Perspectives on Noncommutative Graphs

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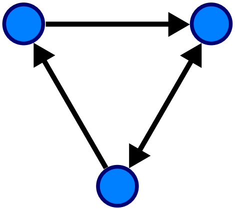
Graphs

A graph consists of a (finite) set of *vertices* V and a collection of *edges* $E \subseteq V \times V$.



$V = \{A, B, C\}$ say, and $E = \{(A, B), (B, C), (C, B), (C, A)\}$.

A graph is *undirected* if $(x, y) \in E \Leftrightarrow (y, x) \in E$. We allow *self-loops*, so $(x, x) \in E$.

Notice that a graph $G = (V, E)$ is exactly a *relation* on the set V. An undirected graph gives a symmetric relation; having a loop on each vertex gives a reflexive relation.

Channels

A channel sends an input message (element of a finite set A) to an output message (element of a finite set B) perhaps with *noise* so that there is a probability that $a \in A$ is mapped to different $b \in B$.

- Input "o" might be sent to "o" or "0" or "a".

$$p(b|a) = \text{probability that } b \text{ is received given that } a \text{ was sent}$$

Define a (simple, undirected) graph structure on A by

$$(a_1, a_2) \text{ an edge when } p(b|a_1)p(b|a_2) > 0 \text{ for some } b.$$

This is the *confusability graph* of the channel.
If we want to communicate with *zero error* then we seek a maximal *independent set* in A.

Quantum Mechanics

- A *state* is a unit vector $|\psi\rangle$ in a (finite dim) Hilbert space H.
- More generally, a *density* is a positive, trace one operator $\rho \in \mathcal{B}(H)$.
- A rank-one density is always of the form $|\psi\rangle\langle\psi|$ for some state ψ.
- (Use Trace duality, so $\omega \in \mathcal{B}(H)^*$ is associated uniquely to $A \in \mathcal{B}(H)$ with $\omega(T) = \mathrm{tr}(AT)$. Then densities are exactly the *states* on $\mathcal{B}(H)$. Here we "overload" the term "state"!)

A *(quantum) channel* is a trace-preserving, completely positive (CPTP) map $\mathcal{B}(H_A) \to \mathcal{B}(H_B)$:

- positive and trace-preserving so it maps densities to densities;
- completely positive so you can tensor with another system and still have positivity.

Stinespring and Kraus

The Stinespring Representation Theorem tells us that any CP map $\mathcal{E} : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ has the form

$$\mathcal{E}(x) = V^* \pi(x) V \qquad (x \in \mathcal{B}(H_A)),$$

where $V : H_B \to K$, and $\pi : \mathcal{B}(H_A) \to \mathcal{B}(K)$ is a $*$-representation.

- Any such π is of the form $\pi(x) = x \otimes 1$ where $K \cong H_A \otimes K'$.
- Take an o.n. basis (e_i) for K' so $V(\xi) = \sum_i K_i^*(\xi) \otimes e_i$ for some operators $K_i : H_A \to H_B$.

We arrive at the *Kraus form*:

$$\mathcal{E}(x) = \sum_i K_i x K_i^* \qquad (x \in \mathcal{B}(H_A)).$$

Trace-preserving when $\sum_i K_i^* K_i = 1$.

Quantum zero-error

We turn $\mathcal{B}(H)$ into a Hilbert space using the trace: $(T|S) = \mathrm{tr}(T^*S)$.
A sensible notion of when densities ρ, σ are distinguishable is when they are orthogonal.
Let $\mathcal{E}(x) = \sum_i K_i x K_i^*$ be a quantum channel. We wish to consider when $\mathcal{E}(\rho) \perp \mathcal{E}(\sigma)$. As $\mathcal{E}$ is positive, this is equivalent to

$$\mathcal{E}(|\psi\rangle\langle\psi|) \perp \mathcal{E}(|\phi\rangle\langle\phi|) \qquad (\psi \in \mathrm{Im}\,\rho, \phi \in \mathrm{Im}\,\sigma).$$

Equivalently

$$\begin{aligned} 0 = \mathrm{tr}\left(\mathcal{E}(|\psi\rangle\langle\psi|)\mathcal{E}(|\phi\rangle\langle\phi|)\right) &= \sum_{i,j} \mathrm{tr}\left(K_i|\psi\rangle\langle\psi|K_i^* K_j|\phi\rangle\langle\phi|K_j^*\right) \\ &= \sum_{i,j} |\langle\psi|K_i^* K_j|\phi\rangle|^2 \end{aligned}$$

which is equivalent to $\langle\psi|K_i^* K_j|\phi\rangle = 0$ for each i, j.

To operator systems

So ψ, ϕ are distinguishable after $\mathcal{E}$ when

$$\langle \psi | T | \phi \rangle = 0 \quad \text{for each} \quad T \in \text{lin}\{K_i^* K_j\}.$$

Set $\mathcal{S} = \text{lin}\{K_i^* K_j\}$ which has properties:

- $\mathcal{S}$ is a linear subspace;
- $T \in \mathcal{S}$ if and only if $T^* \in \mathcal{S}$;
- $1 \in \mathcal{S}$ (as $\sum_i K_i^* K_i = 1$ as $\mathcal{E}$ is CPTP).

That is, $\mathcal{S}$ is an *operator system*, which depends only on $\mathcal{E}$ and not the choice of (K_i).

Theorem (Duan)

For any operator system $\mathcal{S} \subseteq \mathcal{B}(H_A)$ there is some quantum channel $\mathcal{E} : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ giving rise to $\mathcal{S}$.

In the classical case

Given a classical channel from A to B with probabilities $p(b|a)$, define Kraus operators

$$K_{ab} = p(b|a)^{1/2}|b\rangle\langle a| : H_A \to H_B.$$

Here $(\langle a|)$ is the canonical basis of $H_A = \ell^2(A) \cong \mathbb{C}^{|A|}$.

$$\sum_{ab} K_{ab}|c\rangle\langle c|K_{ab}^* = \sum_{ab} p(b|a)|b\rangle\langle a|c\rangle\langle c|a\rangle\langle b| = \sum_{b} p(b|c)|b\rangle\langle b|.$$

So the pure state $|c\rangle\langle c|$ is mapped to the combination of pure states which can be received, given that message c is sent.

$$\begin{aligned}\mathcal{S} &= \text{lin}\{K_{ab}^* K_{cd}\} = \text{lin}\{p(b|a)^{1/2} p(d|c)^{1/2}|a\rangle\langle b|d\rangle\langle c|\} \\ &= \text{lin}\{|a\rangle\langle c| : a \sim c\}\end{aligned}$$

Thus $\mathcal{S}$ is directly linked to the confusability graph of the channel.

Quantum relations

Simultaneously, and motivated more by "noncommutative geometry":

Definition (Weaver)

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. A *quantum relation* on M is a weak*-closed subspace $S \subseteq \mathcal{B}(H)$ with $M'SM' \subseteq S$. We say that the relation is:

1. *reflexive* if $M' \subseteq S$;
2. *symmetric* if $S^* = S$ where $S^* = \{x^* : x \in S\}$;
3. *transitive* if $S^2 \subseteq S$ where $S^2 = \overline{\text{lin}}^{w^*}\{xy : x, y \in S\}$.

When $M = \ell^\infty(X) \subseteq \mathcal{B}(\ell^2(X))$ there is a bijection between the usual meaning of "relation" on X and quantum relations on M, given by

$$S = \overline{\text{lin}}^{w^*}\{e_{x,y} : x \sim y\}.$$

Quantum graphs

As a graph on a (finite) vertex set V is simply a relation, and as:

- undirected graphs corresponds to symmetric relations;
- a reflexive relation corresponds to having a “loop” at every vertex.

Definition (Weaver)

A *quantum graph* on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is a reflexive, symmetric quantum relation. That is, a unital, self-adjoint, weak*-closed subspace $S \subseteq \mathcal{B}(H)$, which is an M'-bimodule ($M'SM' \subseteq S$).

If $M = \mathcal{B}(H)$ with H finite-dimensional, then as $M' = \mathbb{C}$, a quantum graph is just an operator system: that is, exactly what we had before! [Duan, Severini, Winter; Stahlke]

Adjacency matrices

Given a graph $G = (V, E)$ consider the $\{0, 1\}$-valued matrix A with

$$A_{i,j} = \begin{cases} 1 & : (i,j) \in E, \\ 0 & : \text{otherwise}, \end{cases}$$

the *adjacency matrix* of G.

- A is idempotent for the *Schur product*;
- G is undirected if and only if A is self-adjoint;
- A has 1s down the diagonal when G has a loop at every vertex.

We can think of A as an operator on $\ell^2(V)$. This is the GNS space for the C^*-algebra $\ell^\infty(V)$ for the state induced by the uniform measure.

General C^*-algebras

Let B be a finite-dimensional C^*-algebra, and let φ be a faithful state on B, with GNS space $L^2(B)$. Thus B bijects with $L^2(B)$ as a vector space, and so we get:

- The multiplication on B induces a map $m : L^2(B) \otimes L^2(B) \to L^2(B)$;
- The unit in B induces a map $\eta : \mathbb{C} \to L^2(B)$.

We get an analogue of the Schur product:

$$x \bullet y = m(x \otimes y)m^* \qquad (x, y \in \mathcal{B}(L^2(B))).$$

Quantum adjacency matrix

Definition (Many authors)

A *quantum adjacency matrix* is a self-adjoint $A \in \mathcal{B}(L^2(B))$ with:

- $m(A \otimes A)m^* = A$ (so Schur product idempotent);
- $(1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1) = A$;
- $m(A \otimes 1)m^* = \mathrm{id}$ (a "loop at every vertex");

The middle axiom is a little mysterious: it roughly corresponds to "undirected".

I want to sketch why this definition is equivalent to the previous notion of a "quantum graph".

Subspaces to projections

Fix a finite-dimensional C^*-algebra (von Neumann algebra) M. A "quantum graph" is either:

- A subspace of $\mathcal{B}(H)$ (where $M \subseteq \mathcal{B}(H)$) with some properties; or
- An operator on $L^2(M)$ with some properties.

How do we move between these?

$S \subseteq \mathcal{B}(H)$ is a bimodule over M'. As H is finite-dimensional, $\mathcal{B}(H)$ is a Hilbert space for

$$(x|y) = \mathrm{tr}(x^* y).$$

Then $M \otimes M^{\mathrm{op}}$ is represented on $\mathcal{B}(H)$ via

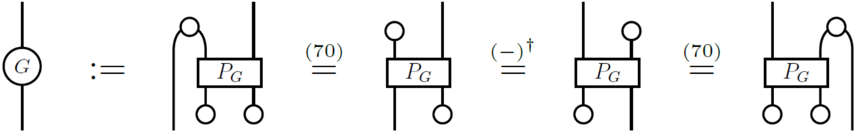
$$\pi : M \otimes M^{\mathrm{op}} \to \mathcal{B}(\mathcal{B}(H)); \quad \pi(x \otimes y) : T \mapsto xTy.$$

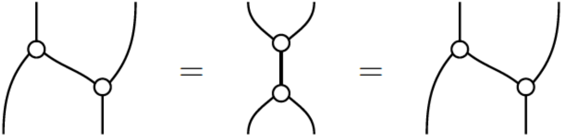
- The commutant of $\pi(M \otimes M^{\mathrm{op}})$ is naturally $M' \otimes (M')^{\mathrm{op}}$.
- An M'-bimodule of $\mathcal{B}(H)$ corresponds to an $M' \otimes (M')^{\mathrm{op}}$-invariant subspace of the Hilbert space $\mathcal{B}(H)$;
- Which corresponds to a *projection* in $M \otimes M^{\mathrm{op}}$.

Operators to algebras

So how can we relate:

- Operators $A \in \mathcal{B}(L^2(M))$;
- Projections in $M \otimes M^{\mathrm{op}}$?





[Musto, Reutter, Verdon]

Operators to algebras 2

Recall the GNS construction for a *tracial* state ψ on M:

$$\Lambda : M \to L^2(M); \quad (\Lambda(x)|\Lambda(y)) = \psi(x^* y).$$

As $L^2(M)$ is finite-dimensional, every operator on $L^2(M)$ is a linear combination of rank-one operators of the form

$$\theta_{\Lambda(a),\Lambda(b)} : \xi \mapsto (\Lambda(a)|\xi)\Lambda(b) \qquad (\xi \in L^2(M)).$$

Define a bijection

$$\Psi : \mathcal{B}(L^2(M)) \to M \otimes M^{\mathrm{op}}; \quad \theta_{\Lambda(a),\Lambda(b)} = b \otimes a^*,$$

and extend by linearity!

Operators to algebras 3

$$\Psi : \mathcal{B}(L^2(M)) \to M \otimes M^{\mathrm{op}}; \quad \theta_{\Lambda(a),\Lambda(b)} = b \otimes a^*,$$

- Ψ is a homomorphism for the "Schur product" $A_1 \bullet A_2 = m(A_1 \otimes A_2)m^*$;
- $A \mapsto (1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1)$ corresponds to the anti-homomorphism $\sigma : a \otimes b \mapsto b \otimes a$;
- $A \mapsto A^*$ corresponds to $e \mapsto \sigma(e)^*$.

Conclude: A quantum adjacency matrix corresponds to a projection e with $\sigma(e) = e$. BUT: There is no clean one-to-one correspondence between the axioms.

KMS States

Any faithful state ψ is KMS: there is an automorphism σ' of M with

$$\psi(ab) = \psi(b\sigma'(a)) \qquad (a, b \in M).$$

Indeed, there is $Q \in M$ positive and invertible with

$$\psi(a) = \mathrm{tr}(Qa) \qquad \sigma'(a) = QaQ^{-1}.$$

Theorem (D.)

Twisting our bijection Ψ using σ' allows us to establish a bijection between:

- *Quantum adjacency operators $A \in \mathcal{B}(L^2(M))$;*
- *projections $e \in M \otimes M^{\mathrm{op}}$ with $e = \sigma(e)$ and $(\sigma' \otimes \sigma')(e) = e$;*
- *self-adjoint M'-bimodules $S \subseteq \mathcal{B}(H)$ with $QSQ^{-1} = S$.*

So this is *more restrictive* than the tracial case.

Towards homomorphisms: Pushforwards

skip? Let M, N be finite-dimensional von Neumann algebras, and again let $\theta : M \to N$ be a UCP map (Notice I have changed convention!) with Kraus form

$$\theta(x) = \sum_{i=1}^{n} b_i^* x b_i.$$

Letting $M \subseteq \mathcal{B}(H_M), N \subseteq \mathcal{B}(H_N)$ and given $S \subseteq \mathcal{B}(H_N)$ a quantum graph/relation over N, define

$$\overrightarrow{S} = \mathrm{lin}\{b_i x b_j^* : x \in S\} \subseteq \mathcal{B}(H_M),$$

the "pushforward". [Weaver]

Notice that $\overrightarrow{S}$ need not be unital, but it is always self-adjoint.

Proposition (D.)

The pushforward $\overrightarrow{S}$ is a quantum relation over M. That is, $\overrightarrow{S}$ is automatically an M'-bimodule.

The classical case

Given classical graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, a function $f : V_G \to V_H$ defines a $*$-homomorphism (so certainly a UCP map)

$$\theta : C(V_H) \to C(V_G); \quad a \mapsto a \circ f \quad (a \in C(V_H)).$$

Let G induce $S_G \subseteq \mathcal{B}(\ell^2(V_G))$, that is,

$$S_G = \operatorname{lin}\{e_{u,v} : (u, v) \in E_G\}$$

the span of matrix units supported on the edges. Then

$$\overrightarrow{S_G} = \operatorname{lin}\{e_{f(u),f(v)} : (u, v) \in E_G\}$$

and so $\overrightarrow{S_G} \subseteq S_H$ exactly when f is a graph homomorphism.

Homomorphisms

[Stahkle] defines $\theta : M \to N$ to be a *homomorphism* between S_1 and S_2 when $\overrightarrow{S_2} \subseteq S_1$. [Weaver] calls this a *CP-morphism*.

Theorem (Stahkle)

Let $\theta : C(V_H) \to C(V_G)$ be a UCP map giving a homomorphism G to H (that is, with $\overrightarrow{S_G} \subseteq S_H$). Then there is some map $f : V_G \to V_H$ which is a (classical) graph homomorphism.

- In general θ need not be directly related to f.
- However, often we just care about the *existence* of a homomorphism.
- E.g. a k-colouring of G corresponds to some homomorphism $G \to K_k$, the complete graph.

Isomorphisms

We return to a finite-dimensional von Neumann algebra M equipped with a faithful state ψ, and a quantum adjacency matrix A, an operator on $L^2(M) = L^2(M, \psi)$.

An *isomorphism* of A is a $*$-automorphism θ of M which preserves the state ψ, and which commutes with A. This means either:

- Think of A as a map on M, so simply $A \circ \theta = \theta \circ A$; or
- θ preserves ψ, so induces a unitary operator

$$\widehat{\theta} : L^2(M) \to L^2(M); \quad \Lambda(a) \mapsto \Lambda(\theta(a)).$$

  Then require that $\widehat{\theta} A = A \widehat{\theta}$.

Isomorphisms of operator bimodules

What can we say about an M'-bimodule $S \subseteq \mathcal{B}(H)$?

- Not every automorphism of M lifts to $\mathcal{B}(H)$;
- Seems we get dependence on H here.

Does all work if $H = L^2(M)$: then we can define an automorphism of S to be a $*$-automorphism of $\mathcal{B}(H)$ which restricts to a ψ-persevering automorphism of M, and which restricts to a bijection on S.

In the classical case of a graph (V_G, E_G), with $M = C(V_G)$ and $A = A_G$ and $S = S_G$ on $L^2(M) = \ell^2(V_G)$, we obtain the usual meaning of a graph isomorphism: a permutation of V_G which doesn't change E_G.

Quantum group (co)actions

An (right) action of a (finite/compact) group G on a space/set X is a map

$$X \times G \to X.$$

So we get a $*$-homomorphism

$$\alpha : C(X) \to C(X) \otimes C(G),$$

Consider $(C(G), \Delta)$ as a compact quantum group.

- $(\mathrm{id} \otimes \Delta)\alpha = (\alpha \otimes \mathrm{id})\alpha$ corresponds to $x \cdot st = (x \cdot s) \cdot t$;
- $\mathrm{lin}\{\alpha(b)(1 \otimes a) : a \in C(G), b \in C(X)\}$ is dense in $C(X) \otimes C(G)$ corresponds to $x \cdot e = x$.

Definition (Podleś)

A (right) coaction of a compact quantum group (A, Δ) on a C^*-algebra B is a unital $*$-homomorphism $\alpha : B \to B \otimes A$ with these two conditions.

Coactions on ℓ^∞_n

Fix a compact quantum group (A,Δ).

- The algebra ℓ^∞_n is spanned by projections $(e_i)_{i=1}^n$.
- So $\alpha:\ell^\infty_n\rightarrow\ell^\infty_n\otimes A$ is determined by (u_{ij}) in A with

$$\alpha(e_i)=\sum_{j=1}^n e_j\otimes u_{ji}.$$

- α is a $*$-homomorphism $\Leftrightarrow$ each u_{ji} a projection and $u_{ji}u_{jk}=\delta_{ik}u_{ji}$;
- α is unital $\Leftrightarrow\sum_i u_{ji}=1$;
- α satisfies the coaction equation $\Leftrightarrow\Delta(u_{ji})=\sum_k u_{jk}\otimes u_{ki}$;
- α satisfies the Podleś density condition $\Leftrightarrow\sum_i u_{ji}=1$.
- General Theory $\implies\sum_j u_{ji}=1$. So (u_{ij}) is a *magic unitary*.

(Co)actions on (classical) graphs

Recall that a permutation θ gives an automorphism of a graph G when

$$P_\theta A_G = A_G P_\theta.$$

Here A_G is the adjacency matrix of G, which we can think of as also a linear map $\ell^\infty_n \to \ell^\infty_n$.
So $\mathrm{Aut}(G)$ acts in a way which preserves A_G:

$$\alpha : \ell^\infty_n \to \ell^\infty_n \otimes C(\mathrm{Aut}(G)); \quad \alpha A_G = (A_G \otimes \mathrm{id})\alpha.$$

Definition (Banica)

The *quantum automorphism group* of G is the maximal compact quantum group $\mathrm{QAut}(G)$ with a coaction satisfying

$$\alpha : \ell^\infty_n \to \ell^\infty_n \otimes \mathrm{QAut}(G); \quad \alpha A_G = (A_G \otimes \mathrm{id})\alpha.$$

Equivalently, the underlying magic unitary $U = (u_{ij})$ has to commute with the adjacency matrix A_G. This allows us to construct $\mathrm{QAut}(G)$ as a quotient of S_n^+.

Unitary implementations

Given a coaction $\alpha : \ell^\infty(V) \to \ell^\infty(V) \otimes A$ of (A, Δ) on $\ell^\infty(V)$, we saw before that α gives rise to a magic unitary $u = (u_{ij})_{i,j\in V}$,

$$\alpha(e_i) = \sum_{j\in V} e_j \otimes u_{ji} \qquad (i \in V).$$

This magic unitary "implements" the coaction α in a very simple way:

Lemma

Let $\ell^\infty(V) \subseteq \mathcal{B}(\ell^2(V))$. Then

$$\alpha(x) = u(x \otimes 1)u^* \qquad (x \in \ell^\infty(V)).$$

Coactions on operator bimodules

$$\alpha(x) = u(x \otimes 1)u^* \qquad (x \in \ell^\infty(V) \subseteq \mathcal{B}(\ell^2(V))).$$

It hence make sense...

Definition

α is a coaction on $\mathcal{S} \subseteq \mathcal{B}(\ell^2(V))$ exactly when $u(x \otimes 1)u^* \in \mathcal{S} \otimes A$ for each $x \in \mathcal{S}$.

One can check (non-trivially) that we then get the following.

Theorem (Eifler)

If a graph G is associated to the $\ell^\infty(V)$-operator bimodule $\mathcal{S}$, then a coaction of (A, Δ) on $\ell^\infty(V)$ gives a coaction on G if and only if it gives a coaction on $\mathcal{S}$.

Coactions on quantum adjacency operators-

There is now a clear definition:

Definition (Brannan, Chirvasitu, Eifler, Harris, Paulsen, Su, Wasilewski)

Let A_G be a quantum adjacency operator on (B, ψ). We say that (A, Δ) coacts on A_G when $\alpha : B \to B \otimes A$ is a coaction, which preserves ψ, and with $(A_G \otimes \mathrm{id})\alpha = \alpha A_G$.

- Here we regard A_G as a linear map on B.
- That α preserves ψ allows us to define a unitary $U \in \mathcal{B}(L^2(B)) \otimes A$ which implements α, as $\alpha(x) = U(x \otimes 1)U^*$.
- [Indeed, one way to prove Wang's theorem is to start with such a U and impose certain conditions on it (compare Compact Quantum Matrix Groups).]
- Then, equivalently, we require that U and $A_G \otimes 1$ commute.

Coactions on operator bimodules

A coaction α which preserves ψ gives a unitary U (which is a *corepresentation*) and it is then easy to see that

$$\alpha_U : \mathcal{B}(L^2(B)) \to \mathcal{B}(L^2(B)) \otimes A; \quad x \mapsto U(x \otimes 1)U^*$$

is a coaction (which extends α).

Might this leave $\mathcal{S} \subseteq \mathcal{B}(L^2(M))$ invariant if and only if U commutes with A_G?

- No, as the "trivial quantum graph" is $\mathcal{S} = B'$, which should always be invariant, but α_U leaves B invariant, not B'.
- Instead, we can use the *modular conjugation* J and *antipode* to form a "commutant" coaction α'_U; or equivalently, look at α_U but work with

$$\mathcal{S}' := \{JTJ : T \in \mathcal{S}\}.$$

Theorem (D.)

α leaves A_G invariant if and only if α_U leaves $\mathcal{S}'$ invariant.