Multiplers of the Fourier algebra and non-commutative L^p spaces

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Multipliers

Suppose that A is an algebra: how might we embed A into a unital algebra B?

- Could use the unitisation: $A \oplus \mathbb{C}1$.
- Natural to ask that A is an *ideal* in B.
- But we don't want B to be too large: the natural condition is that A should be *essential* in B: if $I \subseteq B$ is an ideal then $A \cap I \neq \{0\}$.
- For *faithful* A, this is equivalent to: if $b \in B$ and $aba' = 0$ for all $a, a' \in A$, then $b = 0$.
- Turns out there is a maximal such B, called the *multiplier algebra* of A, written $M(A)$. Maximal in the sense that if $A \trianglelefteq B$, then $B \rightarrow M(A)$. Clearly $M(A)$ is unique.

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How to build $M(A)$

We define $M(A)$ to be the collection of maps $L, R : A \to A$ with

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad aL(b) = R(a)b \qquad (a, b \in A).$$

- If A is faithful (which we shall assume from now on) then we only need the third condition.
- $M(A)$ is a vector space, and an algebra for the product $(L, R)(L', R') = (LL', R'R)$.
- Each $a \in A$ defines a pair $(L_a, R_a) \in M(A)$ by $L_a(b) = ab$ and $R_a(b) = ba$.
- The homomorphism $A \to M(A); a \mapsto (L_a, R_a)$ identifies A with an essential ideal in $M(A)$.
- If A is a Banach algebra, then natural to ask that L and R are bounded; but this is automatic by using the Closed Graph Theorem.

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Multipliers of C*-algebras

Let A be a C*-algebra acting non-degenerately on a Hilbert space H. Then we have that

$$M(A) = \{T \in \mathcal{B}(H) : Ta, aT \in A \ (a \in A)\}.$$

- Each such T does define a multiplier in the previous sense: let $L(a) = Ta$ and $R(a) = aT$.
- Conversely, a bounded approximate identity argument allows you to build $T \in \mathcal{B}(H)$ given $(L, R) \in M(A)$. Indeed, let $T = \lim L(e_\alpha)$, in the weak operator topology, say.
- If $A = C_0(X)$ then $M(A) = C^b(X) = C(\beta X)$, so $M(A)$ is a non-commutative Stone-Čech compactification.

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Locally compact groups

Let G be a locally compact group, equipped with a left invariant Haar measure. Examples include:

- Any *discrete* group with the counting measure.
- Any *compact* group, where the Haar measure is normalised to be a probability measure.
- The real line $\mathbb{R}$ with Lebesgue measure.
- Various non-compact Lie groups give interesting examples.

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Group algebras

Turn $L^1(G)$ into a Banach algebra by using the convolution product:

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)\, dt.$$

- We can also convolve finite measures.
- Identify $M(G)$ with $C_0(G)^*$, then

$$\langle \mu * \lambda, F \rangle = \int \int F(st)\, d\mu(s)\, d\lambda(t) \qquad (\mu, \lambda \in M(G), F \in C_0(G)).$$

- [Wendel] Then we have that

$$M(L^1(G)) = M(G),$$

where for each $(L, R) \in M(L^1(G))$, there exists $\mu \in M(G)$,

$$L(a) = \mu * a, \quad R(a) = a * \mu \qquad (a \in L^1(G)).$$

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Representing $M(G)$

This is an idea which goes back to [Young].

- For $1 < p < \infty$, $L^1(G)$ acts by convolution on $L^p(G)$.
- We can extend this to a convolution action of $M(G)$.
- Let $p_n \to 1$, and let $E = \bigoplus_n L^{p_n}(G)$ (say in the ℓ^2 sense, so that E is reflexive).
- Then $L^1(G)$ and $M(G)$ act on E.
- Young observed that the resulting homomorphism $\theta : L^1(G) \to \mathcal{B}(E)$ is an isometry.
- The same is true for $\theta : M(G) \to \mathcal{B}(E)$, which is also weak*-continuous (why I want E reflexive).
- We actually get that

$$\theta(M(G)) = \big\{ T \in \mathcal{B}(E) : T\theta(f), \theta(f)T \in \theta(L^1(G)) \ (f \in L^1(G)) \big\}.$$

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The Fourier transform

If G is abelian, then we have the dual group

$$\hat{G} = \{\chi : G \to \mathbb{T} \text{ a continuous homomorphism}\}.$$

Also we have the Fourier Transform

$$\mathcal{F} : L^1(G) \to C_0(\hat{G}) \quad \text{also} \quad L^2(G) \cong L^2(\hat{G}).$$

- The image $\mathcal{F}(L^1(G))$ is the *Fourier algebra* $A(\hat{G})$.
- As $L^1(G) = L^2(G) \cdot L^2(G)$ (pointwise product) we see that $A(\hat{G}) = L^2(G) * L^2(G) = L^2(\hat{G}) * L^2(\hat{G})$ (convolution).
- $\mathcal{F}$ extends to $M(G)$, and the image is $B(\hat{G}) \subseteq C^b(G)$, the *Fourier-Stieltjes algebra*.

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$$\mathcal{F} : L^1(G) \to C_0(\hat{G}) \quad \text{also} \quad L^2(G) \cong L^2(\hat{G}).$$

- The image $\mathcal{F}(L^1(G))$ is the *Fourier algebra* $A(\hat{G})$.
- As $L^1(G) = L^2(G) \cdot L^2(G)$ (pointwise product) we see that $A(\hat{G}) = L^2(G) * L^2(G) = L^2(\hat{G}) * L^2(\hat{G})$ (convolution).
- $\mathcal{F}$ extends to $M(G)$, and the image is $B(\hat{G}) \subseteq C^b(G)$, the *Fourier-Stieltjes algebra*.

Operator algebras

The Fourier transform similarly sets up isomorphisms

$$C_0(G) \cong C_r^*(\hat{G}) \qquad L^\infty(G) \cong VN(\hat{G}).$$

Let $\lambda : G \to \mathcal{B}(L^2(G))$ be the *left-regular representation*,

$$\lambda(s) : f \mapsto g \qquad g(t) = f(s^{-1}t) \qquad (f \in L^2(G), s, t \in G).$$

Integrate this to get a homomorphism $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$.

- $C_r^*(G)$ is the closure of $\lambda(L^1(G))$.
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The Fourier algebra

For a general G, we could hence define $A(G)$ to be:

- the predual of $VN(G)$.
- Or $A(G) = L^2(G) * L^2(G)$.
- We *hope* that these agree and that $A(G)$ is an algebra for the pointwise product.

Remember that a von Neumann algebra always has a *predual*: the space of normal functionals.

As $VN(G) \subseteq \mathcal{B}(L^2(G))$, and $\mathcal{B}(L^2(G))$ is the dual of $\mathcal{T}(L^2(G))$, the trace-class operators on $L^2(G)$, we have a quotient map

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What is the Fourier algebra? [Eymard]

We do have that $A(G) = VN(G)_* = L^2(G) * L^2(G) \subseteq C_0(G)$:

- (Big Machine $\Rightarrow$) $VN(G)$ is in *standard position*, so any normal functional ω on $VN(G)$ is of the form $\omega = \omega_{\xi,\eta}$ for some $\xi, \eta \in L^2(G)$,

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- As $\{\lambda(s) : s \in G\}$ generates $VN(G)$, for $\omega \in VN(G)_*$, if we know what $\langle \lambda(s), \omega \rangle$ is for all s, then we know ω.
- Observe that

$$\begin{aligned} \langle \lambda(s), \omega_{\xi,\eta} \rangle &= \int_G \lambda(s)(\xi)(t)\overline{\eta(t)}\, dt = \int_G \xi(s^{-1}t)\overline{\eta(t)}\, dt \\ &= \int_G \overline{\eta(t)}\check{\xi}(t^{-1}s)\, dt = (\overline{\eta} * \check{\xi})(s). \end{aligned}$$

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Why an algebra? [Takesaki-Tatsumma]

There is a normal $*$-homomorphsm
$\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G) = VN(G \times G)$ which satisfies

$$\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s) = \lambda(s, s).$$

- As Δ is normal, we get a (completely) contractive map $\Delta_* : A(G) \times A(G) \to A(G)$.
- Turns out that Δ_* is associative, because Δ is *coassociative*.
- This obviously induces the pointwise product on $A(G)$, as for $\omega, \sigma \in A(G)$ and $s \in G$,

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Multipliers of the Fourier algebra

- As $A(G)$ is commutative, multipliers of $A(G)$ are simply maps T on $A(G)$ with $T(ab) = T(a)b$.
- As we consider $A(G) \subseteq C_0(G)$, we find that every $T \in MA(G)$ is given by some $f \in C^b(G)$:

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- By duality, each $T \in MA(G)$ induces a map $T^* : VN(G) \to VN(G)$.
- If this is *completely bounded*– that is gives uniformly (in n) bounded maps $1 \otimes T^*$ on $\mathbb{M}_n \otimes VN(G)$– then $T \in M_{cb}A(G)$.
- [Haagerup, DeCanniere] For $f \in MA(G)$, we have that $f \in M_{cb}A(G)$ if and only if $f \otimes 1_K \in MA(G \times K)$ for all compact K (or just $K = SU(2)$).

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Properties of groups via multipliers

Lots of interesting properties of groups are related to how $A(G)$ sits in $M_{cb}A(G)$:

- $A(G)$ has a bounded approximate identity if and only if G is amenable.
- If $A(G)$ has an approximate identity, bounded in $M_{cb}A(G)$, then G is *weakly amenable*.
- For example, this is true for $SO(1, n)$ and $SU(1, n)$.
- Let Λ_G be the minimal bounded (in $M_{cb}A(G)$) for such an approximate identity.
- [Cowling, Haagerup] Then, for $G = Sp(1, n)$, then $\Lambda_G = 2n - 1$.
- [Ozawa] All hyperbolic groups are weakly amenable.

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Non-commutative L^p spaces

Want an abstract way to think about $L^p(G)$:

- We regard $L^\infty = L^\infty(G)$ and $L^1 = L^1(G)$ as spaces of functions on G, so it makes sense to talk about $L^\infty \cap L^1$ and $L^\infty + L^1$.
- We have inclusions $L^\infty \cap L^1 \subseteq L^p \subseteq L^\infty + L^1$.
- Let $\mathcal{S} = \{x + iy : 0 \leq x \leq 1\}$ and $\mathcal{S}_0$ be the interior;
- Let $\mathcal{F}$ be the space of continuous functions $f : \mathcal{S} \to L^\infty + L^1$ which are analytic on $\mathcal{S}_0$;
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Non-commutative L^p spaces

Want an abstract way to think about $L^p(G)$:

- We regard $L^\infty = L^\infty(G)$ and $L^1 = L^1(G)$ as spaces of functions on G, so it makes sense to talk about $L^\infty \cap L^1$ and $L^\infty + L^1$.
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Complex interpolation

- We can apply this procedure to any pair of Banach spaces (E_0, E_1).
- Have to embed E_0 and E_1 into some Hausdorff topological vector space X, which allows us to form $E_0 + E_1$ and $E_0 \cap E_1$.
- Let to $E_\theta = (E_0, E_1)_{[\theta]} = \{f(\theta) : f \in \mathcal{F}\}$, for $0 \leq \theta \leq 1$;
- Previously we had $(L^\infty, L^1)_{[1/p]} = L^p$.
- (Riesz-Thorin) If $T : E_0 + E_1 \to E_0 + E_1$ is linear, and restricts to give maps $E_0 \to E_0$ and $E_1 \to E_1$, then

$$\|T : E_\theta \to E_\theta\| \leq \|T : E_0 \to E_0\|^{1-\theta} \|T : E_1 \to E_1\|^\theta.$$

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For the Fourier algebra

- Suppose for the moment we have a way to make sense of $A(G) + VN(G)$.
- Then we can form $L^p(\hat{G}) = (VN(G), A(G))_{[1/p]}$.
- If G is abelian, then $L^p(\hat{G})$ *is* the L^p space of $\hat{G}$.
- For example, if G is compact, then

$$VN(G) = \prod_{\pi \in \hat{G}} \mathbb{M}_{d(\pi)}, \quad L^p(\hat{G}) = \ell^p - \bigoplus_{\pi} d(\pi)^{1/p} S^p_{d(\pi)},$$

  a direct sum of Schatten-classes.
- $S^p_d = \mathbb{M}_d$ with the norm $\|x\| = \operatorname{trace}(|x|^p)^{1/p}$.
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A hint from operator spaces

Using complex interpolation between a von Neumann algebra and its predual is a well-known way to construct non-commutative L^p spaces. See work of [Kosaki], [Terp] and [Izumi].

- We eventually want to deal with the completely bounded case: we want $L^p(\hat{G})$ to be an operator space.
- We also hope that $L^2(\hat{G})$ is a Hilbert space;
- so it should be *self-dual*;
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Using the right von Neumann algebra

- As $VN(G)$ is in *standard position* on $L^2(G)$, it follows that $VN(G)^{\text{op}}$ is isomorphic to $VN(G)'$, the commutant of $VN(G)$.
- But this is $VN_r(G)$, the von Neumann algebra generated by the right regular representation:

$$\rho(s) : \xi \mapsto \eta, \quad \eta(t) = \xi(ts)\nabla(s)^{1/2} \qquad (s, t \in G, \xi \in L^2(G)).$$

  Here ∇ is the modular function on G.
- If we follow Terp, then we construct $A(G) \cap VN_r(G)$ by identifying $a \in A(G) \cap C_{00}(G)$ with $\rho(\nabla^{-1/2}a) \in VN_r(G)$.
- By doing some work with left Hilbert algebras, we can show that $a \in A(G) \cap VN_r(G)$ if and only if convolution by $\check{a}$ on the right gives a bounded map on $L^2(G)$.

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Building $L^p(\hat{G})$

- Once we have $A(G) \cap VN_r(G)$, we can form $A(G) + VN_r(G)$ (formally, this will be a subspace of the dual of $A(G) \cap VN_r(G)$).
- We use complex interpolation: $L^p(\hat{G}) = (VN_r(G), A(G))_{[1/p]}$.
- If G is abelian, then everything is commutative, and we really do just recover $L^p(\hat{G})$.
- As $A(G) \cap VN_r(G)$ is dense in $L^p(\hat{G})$, we can view $L^p(\hat{G})$ as an abstract Banach space completion of some subspace (actually, ideal) of $A(G)$. So a *function space*.
- Then the $A(G)$ module action is just multiplication of functions!
- This generalises work of [Forrest, Lee, Samei]: they have different constructions for $p > 2$ and $p < 2$, but actually the spaces are isomorphic to $L^p(\hat{G})$ (just via "different" isomorphisms).

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- We use complex interpolation: $L^p(\hat{G}) = (VN_r(G), A(G))_{[1/p]}$.
- If G is abelian, then everything is commutative, and we really do just recover $L^p(\hat{G})$.
- As $A(G) \cap VN_r(G)$ is dense in $L^p(\hat{G})$, we can view $L^p(\hat{G})$ as an abstract Banach space completion of some subspace (actually, ideal) of $A(G)$. So a *function space*.
- Then the $A(G)$ module action is just multiplication of functions!
- This generalises work of [Forrest, Lee, Samei]: they have different constructions for $p > 2$ and $p < 2$, but actually the spaces are isomorphic to $L^p(\hat{G})$ (just via "different" isomorphisms).

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Representing multipliers

- Similarly, $MA(G)$ and $M_{cb}A(G)$ act on $L^p(\hat{G})$ by multiplication.
- So let $p_n \to 1$, and let
$$E = \bigoplus_n L^{p_n}(\hat{G}),$$
say in the ℓ^2 sense (so E is reflexive).
- Thus E is an $A(G)$ module and an $MA(G)$ module.
- The action of $MA(G)$ is weak*-continuous, and
$$MA(G) = \{T \in \mathcal{B}(E) : Ta, aT \in A(G) \ (a \in A(G))\}.$$

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Representing cb multipliers

- The actions of $A(G)$ and $M_{cb}A(G)$ on $L^p(\hat{G})$ are completely contractive.
- We can give the ℓ^2-direct sum of operator spaces a natural operator space structure ([Xu]: use interpolation again!)
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Analogues of the Figa-Talamanca–Herz algebras

- Recall that $A(G) = L^2(G) * L^2(G)\check{}$.
- We could instead define $A_p(G) = L^p(G) * L^{p'}(G)\check{}$, the Figa-Talamanca–Herz algebra (where $1/p + 1/p' = 1$).
- These have similar properties to $A(G)$, although some results are still conjecture: as working away from a Hilbert space can be tricky.
- We've developed a theory of L^p spaces "on the dual side",
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