Asymptotic sequence algebras, and ultrapowers, of Banach algebras

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The Calkin Algebra

Definition (Calkin, 1941)

Let H be a (separable, infinite dimensional) Hilbert space, and denote by $\mathcal{K}(H)$ the compact operators, and $\mathcal{B}(H)$ the bounded operators, on H. The *Calkin Algebra* is $\mathcal{C}(H) = \mathcal{B}(H)/\mathcal{K}(H)$.

- It is well-known that $\mathcal{K}(H)$ is the only proper closed two-sided ideal in $\mathcal{B}(H)$.
- So $\mathcal{C}(H)$ is simple.
- $\mathcal{C}(H)$ is a C^*-algebra, and so admits a faithful representation on some Hilbert space K (K cannot be separable). It was the first C^*-algebra which does not obviously arise as a subalgebra of $\mathcal{B}(K)$. [Calkin proved this before the GNS theory was available!]
- Does $\mathcal{C}(H)$ have outer automorphisms (not arising from a unitary)? This is independent of ZFC.

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Reframe using multipliers

Given a (faithful Banach) C^*-algebra A, and an ideal $I \lhd A$, we say that I is *essential* if $a \in A, aI + Ia = \{0\}$ implies $a = 0$.

The Multiplier Algebra of A, denoted $M(A)$, is the largest C^*-algebra which contains A as an essential ideal.

More concretely, if $A \subseteq \mathcal{B}(H)$ then

$$M(A) \cong \{T \in \mathcal{B}(H) : Ta, aT \in A \ (a \in A)\}.$$

We think of $M(A)$ as being the "maximal unitisation" of A. For example:

- $M(C_0(X)) = C(\beta X)$ the Stone–Čech compactification of X.
- $M(\mathcal{K}(H)) = \mathcal{B}(H)$.

We call $\mathcal{C}(A) := M(A)/A$ the "Corona" of A.

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Asymptotic sequence algebras

Given a (Banach) C^*-algebra A let $c_0(A)$ be the space of sequences (a_n) in A with $\lim_n \|a_n\| = 0$, endowed with the pointwise algebra operations:

$$(a_n) + (b_n) = (a_n + b_n), \qquad (a_n)(b_n) = (a_n b_n).$$

The multiplier algebra of $c_0(A)$ is $\ell^\infty(A)$, the space of all bounded sequences.
The corona of $c_0(A)$ is the "asymptotic sequence algebra"

$$\mathrm{Asy}(A) = \ell^\infty(A)/c_0(A).$$

(We can also let A vary, leading to $\mathrm{Asy}((A_n)) = \ell^\infty((A_n))/c_0((A_n))$.)

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Typical application

Here A, H, K are separable.

Theorem (Voiculescu)

Let $A \subseteq \mathcal{B}(H)$ and let $\pi : A \to \mathcal{B}(K)$ be a non-degenerate representation with $\pi(A \cap \mathcal{K}(H)) = \{0\}$. Then there is a sequence of unitaries $u_n : H \oplus K \to H$ with:

1. $\lim_n \|a - u_n(a \oplus \pi(a))u_n^*\| = 0$ *for* $a \in A$;
2. $a - u_n(a \oplus \pi(a))u_n^* \in \mathcal{K}(H)$ *for* $a \in A$.

So id and $\mathrm{id} \oplus \pi$ are unitarily equivalent "in the limit".

Typical application continued

Let $\pi : \mathcal{B}(K) \to \mathcal{C}(K) = \mathcal{B}(K)/\mathcal{K}(K)$ be the quotient onto the Calkin algebra.

Corollary

Let π_1 and π_2 be representations of A with

$$\ker \pi_1 = \ker \pi_2 = \ker \pi\pi_1 = \ker \pi\pi_2.$$

(That is, the images of π_1 and π_2 contain no non-zero compact operators.)
Then there is a sequence of unitaries (u_n) with $\lim_n \|\pi_1(a) - u_n\pi_2(a)u_n^*\| = 0$ for $a \in A$.

Abstract key idea

Let A, B be C^*-algebras.

Definition

Two maps $\psi, \phi : A \to B$ are *approximately unitarily equivalent* if there is a sequence of unitaries (u_n) in $M(B)$ with

$$\|\phi(a) - u_n \psi(a) u_n^*\| \to 0 \qquad (a \in A).$$

- Let $\iota_B : B \to \ell^\infty(B)$ be the "diagonal embedding" which sends $b \in B$ to the constant sequence (b).
- The above definitions becomes that there is $u \in \ell^\infty(B)$ unitary with $\iota_B(\phi(a)) - u\iota_B(\psi(b))u^* \in c_0(B)$ for $a \in A$.
- Or equivalently $\iota_B(\phi(a)) = u\iota_B(\psi(b))u^*$ in $\mathrm{Asy}(B) = \ell^\infty(B)/c_0(B)$, for $a \in A$.

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Slogan

$\mathrm{Asy}(A)$ is a construct to convert "approximate relations" into "exact relations".

Proposition (Gabe)

Two maps $\psi, \phi : A \to \mathrm{Asy}(B)$ are approximately unitarily equivalent if and only if they are unitarily equivalent.

Proof.

A "diagonal" argument. □

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A bit of set theory

A *filter* $\mathcal{F}$ on a set I is a non-empty collection of subsets of I with:

1. If $A, B \in \mathcal{F}$ then $A \cap B \in \mathcal{F}$;
2. If $A \in \mathcal{F}$ and $A \subseteq B$ then $B \in \mathcal{F}$.
3. $\emptyset \notin \mathcal{F}$ (this ensure $\mathcal{F} \neq 2^I$).

Example

The *Fréchet Filter* is the collection of all cofinite subsets of I; that is $A \in \mathcal{F}$ if and only if $I \setminus A$ is finite.

Let $\mathcal{F}$ be the Fréchet Filter on $\mathbb{N}$. Consider the condition on $(a_n) \in \ell^\infty(A)$ that

$$\forall \epsilon > 0, \quad \{n : \|a_n\| < \epsilon\} \in \mathcal{F}.$$

This is clearly equivalent to $(a_n) \in c_0(A)$.

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Ultrafilters

The collection of filters on a set I is partially ordered by inclusions. Zorn's Lemma ensures that there are maximal filters, which are called *ultrafilters*.

Lemma

A filter $\mathcal{U}$ on I is an ultrafilter if and only if for each $A \subseteq I$ either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

- For example, for $i_0 \in I$ the *principle ultrafilter at i_0* is $\{A \subseteq I : i_0 \in A\}$.
- Use Zorn's Lemma to find a maximal filter which contains the Fréchet Filter. This ultrafilter is not principle.

Fix an ultrafilter $\mathcal{U}$. If $(a_i)_{i \in I}$ is a bounded sequence in $\mathbb{R}$ then a compactness argument shows that there is a (unique) $t \in \mathbb{R}$ such that

$$\forall \epsilon > 0, \quad \{i : |a_i - t| < \epsilon\} \in \mathcal{U}.$$

Write $t = \lim_{i \to \mathcal{U}} a_i$.

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For any filter $\mathcal{F}$ define $(A)_\mathcal{F}$ to be the quotient of $\ell^\infty(A)$ by those sequences (a_n) with

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Definition

Let $\mathcal{U}$ be a non-principle ultrafilter (on $\mathbb{N}$). The *ultrapower* of a Banach space E is $(E)_\mathcal{U}$.
Equivalently, this is $\ell^\infty(E)$ with the semi-norm

$$\|(a_n)\| = \lim_{n\to\mathcal{U}} \|a_n\|.$$

One can show that $(E)_\mathcal{U}$ is complete.

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Ultrapowers of Hilbert spaces

Consider defining a sesquilinear form on $(H)_{\mathcal{U}}$ by

$$\big((a_n)\big|(b_n)\big) = \lim_{n\to\mathcal{U}} (a_n|b_n).$$

- This is well-defined as if $(a_n) = 0$ in the quotient $(E)_{\mathcal{U}}$ then $\lim_{n\to\mathcal{U}} \|a_n\| = 0$ and so $\lim_{n\to\mathcal{U}}(a_n|b_n) = 0$ for any (b_n).
- Clearly sesquilinear.
- The induced seminorm is a norm, because of the norm on $(H)_{\mathcal{U}}$
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Contrast this $\mathrm{Asy}(H) = \ell^\infty(H)/c_0(H)$.

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This is joint work with Bence Horváth. Fix a Banach algebra A.

Question

When is $(A)_{\mathcal{U}}$, or $\mathrm{Asy}(A)$, unital?

- If A is unital, under the diagonal embedding $A \to (A)_{\mathcal{U}}$, the unit becomes a unit for $\mathrm{Asy}(A)$.
- Conversely, let $e \in \mathrm{Asy}(A)$ be a unit for A. This has a representative $(e_n) \in \ell^\infty(A)$, which satisfies

$$\lim_n \|e_n a_n - a_n\| = 0, \quad \lim_n \|a_n e_n - a_n\| = 0 \qquad ((a_n) \in \ell^\infty(A)).$$

- By picking (a_n) suitably, this shows that, for example,

$$\lim_n \sup\{\|e_n a - a\| : a \in A, \|a\| \leq 1\} = 0.$$

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$$\lim_n \|e_n a_n - a_n\| = 0, \quad \lim_n \|a_n e_n - a_n\| = 0 \qquad ((a_n) \in \ell^\infty(A)).$$

- By picking (a_n) suitably, this shows that, for example,

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Unital algebras

This is joint work with Bence Horváth. Fix a Banach algebra A.

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$$\lim_n \sup\{\|e_n a - a\|, \|ae_n - a\| : a \in A, \|a\| \leq 1\} = 0.$$

- Extract a subsequence (e_n) with $\|e_n a - a\|, \|ae_n - a\| \leq \frac{1}{n}\|a\|$ for $a \in A$.
- We can also arrange that e.g. $\|e_n\| \leq 2\|(a_n)\|_{\mathrm{Asy}} = K$ say.
- Thus $\|e_n - e_m\| \leq \|e_n - e_n e_m\| + \|e_n e_m - e_m\| \leq K(\frac{1}{m} + \frac{1}{n})$.
- So (e_n) is Cauchy in A, so converges in A, say to e. Clearly e is a unit.

The argument for an ultrapower is similar, just with more bookkeeping.

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Some model theory 1

[Health warning: I am not a model theorist!]

Classical model theory deals with "models" of theories in a formal language.

Example

What is a group? The "language" is usually taken to be the binary product $(\_) \times (\_)$, the unary inverse $(\_)^{-1}$, and a distinguished constant 1.

A "formula" is constructed inductively using the language and first order logic (so $\forall, \exists$, and, or, not).

A "structure" is a set G with an "interpretation" of the product, inverse and 1 (so just a binary map and a unary map, and a constant $1 \in G$). A formula may or may not be true in the structure.

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Some model theory 2

Example

The "theory" of groups is the usual group axioms:

1. $\forall g \forall h \forall k,\ g \times (h \times k) = (g \times h) \times k$;
2. $\forall g,\ g^{-1} \times g = g \times g^{-1} = 1$;
3. $\forall g,\ g \times 1 = 1 \times g = g$.

A structure G that satisfies these axioms is indeed a group.

There is a notion of ultrapower; Łoś's Theorem then tells us that a formula is true in an ultrapower if and only if it is true in the original structure.

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Analysis is not a first-order theory; so model theory doesn't apply, right?

To get around this, one can consider "metric model theory". The language is now:

- A collection of "domains" (which will be bounded subsets of a metric space) and a privileged "relation" d (which will be the metric);
- Functions (which will be uniformly continuous functions) together with a uniform continuity modulus, one for each possible choice of domain;
- Relations (uniformly continuous functions into a bounded subset of the reals)

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Metric Model theory: C^*-algebras

The language of C^*-algebras will be:

- Domains B_n which will be the ball of radius $n \in \mathbb{N}$, and metric $d(a, b) = \|a - b\|$;
- A constant (a constant function) $0 \in B_1$;
- For every $\lambda \in \mathbb{C}$ a function $B_n \to B_m$ which will be scalar multiplication;
- A unary function $* : B_n \to B_n$ (which will be involution);
- Binary functions $+$ and $.$ (from suitable B_n to B_m) which will be addition and multiplication.

A "structure" is then just a metric space with subsets B_n and functions, which only need to satisfy that the functions have the correct uniform continuity bounds.

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Metric Model theory: C^*-algebras cont.

We can now write down the "axioms" to be a C^*-algebra:

- Axioms to be a vector space over $\mathbb{C}$;
- To be a $\mathbb{C}$-algebra;
- Axioms for the involution;
- $d(x, y) = d(x - y, 0)$ (we define $\|x\| = d(x, 0)$).
- $\|xy\| \leq \|x\|\|y\|$ and $\|\lambda x\| = |\lambda|\|x\|$;
- $\|x^*x\| = \|x\|^2$;
- $\sup_{a \in B_1} \|a\| \leq 1$.

Where did sup come from? We cannot use first-order logic; the formulas are built inductively using relations and functions from the language, together with uniformly continuous functions on $\mathbb{R}^n$, and sup and inf.

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These axioms are *not* enough to ensure that B_1 is equal to the ball $\{a \in A : \|a\| \leq 1\}$. To get this, we have to play some tricks by forcing $*$-polynomials to have the correct domains and codomains: see Farah, Hart, Sherman.

Can also perform another trick for a Banach space / algebra.

- We can form ultrapowers; these agree with our previous notion.
- Łoś's Theorem still holds.

So we can immediately show that $(A)_\mathcal{U}$ is unital if and only if A is unital, right?

Well, we cannot quantify with $\exists$ or $\forall$, so this cannot work:

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Back to unital algebras

Proposition

A Banach algebra A is unital if and only if

$$\inf_{e \in B_1} \sup_{a \in B_1} \max(\|ea - a\|, \|ae - a\|) = 0,$$

where B_1 is the unit ball of A.

Proof.

As before, extract a Cauchy sequence (e_n). □

We can then apply Łoś's Theorem to this. Moral is that we don't actually gain much from the abstract theory: just the ultrafilter bookkeeping is taken care of.

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Ring-theoretic infiniteness

Definition

$p \in A$ is an *idempotent* if $p^2 = p$.
Two idempotents p, q are *equivalent*, written $p \sim q$, if there are $a, b \in A$ with $p = ab$ and $q = ba$.

[If $q \sim r$, say $q = cd, r = dc$, then $p = p^2 = abab = aqb = (ac)(db)$ and $(db)(ac) = dqc = dcdc = r^2 = r$ so $p \sim r$.]

Definition

Let A be a unital algebra. A is *Dedekind finite* if $p \sim 1$ implies $p = 1$.

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For C^*-algebras

For C^*-algebras:

- We typically only consider self-adjoint idempotents $p = p^* = p^2$, called *projections*.
- The equivalence we typically use is *Murray–von Neumann equivalence*, which is that $p = u^*u$ and $q = uu^*$. This implies that u is a partial isometry. We write $p \approx q$.

These are actually the same concepts as we have defined.

- For any idempotent p there is a projection q with $p \sim q$. In fact, we can choose q with $pq = q$ and $qp = p$.
- If p, q are projections with $p \sim q$ then also $p \approx q$.
- Suppose A is a Dedekind-finite C^*-algebra. If $p^2 = p \sim 1$ then there is a projection q with $q \sim p$, so also $q \sim 1$ so $q \approx 1$ so $q = 1$. Then $1 = q = pq = p$, so A is Dedekind-finite in our sense.

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For asymptotic sequence algebras

Theorem

Let A be a unital Banach algebra. If A is Dedekind-finite then so is $\mathrm{Asy}(A)$.

Proof.

Let $p^2 = p \sim 1$ in $\mathrm{Asy}(A)$. We need to show that $p = 1$.

Let $(x_n) \in \ell^\infty(A)$ be a representative of p. Of course, (x_n) will not be an idempotent in general. □

Lemma

Let $a \in A$ with $\|a^2 - a\| = t < 1/4$. There is $p = p^2$ with

$$\|a - p\| \leq f_{\|a\|}(t) = \left(\|a\| + \tfrac{1}{2}\right)\left((1 - 4t)^{-1/2} - 1\right).$$

Further, if $ab = ba$ then also $pb = bp$.

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The proof

$$1 \sim p = p^2 \in \mathrm{Asy}(A).$$

- $p = (x_n) + c_0(A)$ so for large enough n, $\|x_n^2 - x_n\|$ is small. So there is $p_n = p_n^2$ close to x_n.
- Then (p_n) is another representative of p, and now (p_n) is an idempotent in $\ell^\infty(A)$.
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The proof cont.

$$(a_n b_n - p_n) \in c_0(A) \qquad (b_n a_n - 1) \in c_0(A).$$

We established that with $u_n = b_n a_n$ we have $q_n = a_n u_n^{-1} b_n = 1$ eventually.

Now compute:

$$\begin{aligned} \|1 - p_n\| = \|q_n - p_n\| &= \|a_n u_n^{-1} b_n - a_n b_n\| + \|a_n b_n - p_n\| \\ &\leq \|a_n\| \|u_n^{-1} - 1\| \|b_n\| + \|a_n b_n - p_n\|, \end{aligned}$$

which is small for large n.

Thus $(1 - p_n) \in c_0(A)$ so $p = 1$ in $\text{Asy}(A)$.

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The converse?

We could consider the ultrapower case, and try to use Łoś's Theorem.

That A is Dedekind finite is the claim that

$$\forall\, a, b \in A,\ \ ab = 1 \implies ba = 1.$$

[Indeed, if $ab = 1$ then $p = ba$ is an idempotent with $p \sim 1$. Conversely, if $p^2 = p \sim 1$ then $p = ba$ and $1 = ab$ for some a, b.]

The problem is that we can only "quantify" over bounded balls, and we cannot use $\forall$ or $\implies$. So the analogy breaks a little...

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Dedekind-infinite

Definition

Say that A is *Dedekind-infinite* if it is not Dedekind-finite. Define

$$C_{\mathrm{DI}}(A) = \inf\left\{\|a\|\|b\| : a, b \in A, ab = 1, ba \neq 1\right\}.$$

Set $C_{\mathrm{DI}}(A) = \infty$ if A is Dedekind-finite.

Remark

Given such a, b set $p = ba$ so $p^2 = p$ and hence $p^n = p$ for all n, and so either $p = 0$ or $\|p\| \geq 1$. As $1 - p$ is also an idempotent, also $\|1 - p\| \geq 1$ (as $p \neq 1$).

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Dedekind-infinite passes to sequence algebras

Theorem

Let (A_n) be a sequence of unital Banach algebras with $C_{DI}(A_n) \leq K$ for all n. Then $\mathrm{Asy}((A_n))$ is Dedekind-infinite.

Proof.

Easy: for each n there is a "witness" $a_n b_n = 1, b_n a_n \neq 1$ and $\|a_n\|\|b_n\| \leq K$. By the remark, $\|b_n a_n - 1\| \geq 1$. Rescale so that $\|a_n\| = \|b_n\|$. Then $a = (a_n), b = (b_n)$ define classes in $\mathrm{Asy}((A_n))$ with $ab = 1$ but $(b_n a_n - 1) \notin c_0((A_n))$ so $ba \neq 1$. □

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Let (A_n) *be a sequence of unital Banach algebras with* $C_{DI}(A_n) \leq K$ *for all* n*. Then* $\mathrm{Asy}((A_n))$ *is Dedekind-infinite.*

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For C^*-algebras

Corollary

If A is a C^*-algebra then A is Dedekind-finite if and only if $\mathrm{Asy}(A)$ is.

Proof.

We can use the C^*-algebra form of Dedekind-finite, so we can assume $b = a^*$ is a partial isometry. Thus, if A is Dedekind-infinite, then $C_{\mathrm{DI}}(A) = 1$. □

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Counter-example for Banach algebras

Maybe we have that A is Dedekind-finite, or $D_{\mathrm{D}I}(A) \leq K$ for some absolute constant K (which is true for C^*-algebras).

Of course not!

Our counter-example will be a *weighted-semigroup* algebra. Let C be the bicyclic semigroup, so S has generators α, β with $\alpha\beta = 1$ and no other relations.

[So C is all *reduced words* which are of the form $\beta^n\alpha^m$ with $n, m \in \mathbb{Z}_{\geq 0}$. Exercise to the reader to work out the multiplication.]

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Semigroup algebras

Let S be a semigroup.

The (classical) semigroup algebra is $\ell^1(S)$, all families $a = (a_s)_{s\in S}$ of complex numbers, with $\|a\| = \sum_s |a_s| < \infty$, and convolution product. Write $a = \sum_s a_s \delta_s$ where (δ_s) the basis (in the Banach space sense) of $\ell^1(S)$, and set

$$\delta_s * \delta_t = \delta_{st} \quad \text{so} \quad a * b = \Big(\sum_{\{s,t\in S: st=r\}} a_s b_t \Big)_{r\in S}.$$

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Weights

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A *weight* on a semigroup is $\omega : S \to (0,\infty)$ with $\omega(st) \leq \omega(s)\omega(t)$.

We shall in fact use the rather trivial weights $\omega_n(s) = n$ for $s \neq 1$ and $\omega_s(1) = 1$, for $n \in \mathbb{N}$. We shall in particular assume that $\omega(s) \geq 1$ for all s.

The *weighted semigroup algebra* is $\ell^1(S,\omega)$, which is those $a \in \ell^1(S)$ with $\|a\|_\omega = \sum_s |a_s|\omega(s) < \infty$. The condition on the weight ensures that $\ell^1(S,\omega)$ is an algebra.

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Let $a, b \in \ell^1(S,\omega)$ with $ab = 1$ and $ba \neq 1$. Then

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Forwards

We also look at:

- Proper Infiniteness: there are $p, q \in A$ idempotents which are *orthogonal* ($pq = qp = 0$) and $p \sim 1, q \sim 1$.
- Stable Rank One: (which has a complicated, but well-motivated, definition, but is equivalent to) the group of invertible elements is dense in A. (This implies being Dedekfind-finite).

The common theme is again norm-control, or lack thereof in the Banach algebra setting.

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Sources

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