m \|T\|$ *for each* $n, m \in \mathbb{N}$ *with* $n \neq m$*, and each* $T \in M$*;*

(3) $\mathrm{supp}(x_n) \subseteq \{n_1 + 1, n_1 + 2, \ldots\}$ *for each* $n \in \mathbb{N}$*.*

Proof. As M has a compact unit ball, let $(T_n)_{n=1}^{\infty}$ be a dense sequence in $\{T \in M : \|T\| = 1\}$. Then for T_1, we can find x_1 in l^p with finite support, $\|x_1\| = 1$, $\min \mathrm{supp}(x_1) > n_1$ and $(1 + \varepsilon_1)\|T_1(x_1)\| > |\langle \lambda, T_1\rangle|$. We can do this because, using the fact that $\lambda \in \mathcal{A}(l^p)^\circ$, $|\langle \lambda, T_1\rangle| = |\langle \lambda, T_1 Q_{n_1}\rangle| \leq \|T_1 Q_{n_1}\|$. Then using Lemma 3.5 we can find $r_1 \in \mathbb{N}$ so that $\|Q_{r_1} T(x_1)\| < \frac{1}{2}\varepsilon_1 \|T\|$ for each $T \in M$.

Assume inductively that we have found $(x_i)_{i=1}^{k} \subset l^p$ of norm one and with pairwise-disjoint support, and $0 = r_0 < r_1 < r_2 < \cdots < r_k$ so that:

(1) for $1 \le i \le k$, $|\langle \lambda, T_i \rangle| \le (1+\varepsilon_1)\|T_i(x_i)\|$;

(2) for $1 \le i \le k$ and $T \in M$, $\|Q_{r_i} T(x_i)\| < \frac{1}{2}\varepsilon_i \|T\|$;

(3) for $1 \le i \le k$ and $T \in M$, $\|P_{r_{i-1}} T(x_i)\| < \frac{1}{2}\varepsilon_i \|T\|$.

We shall show how to choose x_{k+1} and r_{k+1}. By Lemma 3.5 we can find $m \in \mathbb{N}$ so that $\|P_{r_k} T Q_m(x)\| < \frac{1}{2}\varepsilon_{k+1}\|T\|\|x\|$ for each $T \in M$ and each $x \in l^p$. We may suppose that $m > \max \operatorname{supp}(x_k)$, so as

$$|\langle \lambda, T_{k+1} \rangle| = |\langle \lambda, T_{k+1} Q_m \rangle| \le \|T_{k+1} Q_m\|,$$

we can find a unit vector $x_{k+1} \in l^p$ with finite support, $\min \operatorname{supp}(x_{k+1}) > m$, and $|\langle \lambda, T_{k+1} \rangle| \le (1+\varepsilon_1)\|T_{k+1}(x_{k+1})\|$. Then, by our choice of m,

$$\|P_{r_k} T(x_{k+1})\| < \frac{1}{2}\varepsilon_{k+1}\|T\| \qquad (T \in M).$$

By Lemma 3.5 we can find r_{k+1} so that, for $T \in M$, we have $\|Q_{r_{k+1}} T(x_{k+1})\| < \frac{1}{2}\varepsilon_{k+1}\|T\|$.

So by induction we can find a block basis $(x_n)_{n=1}^{\infty}$ and $0 = r_0 < r_1 < r_2 < \cdots$ with the above properties. For each $n \in \mathbb{N}$, set $A_n = \{i : r_{n-1} < i \le r_n\}$. Then, for $T \in M$, we have

$$\|P_{\mathbb{N}\setminus A_n} T(x_n)\| \le \|P_{r_{n-1}} T(x_n)\| + \|Q_{r_n} T(x_n)\| < \varepsilon_n \|T\|$$

and, if $n < m$,

$$\|P_{A_n} T(x_m)\| \le \|P_{r_n} T(x_m)\| \le \|P_{r_{m-1}} T(x_m)\| < \tfrac{1}{2}\varepsilon_m \|T\| < \varepsilon_m \|T\|,$$

while, if $n > m$, we have,

$$\begin{aligned} \|P_{A_n} T(x_m)\| &\le \|Q_{r_{n-1}} T(x_m)\| \le \|Q_{r_m} T(x_m)\| \\ &\le \|T(x_m)\| - \|P_{r_m} T(x_m)\| \\ &< \tfrac{1}{2}\varepsilon_m \|T(x_m)\| < \varepsilon_m \|T\|, \end{aligned}$$

as required.

Finally, let $T \in M$. Then, for each $\delta > 0$, there exists an $n \in \mathbb{N}$ so that $\|T - T_n\| < \delta$, and thus

$$\begin{aligned} |\langle \lambda, T \rangle| &< |\langle \lambda, T_n \rangle| + \delta \leq (1+\varepsilon_1)\|T_n(x_n)\| + \delta \\ &\leq (1+\varepsilon_1)\|T(x_n)\| + \delta(2+\varepsilon_1). \end{aligned}$$

As this holds for each $\delta > 0$, we see that $|\langle \lambda, T \rangle| \leq (1+\varepsilon_1)\sup_n \|T(x_n)\|$. □

We can now prove our key result, which tells us that any member of $\mathcal{A}(l^p)^\circ$ can be approximated, on a finite-dimensional subspace of $\mathcal{B}(l^p)$, by an elementary tensor in $l^p \widehat{\otimes} l^q$ (recalling that $p^{-1} + q^{-1} = 1$).

Theorem 3.7 *Let $\lambda \in \mathcal{A}(l^p)^\circ$, $M \subset \mathcal{B}(l^p)$ be a finite-dimensional subspace and $\varepsilon > 0$. Then we can find $x \in l^p$ and $\mu \in l^q$ with $\|x\| < \|\lambda\|^{1/p}(1+\varepsilon)^{1/p}$ and $\|\mu\| < \|\lambda\|^{1/q}(1+\varepsilon)^{1/q}$, and such that $|\langle \lambda, T \rangle - \langle \mu, T(x) \rangle| < \varepsilon\|\lambda\|\|T\|$ for each $T \in M$.*

Proof. We can find n_1 so that $\|TQ_{n_1}\| < \frac{1}{2}\varepsilon\|T\|$ for each $T \in M \cap \mathcal{A}(l^p)$. This follows by a compactness arguement, similar to those used above. Let $\widehat{M} \subseteq M$ be a subspace of M so that $\widehat{M} \cap \mathcal{A}(l^p) = \{0\}$ and $M = \widehat{M} \oplus (M \cap \mathcal{A}(l^p))$. Let (ε_n) be a sequence of positive reals so that $\sum_{n=1}^\infty \varepsilon_n < \varepsilon/3$. If the result is true in the special case that $\|\lambda\| = 1$, then we can find x and μ with $\|x\| < (1+\varepsilon)^{1/p}$ and $\|\mu\| < (1+\varepsilon)^{1/q}$ and with $|\|\lambda\|^{-1}\langle \lambda, T \rangle - \langle \mu, T(x) \rangle| < \varepsilon\|T\|$ for each $T \in M$. Then let $\hat{x} = \|\lambda\|^{1/p}x$ and $\hat{\mu} = \|\lambda\|^{1/q}\mu$ so that $\|\hat{x}\| < \|\lambda\|^{1/p}(1+\varepsilon)^{1/p}$ and $\|\hat{\mu}\| < \|\lambda\|^{1/q}(1+\varepsilon)^{1/q}$ and, for each $T \in M$, we have $|\langle \lambda, T \rangle - \langle \hat{\mu}, T(\hat{x}) \rangle| < \varepsilon\|\lambda\|\|T\|$, as required. Thus we may suppose henceforth that $\|\lambda\| = 1$.

We can use Proposition 3.6, applied to $\widehat{M}$, to find sequences (x_n) and (A_n).

Let $l^1(l^p)$ be the Banach space of all absolutely-summable sequences of vectors in l^p with the l^1 norm, so that

$$l^1(l^p) = \left\{ (y_n)_{n=1}^\infty \subset l^p : \|(y_n)\| := \sum_{n=1}^\infty \|y_n\| < \infty \right\},$$

and let $l^\infty(l^p)$ have a similar definition. Then $l^1(l^q)' = l^\infty(l^p)$. Let

$$X = \{(T(x_n))_{n=1}^\infty : T \in \widehat{M}\} \subset l^\infty(l^p),$$

so that X is a finite-dimensional subspace of $l^\infty(l^p)$. Define $\Phi \in X'$ by

$$\langle \Phi, (T(x_n))\rangle = \langle \lambda, T\rangle \qquad (T \in \widehat{M}).$$

Because $|\langle \lambda, T\rangle| \leq (1+\varepsilon_1)\|(T(x_n))\|_\infty$, we have $\|\Phi\| \leq 1+\varepsilon_1$. Then, by Proposition 3.4, as X is finite-dimensional, we can find $(\mu_n) \in l^1(l^q)$ so that $\sum_{n=1}^\infty \|\mu_n\| \leq 1+\varepsilon_1+\varepsilon_2 < 1+\varepsilon$ and $\langle \Phi, (T(x_n))\rangle = \sum_{n=1}^\infty \langle \mu_n, T(x_n)\rangle$ for each $T \in \widehat{M}$.

For each $n \in \mathbb{N}$, set $\hat{\mu}_n = P_{A_n}(\mu_n)$, and set

$$x = \sum_{n=1}^\infty x_n \|\hat{\mu}_n\|^{1/p} \quad \text{and} \quad \mu = \sum_{n=1}^\infty \hat{\mu}_n \|\hat{\mu}_n\|^{-1+1/p},$$

so that

$$\|x\| = \left(\sum_{n=1}^\infty \|\hat{\mu}_n\|\right)^{1/p} < (1+\varepsilon)^{1/p} \quad , \quad \|\mu\| = \left(\sum_{n=1}^\infty \|\hat{\mu}_n\|\right)^{1/q} < (1+\varepsilon)^{1/q}.$$

Then, for $T \in \widehat{M}$, we have

$$\langle \mu, T(x)\rangle = \sum_{n=1}^\infty \sum_{m=1}^\infty \langle P_{A_n}(\mu_n), T(x_m)\rangle.$$

By condition (2) in Proposition 3.6, for each $T \in \widehat{M}$, we have

$$\begin{aligned}\left|\sum_{n\neq m} \langle P_{A_n}(\mu_n), T(x_m)\rangle\right| &\leq \sum_{n=1}^\infty \left|\langle \mu_n, \sum_{m\neq n} P_{A_n}(T(x_m))\rangle\right| \\ &\leq \sum_{n=1}^\infty \|\mu_n\| \sum_{m=1}^\infty \varepsilon_m \|T\| \leq \|T\| \left(\sum_{m=1}^\infty \varepsilon_m\right)\left(\sum_{n=1}^\infty \|\mu_n\|\right) < \tfrac{1}{3}\varepsilon(1+\varepsilon_1+\varepsilon_2)\|T\|.\end{aligned}$$

Then, again by condition (2), for $T \in \widehat{M}$, we have

$$\begin{aligned}\left|\langle \lambda, T\rangle - \sum_{n=1}^\infty \langle \hat{\mu}_n, T(x_n)\rangle\right| &\leq \sum_{n=1}^\infty \|\mu_n\|\|P_{A_n}(T(x_n)) - T(x_n)\| \\ &< \sum_{n=1}^\infty \varepsilon_n \|\mu_n\|\|T\| < \|T\| \left(\sup_n \|\mu_n\|\right)\left(\sum_{n=1}^\infty \varepsilon_n\right) < \tfrac{1}{3}\varepsilon(1+\varepsilon_1+\varepsilon_2)\|T\|.\end{aligned}$$

Consequently, if $T \in \widehat{M}$, then

$$|\langle \lambda, T\rangle - \langle \mu, T(x)\rangle| < \tfrac{2}{3}\varepsilon(1+\varepsilon_1+\varepsilon_2)\|T\|,$$

and we may suppose that $\frac{2}{3}\varepsilon(1+\varepsilon_1+\varepsilon_2) < \varepsilon$. Finally, if $T \in M \cap \mathcal{A}(l^p)$, then, by the choice of n_1, we have

$$\begin{aligned}|\langle \mu, T(x)\rangle| &\leq \sum_{n=1}^{\infty} |\langle P_{A_n}(\mu_n), T(x_n)\rangle| \leq \sum_{n=1}^{\infty} \|\mu_n\|\|TQ_{n_1}\| \\ &< \tfrac{1}{2}\varepsilon(1+\varepsilon_1+\varepsilon_2)\|T\| < \varepsilon\|T\|,\end{aligned}$$

as required, since $\langle \lambda, T\rangle = 0$ and $\|\lambda\| = 1$. □

Theorem 3.8 *For $p \in (1,\infty)$, the map $\phi : (l^p)_{\mathcal{U}}\widehat{\otimes}(l^q)_{\mathcal{U}} \to \mathcal{B}(l^p)'$ is surjective for a suitable ultrafilter $\mathcal{U}$. In fact, for $\lambda \in \mathcal{B}(l^p)'$, we can find $\sigma \in (l^p)_{\mathcal{U}}\widehat{\otimes}(l^q)_{\mathcal{U}}$ with $\phi(\sigma) = \lambda$ and $\|\sigma\| = \|\lambda\|$.*

Proof. Let I be the collection of finite-dimensional subspaces of $\mathcal{B}(l^p)$, partially ordered by inclusion. Let $\mathcal{U}$ be an ultrafilter on I which refines the order filter, so that, if $M \in I$, then $\{N \in I : M \subseteq N\} \in \mathcal{U}$.

Pick $\lambda \in \mathcal{A}(l^p)^\circ$ and, for $M \in I$, let $x_M \in l^p$ and $\mu_M \in l^q$ be given by Theorem 3.7 applied with $\varepsilon_M = (\dim M)^{-1}$. Then $\|x_M\| < (1+\varepsilon_M)^{1/p}\|\lambda\|^{1/p}$ and $\|\mu_M\| < (1+\varepsilon_M)^{1/q}\|\lambda\|^{1/q}$, so that if we set $x = (x_M)$ and $\mu = (\mu_M)$ then $x \in (l^p)_{\mathcal{U}}$, $\mu \in (l^q)_{\mathcal{U}}$, and

$$\|x\|\|\mu\| = \lim_{M\in\mathcal{U}} \|x_M\|\|\mu_M\| \leq \lim_{M\in\mathcal{U}} (1+\varepsilon_M) = \|\lambda\|.$$

Then, for each $T \in \mathcal{B}(l^p)$, we have

$$|\langle \lambda, T\rangle - \langle \phi(x\otimes\mu), T\rangle| = |\langle \lambda, T\rangle - \lim_{M\in\mathcal{U}} \langle \mu_M, T(x_M)\rangle| < \lim_{M\in\mathcal{U}} \varepsilon_M\|\lambda\|\|T\| = 0,$$

so that $\phi(x\otimes\mu) = \lambda$, and hence $\|x\|\|\mu\| = \|\lambda\|$.

Let $\lambda \in \mathcal{B}(l^p)'$. Then let $\lambda = \hat{\lambda} + \tau$ where $\tau = \kappa'_{\mathcal{A}(l^p)}(\lambda) \in l^p\widehat{\otimes}l^q$ and $\hat{\lambda} = \lambda - \tau \in \mathcal{A}(l^p)^\circ$. Then we can find $x_0 \in (l^p)_{\mathcal{U}}$ and $\mu_0 \in (l^q)_{\mathcal{U}}$ with $\|x_0\|\|\mu_0\| = \|\hat{\lambda}\|$ and $\phi(x_0\otimes\mu_0) = \hat{\lambda}$. We see that

$$\phi(\rho(\tau) + x_0\otimes\mu_0) = \lambda \quad , \quad \|\rho(\tau) + x_0\otimes\mu_0\| \leq \|\tau\| + \|\hat{\lambda}\|.$$

For each $\varepsilon > 0$, we can find $S \in \mathcal{F}(l^p)$ and $N \in \mathbb{N}$ so that $\|S\| = 1$, $P_N S P_N = S$, $|\langle \tau, S \rangle| > \|\tau\| - \varepsilon$, and $|\langle Q_N R Q_N, \tau \rangle| < \varepsilon \|R\|$ for $R \in \mathcal{B}(l^p)$. Next, we can find $T \in \mathcal{B}(l^p)$ with $\|T\| = 1$ and $|\langle \hat{\lambda}, Q_N T Q_N \rangle| = |\langle \hat{\lambda}, T \rangle| > \|\hat{\lambda}\| - \varepsilon$. Then, for each $x \in l^p$, we have

$$\begin{aligned} \|S(x) + Q_N T Q_N(x)\| &= (\|P_N S P_N(x)\|^p + \|Q_N T Q_N(x)\|^p)^{1/p} \\ &\leq (\|S\|^p \|P_N(x)\|^p + \|Q_N T Q_N\|^p \|Q_N(x)\|^p)^{1/p} \\ &\leq \|x\| \max\{\|S\|, \|T\|\} = \|x\|. \end{aligned}$$

Thus $\|S + Q_N T Q_N\| \leq 1$, and so

$$\|\lambda\| = \|\tau + \hat{\lambda}\| \geq |\langle \tau + \hat{\lambda}, S + Q_N T Q_N \rangle| > \|\tau\| + \|\hat{\lambda}\| - 3\varepsilon.$$

As $\varepsilon > 0$ was arbitrary, we see that

$$\|\tau\| + \|\hat{\lambda}\| \leq \|\lambda\| = \|\phi(\rho(\tau) + x_0 \otimes \mu_0)\| \leq \|\rho(\tau) + x_0 \otimes \mu_0\| \leq \|\tau\| + \|\hat{\lambda}\|,$$

and so we must have $\|\lambda\| = \|\rho(\tau) + x_0 \otimes \mu_0\|$, as required. □

We can thus identify $\mathcal{B}(l^p)'$ with a quotient of $(l^p)_{\mathcal{U}} \widehat{\otimes} (l^q)_{\mathcal{U}}$, and hence the map $\phi' : \mathcal{B}(l^p)'' \to \mathcal{B}((l^p)_{\mathcal{U}})$ is an isometry onto its range.

3.2 Systems of projections

Let $\mathcal{W}$ be an ultrafilter on $\mathbb{N}$, and partially order $\mathcal{W}$ by reverse inclusion (so that $A \leq B$ if and only if $B \subseteq A$). Then, as $\mathcal{W}$ is a filter, $\mathcal{W}$ is a directed set with this order, and so we can let $\mathcal{V}$ be an ultrafilter on $\mathcal{W}$ refining the order filter. Hence for each $A \in \mathcal{W}$ we have $V_A = \{B \in \mathcal{U} : B \subseteq A\} \in \mathcal{V}$.

For $A \subseteq \mathbb{N}$, recall the definition of P_A from above:

$$P_A(e_n) = \begin{cases} e_n & (n \in A), \\ 0 & (n \notin A). \end{cases}$$

Let $\mathcal{U}$ be some ultrafilter on $\mathbb{N}$, and define $\psi \in \mathcal{B}((l^p)_\mathcal{U})$ by,

$$\psi(x) = \operatorname*{weak-lim}_{A \in \mathcal{V}} P_A(x) \qquad (x = (x_i) \in (l^p)_\mathcal{U}).$$

Lemma 3.9 *The map ψ is a projection onto the subspace*

$$\{x \in (l^p)_\mathcal{U} : P_A(x) = x \ (A \in \mathcal{W})\}.$$

Proof. If $\mu \in (l^q)_\mathcal{U}$ and $B \in \mathcal{W}$, then

$$\begin{aligned} \langle \mu, P_B\psi(x)\rangle &= \lim_{A \in \mathcal{V}} \langle P'_B(\mu), P_A(x)\rangle = \lim_{A \in \mathcal{V}} \langle \mu, P_{B\cap A}(x)\rangle \\ &= \lim_{A \in \mathcal{V}} \langle \mu, P_A(x)\rangle = \langle \mu, \psi(x)\rangle, \end{aligned}$$

so that $P_B \circ \psi = \psi$, and hence $\psi \circ \psi = \psi$. If $x \in (l^p)_\mathcal{U}$ with $P_A(x) = x$ for each $A \in \mathcal{W}$, then clearly $\psi(x) = x$, so we are done. □

Lemma 3.10 *For each $x \in (l^p)_\mathcal{U}$, the limit $\lim_{A\in\mathcal{V}} P_A(x)$ exists (we only know a priori that the limit exists in the weak topology, not the norm topology).*

Proof. Let C be the convex hull of $\{P_A(x) : A \in \mathcal{W}\}$, so that the norm and weak closures of C coincide. Thus for each $\varepsilon > 0$ we can find a convex combination $S = \sum_{i=1}^n \lambda_i P_{A_i}$ so that $\|S(x) - \psi(x)\| < \varepsilon$. Let $A = A_1 \cap \cdots \cap A_n$, so that $A \in \mathcal{W}$, and $P_A(S(x)) = \sum_{i=1}^n \lambda_i P_A P_{A_i}(x) = P_A(x)$. Then

$$\|P_A(x) - \psi(x)\| = \|P_A(S(x)) - P_A(\psi(x))\| < \|P_A\|\varepsilon = \varepsilon.$$

Hence for each $B \in V_A$, we have

$$\|P_B(x) - \psi(x)\| = \|P_B(P_A(x)) - P_B(\psi(x))\| \leq \|P_A(x) - \psi(x)\| < \varepsilon.$$

Hence $\{B \in \mathcal{W} : \|P_B(x) - \psi(x)\| < \varepsilon\} \supseteq V_A \in \mathcal{V}$, so that $\psi(x) = \lim_{A\in\mathcal{V}} P_A(x)$. □

When E and F are Banach spaces and $\varepsilon > 0$, a map $T \in \mathcal{B}(E, F)$ is said to be a $(1+\varepsilon)$-isomorphism if T is an isomorphism onto its range, and $(1-\varepsilon)\|x\| \leq \|T(x)\| \leq (1+\varepsilon)\|x\|$ for each $x \in E$.

For $n \in \mathbb{N}$ and $p \in [1, \infty]$, let l^p_n be $\mathbb{C}^n$ with the l^p norm. If $A \subseteq \mathbb{N}$, then $l^p(A)$ is the subspace of l^p consisting of vectors x with $\mathrm{supp}(x) \subseteq A$. If $|A| < \infty$, then $l^p(A)$ is isometrically isomorphic to $l^p_{|A|}$.

By a result of Dvoretsky (see, for example, [6]) we know that for any Banach space E, $\varepsilon > 0$ and $n \in \mathbb{N}$, we can find a $(1+\varepsilon)$-isomorphism $T : l^2_n \to E$.

Choose an increasing sequence (n_k) of integers, and let $N_0 = 0$, $N_1 = n_1$, $N_{i+1} = N_i + n_{i+1}$ and $A_k = \{i : N_{k-1} < i \leq N_k\}$. Then we can find a linear map $T : l^p \to l^p$ which maps $\mathrm{lin}\{e_i : i \in A_k\}$ to a $(1+\frac{1}{k})$-isomorphic copy of $l^2_{n_k}$, say $w_i = T(e_i)$. By this, we mean that if $(a_i)_{i \in A_k}$ is a sequence of scalars, then

$$\frac{k-1}{k}\left(\sum_{i \in A_k} |a_i|^2\right)^{1/2} \leq \left\|\sum_{i \in A_k} a_i w_i\right\| \leq \frac{k+1}{k}\left(\sum_{i \in A_k} |a_i|^2\right)^{1/2}.$$

Further, we may assume that, when $k \neq l$, the sets $\{w_i : i \in A_k\}$ and $\{w_i : i \in A_l\}$ are disjointly supported in l^p. That is, if $i \in A_l$ and $j \in A_k$, then $\mathrm{supp}(w_i) \cap \mathrm{supp}(w_j) = \emptyset$.

In the case where $1 < p < 2$ and (a_k) is a sequence of scalars, we have

$$\begin{aligned}\left\|T\left(\sum_k a_k e_k\right)\right\| &= \left\|\sum_k \sum_{i \in A_k} a_i w_i\right\| = \left(\sum_k \left\|\sum_{i \in A_k} a_i w_i\right\|^p\right)^{1/p} \\ &\leq \left(\sum_k \left(\frac{k+1}{k}\right)^p \left(\sum_{i \in A_k} |a_i|^2\right)^{p/2}\right)^{1/p} \leq 2\|(a_k)\|_p. \qquad (1)\end{aligned}$$

Thus $T \in \mathcal{B}(l^p)$ with $\|T\| \leq 2$.

Now fix $p \in (1,2)$ and form T as above (where we shall choose (n_k) later). For each $A \subseteq \mathbb{N}$, let

$$\mathrm{ud}(A) = \limsup_{k\to\infty} \frac{|A \cap A_k|}{|A_k|},$$

and let $\mathcal{F} = \{A \subseteq \mathbb{N} : \mathrm{ud}(\mathbb{N} \setminus A) = 0\}$. Then $\mathcal{F}$ is a filter on $\mathbb{N}$; let $\mathcal{W}$ be an ultrafilter on $\mathbb{N}$ refining $\mathcal{F}$. By Theorem 3.8, there is an ultrafilter $\mathcal{U}$, on some suitable index set I, such that $\phi_{\mathcal{U}} : (l^p)_{\mathcal{U}}\widehat{\otimes}(l^q)_{\mathcal{U}} \to \mathcal{B}(l^p)'$ is surjective and such that $\phi'_{\mathcal{U}}$ is an isometric isomorphism onto its range. Define

$$\Phi = \operatorname*{weak^*\text{-}\lim}_{A\in\mathcal{V}} TP_A \in \mathcal{B}(l^p)''.$$

Recall the definition of ψ from section 3.2.

Lemma 3.11 *We have $\phi'_{\mathcal{U}}(\Phi) = T \circ \psi$ and $\Phi \neq 0$.*

Proof. Choose $x \in (l^p)_{\mathcal{U}}$, and let $y = \psi(x) = \lim_{A\in\mathcal{V}} P_A(x)$ (the limit exists by Lemma 3.10), so that, if $\mu \in (l^q)_{\mathcal{U}}$, we have

$$\langle \mu, \phi'_{\mathcal{U}}(\Phi)(x)\rangle = \lim_{A\in\mathcal{V}} \langle \mu, TP_A(x)\rangle = \langle T'(\mu), y\rangle = \langle \mu, T(\psi(x))\rangle.$$

Thus $\phi'_{\mathcal{U}}(\Phi) = T \circ \psi$. Actually, we have also shown that $\phi'_{\mathcal{V}}(\Phi) = T \circ \psi$ in $\mathcal{B}((l^p)_{\mathcal{V}})$.

Now let $\alpha : \mathcal{W} \to \mathbb{N}$ be such that $\alpha(A) \in A$ for each $A \in \mathcal{W}$. Then let $x_A = e_{\alpha(A)}$ so that $x = (x_A) \in (l^p)_{\mathcal{V}}$. For each $B \in \mathcal{W}$, we have

$$\{A \in \mathcal{W} : P_B(x_A) = x_A\} = \{A \in \mathcal{W} : \alpha(A) \in B\} \supseteq \{A \in \mathcal{W} : A \subseteq B\} \in \mathcal{V},$$

and so $\lim_{A\in\mathcal{V}} \|P_B(x_A) - x_A\| = 0$. Thus $P_B(x) = x$. So, by Lemma 3.9, $\psi(x) = x$, and clearly $T(x) \neq 0$, so that $\phi'_{\mathcal{V}}(\Phi)(x) \neq 0$, and hence $\Phi \neq 0$. □

We shall now show, by contradiction, that this functional Φ (as defined above) is in the radical of $\mathcal{B}(l^p)''$.

Proposition 3.12 *Let E be a super-reflexive Banach space such that there exists a surjection $\phi_{\mathcal{U}} : (E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}} \to \mathcal{B}(E)'$ (for example, $E = l^p$ for $1 < p < \infty$). If $\Phi \notin \operatorname{rad}\mathcal{B}(E)''$, then, for some $\Psi \in \mathcal{B}(E)''$, the operator $\phi'(\operatorname{Id} - \Psi\Phi) \in \mathcal{B}((E)_{\mathcal{U}})$ is not bounded below.*

Proof. As $\Phi \notin \operatorname{rad}\mathcal{B}(E)''$, we can find $\Psi \in \mathcal{B}(E)''$ with $1 \in \operatorname{Sp}(\Psi\Phi)$. Thus, by rescaling Ψ, we may suppose that 1 is in the boundary of $\operatorname{Sp}(\Psi\Phi)$. Thus we can find a sequence (λ_n) in $\mathbb{C}$ so that $\lambda_n \to 1$ and $\lambda_n \operatorname{Id} - \Psi\Phi$ is invertible for each $n \in \mathbb{N}$. Let $U_n = (\lambda_n \operatorname{Id} - \Psi\Phi)^{-1}$, and suppose that (U_n) is a bounded sequence. Then

$$\begin{aligned}\|U_n(\operatorname{Id} - \Psi\Phi) - \operatorname{Id}\| &= \|U_n(\lambda_n \operatorname{Id} - \Psi\Phi) + U_n(\operatorname{Id} - \lambda_n \operatorname{Id}) - \operatorname{Id}\| \\ &= \|U_n\|(1 - \lambda_n) \to 0,\end{aligned}$$

which contradicts the fact that $\operatorname{Id} - \Psi\Phi$ is not invertible. Indeed, we have shown that no subsequence of (U_n) can be bounded.

Let $S_n = \phi'(U_n)\|\phi'(U_n)\|^{-1}$ for each $n \in \mathbb{N}$, so that $\|S_n\| = 1$ for each n, and note that $\|\phi'(U_n)\|^{-1} \to 0$, because ϕ' is an isomorphism onto its range. Then

$$\|\phi'(\operatorname{Id} - \Psi\Phi)S_n\| \leq \|\phi'((\lambda_n \operatorname{Id} - \Psi\Phi)U_n)\|\|\phi'(U_n)\|^{-1} + (1 - \lambda_n) \to 0,$$

so $\phi'(\operatorname{Id} - \Psi\Phi)$ cannot be bounded below. ☐

Let us say that $C \subset \mathbb{N}$ is *B-reasonable* if $|C \cap A_k| \leq B$ for every k. For any r, a vector $x \in l^r$ is B-reasonable if $\operatorname{supp}(x)$ is B-reasonable. For an ultrafilter $\mathcal{U}$, $x \in (l^r)_{\mathcal{U}}$ is B-reasonable if for some representative (x_i) of x, x_i is B-reasonable for every i.

Proposition 3.13 *If $\Phi \notin \operatorname{rad}\mathcal{B}(l^p)''$, then there exists $\Psi \in \mathcal{B}(l^p)''$, $B \in \mathbb{N}$ and a B-reasonable $z \in (l^p)_{\mathcal{U}}$ with the following properties:*

(1) $\|z\| \leq 1$*;*

(2) $P_A(z) = z$ *for each* $A \in \mathcal{W}$*;*

(3) if $\mu^z \in (l^q)_{\mathcal{W}}$ *with* $\langle \mu^z, z\rangle = \|z\|$ *and* $\|\mu^z\| = 1$*, then*

$$|\langle \mu^z, \phi'(\Psi)(T(z))\rangle| > \frac{1}{2}\|\Psi\|^{-1}.$$

Proof. By Proposition 3.12, we can find $\Psi \in \mathcal{B}(l^p)''$ and $x \in (l^p)_{\mathcal{U}}$ with $\|x\| = 1$ and

$$\|\phi'(\Psi\Phi)(x) - x\| = \|(\phi'(\Psi) \circ T \circ \psi)(x) - x\| < \varepsilon,$$

where $\varepsilon > 0$ is to be chosen later. By Lemma 3.10, $\lim_{A\in\mathcal{V}} P_A(x)$ exists; set $y = \lim_{A\in\mathcal{V}} P_A(x)$, so that $\|y\| \leq 1$ and $\|\phi'(\Psi)(T(y)) - x\| < \varepsilon$, and hence also $\|\phi'(\Psi)(T(y))\| > 1 - \varepsilon$.

Choose a representative (y_i) of y with, for each $i \in I$, $\|y_i\| = \|y\|$ and $y_i = \sum_j y_{i,j}e_j$. Then let $\gamma_{i,k} = \left(\sum_{j\in A_k} |y_{i,j}|^p\right)^{1/p}$, and let $\delta_{i,k} = \max_{j\in A_k} |y_{i,j}|$. Then, for each k and i, we have

$$\begin{aligned}\left(\sum_{j\in A_k} |y_{i,j}|^2\right)^{1/2} &= \gamma_{i,k}\left(\sum_{j\in A_k} \frac{|y_{i,j}|^2}{|\gamma_{i,k}|^2}\right)^{1/2} \leq \gamma_{i,k}\left(\sum_{j\in A_k} \frac{|y_{i,j}|^p}{|\gamma_{i,k}|^p}\delta_{i,k}^{2-p}\gamma_{i,k}^{p-2}\right)^{1/2}\\ &= \delta_{i,k}^{1-p/2}\gamma_{i,k}^{p/2}\left(\sum_{j\in A_k} \frac{|y_{i,j}|^p}{|\gamma_{i,k}|^p}\right)^{1/2} = \delta_{i,k}^{1-p/2}\gamma_{i,k}^{p/2}.\end{aligned}$$

Hence, by (1), we have

$$\|T(y_i)\| \leq \left(\sum_k \frac{(k+1)^p}{k^p}\delta_{i,k}^{p(1-p/2)}\gamma_{i,k}^{p^2/2}\right)^{1/p}. \tag{2}$$

Pick $K \in \mathbb{N}$ and choose $B \in \mathbb{N}$ so that $B \geq |A_k|$ for $k \leq K$, and $B^{1/p-1/2} > (K+1)/K\varepsilon$. For each $i \in \mathbb{N}$ choose a B-reasonable set $D_i \subset \mathbb{N}$ so that $\sum_{j\in D_i} |y_{i,j}|^p$ is maximal. For each i let $\hat{y}_i = P_{\mathbb{N}\setminus D_i}(y_i)$, and define $\hat{\gamma}_{i,k}$ and $\hat{\delta}_{i,k}$ for $\hat{y}_i$ in an analogous manner to the definitions of $\gamma_{i,k}$ and $\delta_{i,k}$. Note that, if $B \geq |A_k|$, then $\hat{\gamma}_{i,k} = 0$ for each i. For each i and k, $\hat{\gamma}_{i,k} \leq \gamma_{i,k}$, and we have

$$\gamma_{i,k}^p = \sum_{j\in A_k\cap D_i} |y_{i,j}|^p + \sum_{j\in A_k\setminus D_i} |y_{i,j}|^p \geq B \max_{j\in A_k\setminus D_i} |y_{i,j}|^p = B\hat{\delta}_{i,k}^p,$$

so that $\hat{\delta}_{i,k} \leq B^{-1/p}\gamma_{i,k}$. Thus, by (2),

$$\begin{aligned}\|T(\hat{y}_i)\| &\leq \left(\sum_{k>K} \frac{(k+1)^p}{k^p} \hat{\delta}_{i,k}^{p(1-p/2)} \hat{\gamma}_{i,k}^{p^2/2}\right)^{1/p} \leq \left(\sum_{k>K} \frac{(k+1)^p}{k^p} B^{p/2-1}\gamma_{i,k}^p\right)^{1/p} \\ &= B^{1/2-1/p}\left(\sum_{k>K} \frac{(k+1)^p}{k^p}\gamma_{i,k}^p\right)^{1/p} \leq \frac{K+1}{K}B^{1/2-1/p}\|y_i\| < \varepsilon\end{aligned}$$

by our choice of B.

Let $z = y - \hat{y} = (P_{D_i}(y_i))$, so that z is B-reasonable, and $\|z\| \leq 1$. For each $A \in \mathcal{W}$, we have $y = P_A(y)$, and so

$$\begin{aligned}\|P_A(z) - z\| &= \lim_{i \in \mathcal{U}} \|P_A(P_{D_i}(y_i)) - P_{D_i}(y_i)\| \\ &\leq \lim_{i \in \mathcal{U}} \|P_A(y_i) - y_i\| = \|P_A(y) - y\| = 0.\end{aligned}$$

Now let $\mu^z = (\mu_i^z) \in (l^q)_{\mathcal{U}}$ be such that $\|\mu_i^z\| = 1$ and $\langle \mu_i^z, z_i\rangle = \|z_i\|$ for each i. Then, for each i, $\operatorname{supp}(z_i) = \operatorname{supp}(\mu_i^z)$ so that

$$\langle \mu_i^z, y_i - z_i\rangle = \langle P_{D_i}(\mu_i^z), P_{\mathbb{N}\setminus D_i}(y_i)\rangle = 0.$$

Thus $\langle \mu^z, z\rangle = \langle \mu^z, y\rangle$. For $A \in \mathcal{W}$, as $P_A(z) = z$ we have $P_A(\mu^z) = \mu^z$, and so

$$\|z\| = \langle \mu^z, z\rangle = \langle \mu^z, y\rangle = \lim_{A \in \mathcal{V}} \langle \mu^z, P_A(x)\rangle = \lim_{A \in \mathcal{V}} \langle P_A(\mu^z), x\rangle = \langle \mu^z, x\rangle.$$

Let T_K be T restricted to the subspace of vectors in l^p whose support is contained in $\bigcup_{k>K} A_k$. Then we have $T(z) = T(y - \hat{y}) = T_K(z)$ and $\|T_K\| \leq (K+1)/K$. As $\|\phi'(\Psi)(T(y))\| > 1 - \varepsilon$ and $\|T(\hat{y})\| < \varepsilon$, we have

$$\begin{aligned}\|z\| &\geq \|T_K\|^{-1}\|T_K(z)\| \geq K(K+1)^{-1}(\|T(y)\| - \|T(y-z)\|) \\ &\geq K(K+1)^{-1}(\|\phi'(\Psi)(T(y))\|\|\Psi\|^{-1} - \varepsilon) \\ &\geq K(K+1)^{-1}((1-\varepsilon)\|\Psi\|^{-1} - \varepsilon)\end{aligned}$$

So finally we have

$$\begin{aligned}|\langle \mu^z, \phi'(\Psi)(T(z))\rangle| &\geq |\langle \mu^z, \phi'(\Psi)(T(y))\rangle| - \|\mu^z\|\|\Psi\|\|T(z-y)\| \\ &\geq |\langle \mu^z, x\rangle| - |\langle \mu^z, x - \phi'(\Psi)(T(y))\rangle| - \varepsilon\|\Psi\| \\ &\geq \|z\| - \varepsilon - \varepsilon\|\Psi\|.\end{aligned}$$

Thus, for each $\delta > 0$, we can, by a choice of $\varepsilon > 0$ and $K \in \mathbb{N}$, ensure that

$$|\langle \mu^z, \phi'(\Psi)(T(z))\rangle| \geq \|\Psi\|^{-1}(1-\delta).$$

We thus have conclusions (1) and (2), and setting $\delta = 1/2$ we get conclusion (3). □

We shall now study maps from l^2 to l^p, and show how this gives rise to a contradiction with the above proposition.

Lemma 3.14 *Fix $M > 0$ and $\varepsilon > 0$, and let*

$$\delta_k = \delta_k(M, \varepsilon) = \sup_{S_k} \frac{1}{k} |\{1 \leq n \leq k : |\langle S_k(e_n), e_n\rangle| \geq \varepsilon\}| \qquad (k \in \mathbb{N})$$

where S_k varies over $\mathcal{B}(l_k^2, l_k^p)$ with $\|S_k\| \leq M$. Then $\lim_{k\to\infty} \delta_k = 0$ and $(k\delta_k)$ is eventually a decreasing sequence.

Proof. If (δ_k) does not tend to zero for some $M > 0$ and $\varepsilon > 0$, then for some $\delta > 0$, we can find infinitely many values of k for which there exists $S_k \in \mathcal{B}(l_k^2, l_k^p)$ so that $|\{1 \leq n \leq k : |\langle S_k(e_n), e_n\rangle| \geq \varepsilon\}| \geq k\delta$. Move to a subsequence (k_j) for which this is always true. By composing S_{k_j} with a permutation operator, we may suppose that

$$|\langle S_{k_j}(e_n), e_n\rangle| \geq \varepsilon \qquad (j \in \mathbb{N}, 1 \leq n \leq k_j\delta).$$

For each $j \in \mathbb{N}$, let $\alpha_j : l^2 \to l_j^2$ be projection onto the first j co-ordinates, and let $\beta_j : l_j^p \to l^p$ be the natural inclusion. Then $\beta_{k_j} \circ S_{k_j} \circ \alpha_{k_j} \in \mathcal{B}(l^2, l^p)$ for each j. As $\mathcal{B}(l^2, l^p) = \mathcal{K}(l^2, l^p)$ is reflexive, we can define $R = \text{weak-lim}_{j\in\mathcal{U}} \beta_{k_j} \circ S_{k_j} \circ \alpha_{k_j} \in \mathcal{B}(l^2, l^p)$. Then $\|R\| \leq M$, R is compact, and, for each $n \in \mathbb{N}$, we have

$$|\langle R(e_n), e_n\rangle| = \lim_{j\in\mathcal{U}} |\langle S_{k_j}(e_n), e_n\rangle| \geq \varepsilon,$$

because eventually $n \leq k_j\delta$. This clearly contradicts the fact that R is compact, showing that $\lim_{k\to\infty} \delta_k = 0$.

Now fix $k \in \mathbb{N}$, and choose $l \in \mathbb{N}$ so that $k\delta_k \leq l \leq k$. Let $\iota_1 : l_l^2 \to l_k^2$ be the canonical inclusion, and $\iota_2 : l_k^p \to l_l^p$ be the projection onto the first l co-ordinates. Choose $S_k \in \mathcal{B}(l_k^2, l_k^p)$ so that, for $1 \leq i \leq k\delta_k$, we have $|\langle S_k(e_i), e_i\rangle| \geq \varepsilon$. Let $R = \iota_2 \circ S_k \circ \iota_1 \in \mathcal{B}(l_l^2, l_l^p)$, so that $|\langle R(e_i), e_i\rangle| \geq \varepsilon$ for $1 \leq i \leq k\delta_k$. We conclude that $l\delta_l \geq k\delta_k$, and thus that, if k is sufficiently large, $k\delta_k \geq (k+1)\delta_{k+1}$. □

For each $M > 0, \varepsilon > 0$ define $(\delta_k(M, \varepsilon))$ as above, and let

$$\delta(M, \varepsilon) = \inf\{k\delta_k(M, \varepsilon) : k \in \mathbb{N}\} = \lim_{k\to\infty} k\delta_k(M, \varepsilon).$$

As $k\delta_k(M, \varepsilon) \in \mathbb{N}$, eventually $k\delta_k(M, \varepsilon) = \delta(M, \varepsilon)$.

Lemma 3.15 *Let $M > 0, \varepsilon > 0$, $S \in \mathcal{B}(l^2, l^p)$ with $\|S\| \leq M$, $(x_i)_{i=1}^n$ be an orthonormal set in l^2 and $(A_i)_{i=1}^n$ be a pairwise disjoint family of subsets of $\mathbb{N}$. If, for each i, $\|P_{A_i}(S(x_i))\| \geq \varepsilon$, then $n \leq \delta(M, \varepsilon)$.*

Proof. For each i, choose $\mu_i \in l^q$ with $\|\mu_i\| = 1$ and $\langle \mu_i, S(x_i)\rangle = \|P_{A_i}(S(x_i))\|$, so that $\mathrm{supp}(\mu_i) \subseteq A_i$. Choose $U \in \mathcal{B}(l^2)$ with $\|U\| = 1$, and $U(e_i) = x_i$ for $1 \leq i \leq n$, and choose $V \in \mathcal{B}(l^q)$ with $\|V\| = 1$, and $V(e_i) = \mu_i$ for $1 \leq i \leq n$. Let $R = V' \circ S \circ U$ so that $|\langle R(e_i), e_i\rangle| = |\langle \mu_i, S(x_i)\rangle| \geq \varepsilon$. Hence, by Lemma 3.14, for each $k \geq n$, we have $k\delta_k \geq n$, and so $n \leq \delta(M, \varepsilon)$. □

Lemma 3.16 *If the sequence (n_k) is such that $n_k \to \infty$, then, for each $S \in \mathcal{B}(l^p)$, each $B \in \mathbb{N}$ and each $\varepsilon > 0$, we can find $A \in \mathcal{F} \subset \mathcal{W}$ so that for any B-reasonable $x \in l^p$ and $\mu \in l^q$ with $\langle \mu, x\rangle = \|\mu\| = \|x\| = 1$, we have $\sum_{k=1}^\infty |\langle \mu, P_{A_k\cap A} S T P_{A_k\cap A}(x)\rangle| < \varepsilon$.*

Proof. For $k \in \mathbb{N}$, let $T_k = T \circ P_{A_k}$ so, as $l_{n_k}^p$ is canonically isomorphic to $l^p(A_k)$, the image of P_{A_k}, we can view T_k as a map from $l_{n_k}^p$ to l^p. Then, for $x \in l_{n_k}^p$, we have

$$\frac{k-1}{k}\|x\|_2 \leq \|T_k(x)\| \leq \frac{k+1}{k}\|x\|_2,$$

so we can view T_k as an isomorphism from $l^2_{n_k}$ onto its image in l^p. Thus, for each k, let $S_k = S \circ T \circ P_{A_k} : l^2_{n_k} \to l^p$, so that $\|S_k\| \leq 2\|S\|$. Let $m \in \mathbb{N}$ be maximal so that we have $(x_i)_{i=1}^m$ a set of B-reasonable norm one vectors in $l^2_{n_k}$ with disjoint support, and $(B_i)_{i=1}^m$ a set of B-reasonable pairwise disjoint subsets of A_k, so that $\|P_{B_i}(S_k(x_i))\| \geq \varepsilon$. Let $C_k = \bigcup_{i=1}^m \operatorname{supp}(x_i) \cup \bigcup_{i=1}^m B_i \subseteq A_k$.

If $x \in l^2_{n_k}$ is B-reasonable with $C_k \cap \operatorname{supp}(x) = \emptyset$, and $\mu \in l^q$ is B-reasonable with $\operatorname{supp}(\mu) \cap C_k = \emptyset$, then, by the maximality of m,

$$|\langle \mu, S_k(x) \rangle| \leq \|\mu\| \|P_{\operatorname{supp}(\mu)}(S_k(x))\| < \varepsilon \|\mu\| \|x\|.$$

Also, by Lemma 3.15, $m \leq \delta(2\|S\|, \varepsilon)$, so that $|C_k| \leq 2Bm \leq 2B\delta(2\|S\|, \varepsilon)$.

Let $A = \mathbb{N} \setminus \bigcup_{k=1}^\infty C_k$, so that for each k, we have

$$|(\mathbb{N} \setminus A) \cap A_k||A_k|^{-1} = |C_k||A_k|^{-1} \leq 2B\delta(2\|S\|, \varepsilon) n_k^{-1},$$

and thus $\limsup_{k\to\infty} |(\mathbb{N} \setminus A) \cap A_k||A_k|^{-1} = 0$, so that $A \in \mathcal{F}$. For a B-reasonable $x \in l^p$, and $\mu \in l^q$ with $1 = \langle \mu, x \rangle = \|x\| = \|\mu\|$, μ is B-reasonable, and so we have

$$\begin{aligned} \sum_{k=1}^\infty |\langle \mu, P_{A_k \cap A} S T P_{A_k \cap A}(x) \rangle| &= \sum_{k=1}^\infty |\langle \mu, P_{A_k \cap A} S_k P_{A_k \cap A}(x) \rangle| \\ &< \varepsilon \sum_{k=1}^\infty \|P_{A_k \cap A}(\mu)\| \|P_{A_k \cap A}(x)\| \\ &\leq \varepsilon \left(\sum_{k=1}^\infty \|P_{A_k \cap A}(\mu)\|^q \right)^{1/q} \left(\sum_{k=1}^\infty \|P_{A_k \cap A}(x)\|^p \right)^{1/p} \leq \varepsilon, \end{aligned}$$

as required. □

Proposition 3.17 *If the sequence (n_k) increases fast enough, then for $S \in \mathcal{B}(l^p)$, $B \in \mathbb{N}$ and $\varepsilon > 0$, we can find $A \in \mathcal{F}$ so that for any B-reasonable $x \in l^p$ and $\mu \in l^q$ with $\langle \mu, x \rangle = \|x\|$ and $\|\mu\| = 1$, we have $|\langle \mu, P_A S T P_A(x) \rangle| < \varepsilon \|x\|$.*

Proof. First note that it is enough to prove the result in the case where $\|x\| = 1$, for otherwise let $y = \|x\|^{-1}x$, so that $\|y\| = 1$ and $\langle \mu, y \rangle = \|x\|^{-1}\langle \mu, x \rangle = 1$, so that $|\langle \mu, P_A S T P_A(x) \rangle| = \|x\| |\langle \mu, P_A S T P_A(y) \rangle| < \varepsilon \|x\|$ as required. Hence we shall suppose that $\|x\| = 1$.

By (n_k) increasing fast enough, we mean that

$$2^{1+k+n_1+\ldots+n_{k-1}}/n_k \to 0$$

as $k \to \infty$.

If $x = \sum_{i=1}^{\infty} x_i e_i$ and $\mu = \sum_{i=1}^{\infty} \mu_i e_i$ then, for each $i \in \mathbb{N}$, $\mu_i = \overline{x_i}|x_i|^{p-2}$. We then have

$$\begin{aligned}
|\langle \mu, P_A STP_A(x)\rangle| &= \left| \sum_{i,j\in A} \overline{x_j}|x_j|^{p-2} x_i \langle e_j, ST(e_i)\rangle \right| \\
&\le \sum_{k=1}^{\infty} \left| \sum_{l=1}^{\infty} \sum_{i\in A\cap A_k} \sum_{j\in A\cap A_l} \overline{x_j}|x_j|^{p-2} x_i \langle e_j, ST(e_i)\rangle \right| \le \alpha_1 + \alpha_2 + \alpha_3,
\end{aligned}$$

where we shall define α_1, α_2 and α_3 below. Note that, if we can find $A_i \in \mathcal{W}$ so that with $A = A_1$, α_1 is small, and similarly for A_2 and A_3, then setting $A = A_1 \cap A_2 \cap A_3 \in \mathcal{F}$ will ensure that $|\langle \mu, P_A STP_A(x)\rangle|$ is small.

We first ensure that α_1 can be made as small as we like by a choice of $A \in \mathcal{F}$. Indeed,

$$\begin{aligned}
\alpha_1 &= \sum_{k=1}^{\infty} \left| \sum_{l=k+1}^{\infty} \sum_{i\in A\cap A_k} \sum_{j\in A\cap A_l} \overline{x_j}|x_j|^{p-2} x_i \langle e_j, ST(e_i)\rangle \right| \\
&\le B^2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \max_{i\in A\cap A_k, j\in A\cap A_l} |x_j|^{p-1}|x_i||\langle e_j, ST(e_i)\rangle| \\
&\le B^2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \max_{i\in A\cap A_k, j\in A\cap A_l} |\langle e_j, ST(e_i)\rangle| \qquad (3)
\end{aligned}$$

because both x and μ are B-reasonable. Let C be chosen later to be much larger than B. For each $k \in \mathbb{N}$ and $i \in A_k$, let $E_i \subset A_{k+1} \cup A_{k+2} \cup \cdots$ be chosen so that, for each $l > k$, $|E_i \cap A_l| \le 2^{i+l}C$ and $\sum_{j\in E_i} |\langle e_j, ST(e_i)\rangle|^p$ is maximal. Let $A = \mathbb{N} \setminus \bigcup_{i=1}^{\infty} E_i$, so for each k,

$$|(\mathbb{N}\setminus A)\cap A_k| = \left| \bigcup_{i=1}^{N_{k-1}} E_i \cap A_k \right| \le \sum_{i=1}^{N_{k-1}} |E_i \cap A_k| \le C \sum_{i=1}^{N_{k-1}} 2^{i+k} \le C 2^{N_{k-1}+k+1},$$

and so $|(\mathbb{N}\setminus A)\cap A_k||A_k|^{-1} \le C2^{1+k+n_1+\cdots+n_{k-1}}/n_k$. By the assumption on (n_k), we thus have $|(\mathbb{N}\setminus A)\cap A_k||A_k|^{-1} \to 0$ as $n \to \infty$, so that $\mathrm{ud}(\mathbb{N}\setminus A) = 0$, and so $A \in \mathcal{F}$.

Now, for each $k \in \mathbb{N}$, $l > k$, $i \in A \cap A_k$ and $j \in A \cap A_l$ we have $j \in A_l \setminus \bigcup_{r=1}^{N_{l-1}} E_r$, so certainly $j \in A_l \setminus E_i$, and hence

$$\begin{aligned}
(2\|S\|)^p \geq \|ST(e_i)\|^p &= \sum_{s=1}^{\infty} |\langle e_s, ST(e_i)\rangle|^p \\
&= \sum_{s \in A_l \cap E_i} |\langle e_s, ST(e_i)\rangle|^p + \sum_{s \in A_l \setminus E_i} |\langle e_s, ST(e_i)\rangle|^p \\
&\geq \sum_{s \in A_l \cap E_i} |\langle e_s, ST(e_i)\rangle|^p \geq |A_l \cap E_i| |\langle e_j, ST(e_i)\rangle|^p,
\end{aligned}$$

so that $|\langle e_j, ST(e_i)\rangle| \leq 2\|S\|(2^{i+l}C)^{-1/p}$. Thus

$$\begin{aligned}
\alpha_1 &\leq B^2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \max_{i \in A \cap A_k, j \in A \cap A_l} 2\|S\|(2^{i+l}B')^{-1/p} \\
&\leq 2\|S\|B^2 C^{-1/p} \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} 2^{-(N_k+l)/p} \\
&\leq DB^2\|S\|C^{-1/p}
\end{aligned}$$

for some constant D depending on $(n_k)_{k=1}^{\infty}$. Thus, by choosing C sufficiently large, we can make α_1 arbitrarily small, independently of x and μ.

Now we will look at α_2, which is

$$\begin{aligned}
\alpha_2 &= \sum_{k=1}^{\infty} \left| \sum_{l=1}^{k-1} \sum_{i \in A \cap A_k} \sum_{j \in A \cap A_l} \overline{x_j} |x_j|^{p-2} x_i \langle e_j, ST(e_i)\rangle \right| \\
&\leq B^2 \sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty} \max_{i \in A \cap A_k, j \in A \cap A_l} |\langle T'S'(e_i), e_j\rangle|.
\end{aligned}$$

Compare this to (3), and we see that we can use exactly the same argument as above to ensure that α_2 is arbitrarily small.

Finally, we need to show that α_3 can be made small, where

$$\alpha_3 = \sum_{k=1}^{\infty} \left| \sum_{i,j \in A \cap A_k} \overline{x_j} |x_j|^{p-2} x_i \langle e_j, ST(e_i)\rangle \right| = \sum_{k=1}^{\infty} |\langle \mu, P_{A \cap A_k} ST P_{A \cap A_k}(x)\rangle| \,.$$

So by Lemma 3.16, we are done. □

We now put Propositions 3.13 and 3.17 together.

Theorem 3.18 *For $1 < p < 2$, $\mathcal{B}(l^p)''$ is not semi-simple.*

Proof. Choose and fix (n_k) so that Proposition 3.17 can be applied. If $\Phi \notin \operatorname{rad} \mathcal{B}(l^p)''$, then by Proposition 3.13, there exists $\Psi \in \mathcal{B}(l^p)''$ and $z \in (l^p)_{\mathcal{U}}$ with $|\langle \mu^z, \phi'(\Psi)(T(z))\rangle| > 1/2\|\Psi\|$. Using Lemma 3.1 we can find $S \in \mathcal{B}(l^p)$ with $\|S\| \leq \|\Psi\|$ and $\|\phi'(\Psi)(T(z)) - ST(z)\| < \varepsilon$, so that $|\langle \mu^z, ST(z)\rangle| > 1/2\|\Psi\|$ if $\varepsilon > 0$ is sufficiently small. As z is such that $P_A(z) = z$ for every $A \in \mathcal{W}$, we also have $P_A(\mu^z) = \mu^z$ for every $A \in \mathcal{W}$. Thus we have

$$\lim_{A \in \mathcal{V}} |\langle \mu^z, P_A S T P_A(z)\rangle| \geq 1/2\|\Psi\|.$$

However, by Proposition 3.17, for every $\delta > 0$ we can find $A \in \mathcal{F} \subset \mathcal{W}$ so that $|\langle \mu_i^z, P_A S T P_A(z_i)\rangle| < \delta$ for each i. Thus we have

$$|\langle \mu^z, P_A S T P_A(z)\rangle| \leq \delta,$$

and as $\delta > 0$ was arbitrary, we have

$$\lim_{A \in \mathcal{V}} |\langle \mu^z, P_A S T P_A(z)\rangle| = 0.$$

This contradiction shows that actually $\Phi \in \operatorname{rad} \mathcal{B}(l^p)''$ and so $\mathcal{B}(l^p)''$ is not semi-simple. □

4 A generalisation

We can use the same idea as in Lemma 2.3 to find further examples of Banach spaces E such that $\mathcal{B}(E)''$ is not semi-simple.

Proposition 4.1 *Let $\mathcal{A}$ be a unital Banach algebra, and let $p, q \in \mathcal{A}$ be orthogonal idemopotents (that is, $p^2 = p, q^2 = q$ and $pq = qp = 0$) such that $p + q = e_{\mathcal{A}}$. If the subalgebra $p\mathcal{A}p$ is not semi-simple, then $\mathcal{A}$ is not semi-simple.*

Proof. As in Lemma 2.3, we can view $\mathcal{A}$ as a matrix algebra. Let $c \in \operatorname{rad} p\mathcal{A}p$ be non-zero, let $a = pcp \in \mathcal{A}$, and pick $b \in \mathcal{A}$. Then

$$ab = \begin{pmatrix} pcp & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} pbp & pbq \\ qbp & qbq \end{pmatrix} = \begin{pmatrix} pcpbp & pcpbq \\ 0 & 0 \end{pmatrix},$$

so that

$$(ab)^n = \begin{pmatrix} (pcpbp)^n & (pcpbp)^{n-1}(pcpbq) \\ 0 & 0 \end{pmatrix}.$$

As $c \in \operatorname{rad} p\mathcal{A}p$, we see that $\lim_{n\to\infty} \|(pcpbp)^n\|^{1/n} = \lim_{n\to\infty} \|(cbp)^n\|^{1/n} = 0$. We then have

$$\begin{aligned} \|(ab)^n\|^{1/n} &= \|(pcpbp)^n + (pcpbp)^{n-1}(pcpbq)\|^{1/n} \\ &\leq (\|(pcpbp)^n\| + \|(pcpbp)^{n-1}\|\|pcpbq\|)^{1/n} \to 0 \end{aligned}$$

as $n \to 0$. Thus, as b was arbitrary, $a \in \operatorname{rad} \mathcal{A}$, and so $\mathcal{A}$ is not semi-simple. □

Let F and G be Banach spaces, and let $E = F \oplus G$. Then

$$\mathcal{B}(E)'' = \left\{ \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} : \Phi_{11} \in \mathcal{B}(F)'', \Phi_{12} \in \mathcal{B}(G,F)'' \text{ etc.} \right\}.$$

We can thus apply to above proposition to see that if E is a Banach space with complemented subspace F such that $\mathcal{B}(F)''$ is not semi-simple, with respect to one of the Arens products, then $\mathcal{B}(E)''$ is not semi-simple with respect to the same Arens product.

We now set out some results about general L^p-spaces, with the aim of showing that $\mathcal{B}(L^p(\mu))''$ is semi-simple if and only if $L^p(\mu)$ is isomorphic to a Hilbert space.

Proposition 4.2 *Let $\varepsilon > 0$, $p \in (2, \infty)$ and ν be an arbitrary measure, and let (x_n) be a normalised sequence in $L^p(\nu)$ equivalent to the canonical basis of*

l^p. Then there exists a subsequence $(x_{n(i)})$ which is $(1+\varepsilon)$-equivalent to the basis of l^p, and whose closed linear span is $(1+\varepsilon)$-complemented in $L^p(\nu)$.

Proof. This follows from the proof of [9, Theorem 2]; see also the proof of [8, Theorem 10]. □

Proposition 4.3 *Let $p \in [1, \infty)$ and E be a separable subspace of $L^p(\nu)$ for some measure ν. Then E is isometrically isomorphic to a subspace of $L^p[0, 1]$.*

Proof. This is [7, Theorem IV.1.7]. □

Proposition 4.4 *Let $p \in [2, \infty)$ and E be an infinite-dimensional subspace of $L^p[0, 1]$. Then either E is isomorphic to l^2 or, for each $\varepsilon > 0$, E contains a subspace which is $(1+\varepsilon)$-isomorphic to l^p.*

Proof. This is [7, Corollary IV.4.4]. □

Theorem 4.5 *Let $p \in (2, \infty)$, ν be an arbitrary measure, and E be a subspace of $L^p(\nu)$ such that E is not isomorphic to a Hilbert space. Then $\mathcal{B}(E)''$ is not semi-simple.*

Proof. Choose a separable subspace F of E, so that, by Theorem 4.3, F is isometrically isomorphic to a subspace of $L^p[0, 1]$. Then by Proposition 4.4, either F is isomorphic to l^2, or F contains an isomorphic copy of l^p. If the latter, then by Proposition 4.2, F contains a complemented copy of l^p, and so, by an application of Proposition 4.1, $\mathcal{B}(F)''$ is not semi-simple.

So the only case left to consider is when every separable subspace of E is isomorphic to l^2. However, then E is itself isomorphic to a Hilbert space, a contradiction of a hypothesis. □

The class of $\mathfrak{L}^g_{p,\lambda}$ spaces are defined in [4, Section 3.13], for $1 \leq p \leq \infty$, $1 \leq \lambda < \infty$, to be Banach spaces E such that for each finite dimensional subspace M of E, and each $\varepsilon > 0$, we can find $R \in \mathcal{B}(M, l^p_m)$ and $S \in \mathcal{B}(l^p_m, E)$ for some $m \in \mathbb{N}$, so that $SR(x) = x$ for each $x \in M$, and $\|S\|\|R\| \leq \lambda + \varepsilon$. Then E is an-$\mathfrak{L}^g_p$ space if it is an $\mathfrak{L}^g_{p,\lambda}$-space for some λ. In [4, Section 23.2], it is shown that for $1 < p < \infty$, E is an $\mathfrak{L}_p^g$-space if and only if E is isomorphic to a complemented subspace of some $L^p(\mu)$ space. Thus we have the following.

Corollary 4.6 *Let E be a complemented subspace of $L^p(\nu)$ for $1 < p < \infty$ and some measure ν (that is, E is an $\mathfrak{L}_p^g$-space). Then $\mathcal{B}(E)''$ is semi-simple if and only if E is isomorphic to a Hilbert space.*

5 Conclusion

Summing up our results, we have the following.

Theorem 5.1 *Let E be a Banach space such that at least one of the following holds:*

(1) E is reflexive and $E = F \oplus G$ with one of F and G having the AP, $\mathcal{B}(F, G) = \mathcal{K}(F, G)$ and $\mathcal{B}(F, G) \neq \mathcal{K}(F, G)$;
(2) E is a complemented subspace of $L^p(\nu)$, for some measure ν and $1 < p < \infty$, such that E is not isomorphic to a Hilbert space;
(3) E is a closed subspace of $L^p(\nu)$ for some measure ν and $2 < p < \infty$, and E is not isomorphic to a Hilbert space;
(4) E contains a complemented subspace F so that F has property (1), (2) or (3).

Then $\mathcal{B}(E)''$ is not semi-simple. □

In particular, at present the only Banach spaces E for which $\mathcal{B}(E)''$ is semi-simple are those isomorphic to a Hilbert space. We conjecture that $\mathcal{B}(E)''$ is semi-simple only if E is isomorphic to a Hilbert space, at least when E is super-reflexive.

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