Approximation properties and averaging for Drinfeld Doubles

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The plan

This talk will be about a number of inter-linked topics:

1. Locally compact quantum groups.
2. The *approximation property* which is a (big) weakening of the notion of amenability.
3. An *averaging procedure* when we have a compact quantum subgroup.
4. How this all works for Drinfeld doubles.
5. Culminating in a link between approximation properties for Drinfeld doubles and central approximation properties for discrete quantum groups.

Locally compact quantum groups

Abstract object $\mathbb{G}$ with:

- von Neumann algebra $L^\infty(\mathbb{G})$;
- equipped with a coproduct $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$ which is coassociative: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;
- which has weights φ, ψ which are left/right invariant, e.g.

$$\varphi\big((\omega \otimes \mathrm{id})\Delta(x)\big) = \varphi(x)\omega(1) \qquad (x \in \mathcal{M}_\varphi^+, \omega \in L^1(\mathbb{G})^+).$$

From this, one gets:

- $L^1(\mathbb{G})$ becomes a Banach algebra, product induced by Δ;
- GNS for φ gives $L^2(\mathbb{G})$ with $L^\infty(\mathbb{G})$ in standard position;
- a multiplicative unitary W, so $W_{12}W_{13}W_{23} = W_{23}W_{12}$;
- $\Delta(x) = W^*(1 \otimes x)W$ and $C_0(\mathbb{G})$ is the closure of $\{(\mathrm{id} \otimes \omega)(W) : \omega \in \mathcal{B}(L^2(\mathbb{G}))_*\}$; $L^\infty(\mathbb{G})$ is the weak*-closure.

Duality

$$\lambda : L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G})); \quad \omega \mapsto (\omega \otimes \mathrm{id})(W)$$

is a homomorphism. The closure of its image is a C^*-algebra $C_0(\widehat{\mathbb{G}})$.

- There indeed exists $\widehat{\mathbb{G}}$ a LCQG; $L^\infty(\widehat{\mathbb{G}})$ is the weak*-closure.
- There is $\widehat{\varphi}$ so that $L^2(\widehat{\mathbb{G}}) = L^2(\mathbb{G})$ canonically.
- $W \in L^\infty(\mathbb{G}) \overline{\otimes} L^\infty(\widehat{\mathbb{G}})$ and $\widehat{W} = \sigma(W^*)$ where σ is the swap map.

For G a locally compact group, set $L^\infty(\mathbb{G}) = L^\infty(G)$ and

$$\Delta(F)(s,t) = F(st) \qquad (F \in L^\infty(G), s, t \in G),$$

and φ, ψ the left/right Haar integrals.
Then we find that $L^\infty(\widehat{\mathbb{G}}) = VN(G)$ and $C_0(\widehat{\mathbb{G}}) = C_r^*(G)$, and

$$\widehat{\Delta} : \lambda_s \mapsto \lambda_s \otimes \lambda_s,$$

where $\lambda_s \in VN(G)$ is the left translation operator by $s \in G$.

The Fourier algebra

Classicaly, the *Fourier algebra*, $A(G)$, is the (non-closed) subalgebra of $C_0(G)$ formed by coefficients of the left-regular representation. In the quantum group framework, consider

$$\widehat{\lambda} : L^1(\widehat{G}) = VN(G)_* \to C_0(\widehat{\widehat{G}}) = C_0(G).$$

The image, equipped with the norm from $L^1(\widehat{G})$, is exactly $A(G)$.

Definition

We define $A(\mathbb{G}) = \widehat{\lambda}(L^1(\widehat{\mathbb{G}}))$ with the norm from $L^1(\widehat{\mathbb{G}})$, but thought of as a subalgebra of $C_0(\mathbb{G})$.

Amenability

Theorem (Leptin)

G is amenable if and only if $A(G)$ has a bounded approximate identity.

There is a notion of amenability for $\mathbb{G}$ involving an invariant mean which seems a priori weaker, so we define away the issue.

Theorem (Bédos–Tuset)

The following are equivalent and define what it means for $\mathbb{G}$ to be strongly amenable:

1. *$\widehat{\mathbb{G}}$ is co-amenable, meaning that $C_0(\widehat{\mathbb{G}})$ has a bounded counit;*
2. *$A(\mathbb{G}) \cong L^1(\widehat{\mathbb{G}})$ has a bai;*
3. *$A(\mathbb{G}) \cong L^1(\widehat{\mathbb{G}})$ has a bai consisting of states;*
4. *$C_0(\widehat{\mathbb{G}}) = C_0^u(\widehat{\mathbb{G}})$ (the universal and reduced C^*-algebraic quantum groups agree).*

Weakening amenability

To weaken the property of $A(G)$ having a bounded approximate identity, we embed $A(G)$ in a larger algebra of (completely bounded) *multipliers*: functions which multiply elements of $A(G)$ into itself.

Definition

An element $a \in L^\infty(\mathbb{G})$ is a *left multiplier* of $A(\mathbb{G})$ when $aA(\mathbb{G}) \subseteq A(\mathbb{G})$.

So a multiplier induces a map $L\colon L^1(\widehat{\mathbb{G}}) \to L^1(\widehat{\mathbb{G}})$ which satisfies

$$a\widehat{\lambda}(\widehat{\omega}) = \widehat{\lambda}(L(\widehat{\omega})) \quad (\widehat{\omega} \in L^1(\widehat{\mathbb{G}})).$$

Thus $L(\widehat{\omega}_1 \star \widehat{\omega}_2) = L(\widehat{\omega}_1) \star \widehat{\omega}_2$, meaning L is a *left centraliser*.

Definition

a is *completely bounded* if the Banach space adjoint of the associated L is cb as a map $L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\widehat{\mathbb{G}})$.

Multipliers

Theorem (Junge–Neufang–Ruan)

Let $L\colon L^1(\widehat{\mathbb{G}}) \to L^1(\widehat{\mathbb{G}})$ *be a completely bounded left centraliser. Then there is* $a \in L^\infty(\mathbb{G})$ *a multiplier which is associated to* L.

The resulting space $M_{cb}A(\mathbb{G})$ is a Banach algebra for the completely bounded norm. It contains $A(\mathbb{G})$, but the resulting map $A(\mathbb{G}) \to M_{cb}A(\mathbb{G})$ may not be bounded below.

Definition

$\mathbb{G}$ is *weakly amenable* if $A(\mathbb{G})$ has an approximate identity bounded for the cb-multiplier norm.

Weak*-topologies

As $M_{cb}A(\mathbb{G}) \subseteq L^\infty(\mathbb{G})$, any $L^1(\mathbb{G})$ functional induces a functional on $M_{cb}A(\mathbb{G})$.

Definition

Let $Q_{cb}A(\mathbb{G})$ be the closure of such functionals in the dual space $M_{cb}A(\mathbb{G})^*$.

Then it turns out that $Q_{cb}A(\mathbb{G})^*$ is canonically equal to $M_{cb}A(\mathbb{G})$, and so we have a weak*-topology on $M_{cb}A(\mathbb{G})$.

Definition (D.–Krajczok–Voigt)

$\mathbb{G}$ has the *approximation property* when the weak*-closure of $A(\mathbb{G})$ in $M_{cb}A(\mathbb{G})$ contains the identity multiplier.

Previously, a priori stronger (but actually equivalent) definitions due to [Crann], [Kraus–Ruan].

The approximation property

We studied this concept (due to Haagerup–Kraus classically).

Theorem (D.–Krajczok–Voigt)

The AP passes to quantum subgroups. Stable under direct limits and free-products of discrete quantum groups.

Free-product argument makes essential use of [Ricard–Xu] work.

Compact and discrete case

Definition

$\mathbb{G}$ is *compact* when $C_0(\mathbb{G})$ is unital; we write $C(\mathbb{G})$.
$\mathbb{\Gamma}$ is *discrete* when $\widehat{\mathbb{\Gamma}}$ is compact.

Recall that compact quantum groups have a representation theory closely paralleling that for classical compact groups: $\mathsf{Irr}(\widehat{\mathbb{\Gamma}})$ is the set of equivalence classes of irreducible (finite-dimensional) corepresentations of $(C(\widehat{\mathbb{\Gamma}}), \Delta)$.

By duality, this implies a structure for discrete quantum groups:

$$c_0(\mathbb{\Gamma}) = \bigoplus_{\alpha\in\mathsf{Irr}(\widehat{\mathbb{\Gamma}})} \mathbb{M}_{\mathsf{dim}(\alpha)}, \quad \ell^\infty(\mathbb{\Gamma}) = \prod_{\alpha\in\mathsf{Irr}(\widehat{\mathbb{\Gamma}})} \mathbb{M}_{\mathsf{dim}(\alpha)}.$$

Notice that the centres of these algebras can be identified with $c_0(\mathsf{Irr}(\widehat{\mathbb{\Gamma}}))$ and $\ell^\infty(\mathsf{Irr}(\widehat{\mathbb{\Gamma}}))$ respectively.

Central multipliers

In many examples, it turns out that one constructs multipliers on discrete quantum groups which are *central* (in $\mathcal{Z}\ell^\infty(\mathbb{\Gamma})$).

Definition

Two compact quantum groups $\widehat{\mathbb{\Gamma}}_1$ and $\widehat{\mathbb{\Gamma}}_2$ are *monoidally equivalent* if the monoidal C^*-tensor categories $\mathsf{Rep}(\widehat{\mathbb{\Gamma}}_1)$ and $\mathsf{Rep}(\widehat{\mathbb{\Gamma}}_2)$ are isomorphic.

[Freslon], using constructions from [Bichon–De Rijdt–Vaes], showed that central cb multipliers can be transferred between monoidally equivalent discrete quantum groups.

More recently, [Arano–de Laat–Wahl], [Arano–Vaes], [Popa–Vaes] and [Ghosh–Jones] have defined and studied a notion of cb multiplier for abstract rigid monoidal C^*-tensor categories, and shown that this notion agrees with that of central multipliers of $\mathbb{\Gamma}$, when applied to $\mathsf{Rep}(\widehat{\mathbb{\Gamma}})$.

Drinfeld doubles

A useful tool here is that of the Drinfeld Double of a quantum group $\mathbb{G}$.

Definition

$D(\mathbb{G})$ is the locally compact quantum group with $L^\infty(D(\mathbb{G})) = L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\widehat{\mathbb{G}})$ and

$$\Delta_{D(\mathbb{G})}(x) = (\mathrm{id} \otimes \chi \circ \mathrm{ad}(W) \otimes \mathrm{id})(\Delta \otimes \widehat{\Delta}).$$

Here $\chi\colon L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\widehat{\mathbb{G}})\bar{\otimes}L^\infty(\mathbb{G})$ is the tensor swap map, and $\mathrm{ad}(W)(x) = WxW^*$; recall that $W \in L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\widehat{\mathbb{G}})$.

- Much as the crossed-product classifies covariant actions, the Drinfeld double is related to Yetter–Drinfeld coactions.
- I get some intuition by thinking about *bradings*: $\widehat{D(\mathbb{G})}$ is generated by copies of $\mathbb{G}$ and $\widehat{\mathbb{G}}$ which commute non-trivially.

Central multipliers and doubles

We now consider $D(\mathbb{\Gamma}) = \ell^\infty(\mathbb{\Gamma})\bar{\otimes}L^\infty(\widehat{\mathbb{\Gamma}})$.
Remember we have $M_{cb}(\mathsf{Rep}(\widehat{\mathbb{\Gamma}}))$, a space of cb-multipliers on the rigid monoidal C^*-tensor category $\mathsf{Rep}(\widehat{\mathbb{\Gamma}})$. The definition is complicated, but any such multiplier is determined uniquely by a bounded family of scalars $(a_\alpha)_{\alpha\in\mathsf{Irr}(\widehat{\mathbb{\Gamma}})}$. Indeed, $a \mapsto (a_\alpha) \in \mathcal{Z}\ell^\infty(\mathbb{\Gamma})$ is the bijection $M_{cb}(\mathsf{Rep}(\widehat{\mathbb{\Gamma}})) \to \mathcal{Z}M_{cb}(A(\mathbb{\Gamma}))$.

Proposition (D.–Krajczok–Voigt)

The category multipliers $M_{cb}(\mathsf{Rep}(\widehat{\mathbb{\Gamma}}))$ is a dual space. The maps

$$M_{cb}(\mathsf{Rep}(\widehat{\mathbb{\Gamma}})) \to \mathcal{Z}M_{cb}(\mathbb{\Gamma}); \quad a \mapsto (a_\alpha)$$

and

$$\mathcal{Z}M_{cb}(\mathbb{\Gamma}) \to M_{cb}(A(D(\mathbb{\Gamma}))); \quad a \mapsto a \otimes 1,$$

are weak-weak*-continuous isomorphisms.*

Approximation property

Definition

$\mathsf{Rep}(\widehat{\mathbb{\Gamma}})$ has the AP when the identity multiplier is in the weak*-closure of the finitely-supported multipliers in $M_{cb}(\mathsf{Rep}(\widehat{\mathbb{\Gamma}}))$.

Definition

$\mathbb{\Gamma}$ has the AP when the identity multiplier is in the weak*-closure of the finitely-supported multipliers in $M_{cb}(A(\mathbb{\Gamma}))$.

Corollary (D.–Krajczok–Voigt)

$\mathbb{\Gamma}$ has the central AP if and only if $\mathsf{Rep}(\widehat{\mathbb{\Gamma}})$ has the AP. This condition implies that $D(\mathbb{\Gamma})$ has the AP; and if $\mathbb{\Gamma}$ is unimodular, the converse holds.

Notice "finite-support" not "centre of the Fourier algebra".

Averaging

We are motivated by some classical proofs about (non)AP for Lie groups:

- if one has a compact subgroup, then averaging functions (with respect to the Haar probability measure) maps: Fourier algebra elements to Fourier algebra elements; and multipliers to multipliers.
- The same is true for quantum groups!

Definition

We have that $\widehat{\mathbb{\Gamma}} \leqslant \mathbb{G}$ when there is a surjective Hopf $*$-homomorphism

$$\pi\colon C_0^u(\mathbb{G}) \to C^u(\widehat{\mathbb{\Gamma}}).$$

This implies a formally stronger property (an analogue of the Herz restriction theorem):

$$\exists\ \widehat{\pi}\colon \ell^\infty(\mathbb{\Gamma}) \hookrightarrow L^\infty(\widehat{\mathbb{G}}).$$

Averaging cont.

To avoid technicalities, suppose we actually have

$$\pi\colon C_0(\mathbb{G}) \to C(\widehat{\mathbb{\Gamma}}).$$

With $h \in L^1(\mathbb{\Gamma})$ the Haar state, we can consider $h\pi \in C_0(\mathbb{G})^*$. Define

$$\Xi\colon C_0(\mathbb{G}) \to C_0(\mathbb{G}); \quad x \mapsto (h\pi \otimes \mathrm{id} \otimes h\pi)\Delta^2(x).$$

This is a conditional expectation of $C_0(\mathbb{G})$ onto the subalgebra

$$C_0(\widehat{\mathbb{\Gamma}}\backslash\mathbb{G}/\widehat{\mathbb{\Gamma}}) = \big\{x \in C_0(\mathbb{G}) : (\pi\otimes \mathrm{id})\Delta(x) = 1\otimes x,\ (\mathrm{id}\otimes\pi)\Delta(x) = x\otimes 1\big\}.$$

- This extends to a normal map on $L^\infty(\mathbb{G})$.
- It restricts to $A(\mathbb{G})$ and $M_{cb}A(\mathbb{G})$, continuous in the natural norms.

For the Drinfeld double

We have

$$\pi : D(\mathbb{\Gamma}) = \ell^\infty(\mathbb{\Gamma}) \bar{\otimes} L^\infty(\widehat{\mathbb{\Gamma}}) \to L^\infty(\widehat{\mathbb{\Gamma}}); \quad \pi = \epsilon \otimes \mathrm{id},$$

where $\epsilon \in \ell^1(\mathbb{\Gamma})$ is the counit.

- So we can average, and hence consider $L^\infty(\widehat{\mathbb{\Gamma}} \backslash D(\mathbb{\Gamma}) / \widehat{\mathbb{\Gamma}})$ and so forth.
- This space of invariants is exactly equal to $\mathcal{Z}\ell^\infty(\mathbb{\Gamma}) \otimes 1$.
- Similarly $C_0(\widehat{\mathbb{\Gamma}} \backslash D(\mathbb{\Gamma}) / \widehat{\mathbb{\Gamma}}) = \mathcal{Z}c_0(\mathbb{\Gamma}) \otimes 1$.
- Similarly $M_{cb}(A(\widehat{\mathbb{\Gamma}} \backslash D(\mathbb{\Gamma}) / \widehat{\mathbb{\Gamma}})) = \mathcal{Z}M_{cb}(A(\mathbb{\Gamma})) \otimes 1$.

Application

Theorem (D.–Krajczok–Voigt)

$\mathbb{\Gamma}$ has the central AP if and only if $D(\mathbb{\Gamma})$ has the AP. The same is true for strong amenability and weak amenability (and the Haagerup property).

- This is still using "finite support" to define the central APs.
- When $\mathbb{\Gamma}$ is unimodular, there is another "averaging" procedure (given by a Haar-state-invariant conditional expectation). This shows that you can define the central APs using the centre of the Fourier algebra, not finitely supported central elements.
- (But of course, in the unimodular case, you can always just average things to be central anyway!)

What's the invariant Fourier algebra?

We have

$$M_{cb}(A(\widehat{\mathbb{\Gamma}}\backslash D(\mathbb{\Gamma})/\widehat{\mathbb{\Gamma}})) = \mathcal{Z}M_{cb}(A(\mathbb{\Gamma})) \otimes 1$$

and so forth; but not for the Fourier algebra, only

$$A(\widehat{\mathbb{\Gamma}}\backslash D(\mathbb{\Gamma})/\widehat{\mathbb{\Gamma}}) \subseteq \mathcal{Z}A(\mathbb{\Gamma}) \otimes 1.$$

Theorem (D.–Krajczok–Voigt)

We have equality in the above if and only if $\mathcal{Z}c_{00}(\mathbb{\Gamma})$ is dense in $\mathcal{Z}A(\mathbb{\Gamma})$.

Corollary

When $\mathbb{\Gamma}$ is unimodular, we have equality.

A counter-example

Using some calculations of [DeCommer–Freslon–Yamashita] we obtain:

Theorem (D.–Krajczok–Voigt)

With $\mathbb{\Gamma} = \widehat{SU_q(2)}$ *we have that* $A(\widehat{\mathbb{\Gamma}}\backslash D(\mathbb{\Gamma})/\widehat{\mathbb{\Gamma}}) \neq A(\mathbb{\Gamma}) \otimes 1$. *So* $\mathcal{Z}c_{00}(\mathbb{\Gamma})$ *is not dense in* $\mathcal{Z}A(\mathbb{\Gamma})$.

One is meant to finish with a question: Could the equality $A(\widehat{\mathbb{\Gamma}}\backslash D(\mathbb{\Gamma})/\widehat{\mathbb{\Gamma}}) = \mathcal{Z}A(\mathbb{\Gamma}) \otimes 1$ characterise that $\mathbb{\Gamma}$ is unimodular?