Completely positive multipliers

Matthew Daws

Leeds

Lille, Oct 2012

Outline

The von Neumann algebra of a group

Let G be a locally compact group $\implies$ has a Haar measure $\implies$ can form the von Neumann algebra $L^\infty(G)$ acting on $L^2(G)$.
Have lost the product of G. We recapture this by considering the injective, normal $*$-homomorphism

$$\Delta : L^\infty(G) \to L^\infty(G \times G); \quad \Delta(F)(s,t) = F(st).$$

The pre-adjoint of Δ gives the usual convolution product on $L^1(G)$

$$L^1(G) \otimes L^1(G) \to L^1(G); \quad \omega \otimes \tau \mapsto (\omega \otimes \tau) \circ \Delta = \omega \star \tau.$$

The map Δ is implemented by a unitary operator W on $L^2(G \times G)$,

$$W\xi(s,t) = \xi(s, s^{-1}t), \quad \Delta(F) = W^*(1 \otimes F)W,$$

where $F \in L^\infty(G)$ identified with the operator of multiplication by F.

The other von Neumann algebra of a group

Let $VN(G)$ be the von Neumann algebra acting on $L^2(G)$ generated by the left translation operators $\lambda_s, s \in G$.
The predual of $VN(G)$ is the "Fourier algebra" (a la Eymard) $A(G)$, a commutative Banach algebra.
Again, there is $\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G)$ whose pre-adjoint induces the product on $A(G)$,

$$\Delta(\lambda_s) = \lambda_s \otimes \lambda_s.$$

That such a Δ exists follows as

$$\Delta(x) = \hat{W}^*(1 \otimes x)\hat{W} \quad \text{with} \quad \hat{W} = \sigma W^* \sigma,$$

where $\sigma : L^2(G \times G) \to L^2(G \times G)$ is the swap map.

How W governs everything

The unitary W is multiplicative; $W_{12}W_{13}W_{23} = W_{23}W_{12}$; and lives in $L^\infty(G)\overline{\otimes}VN(G)$.

The map

$$L^1(G) \to VN(G); \quad \omega \mapsto (\omega \otimes \iota)(W),$$

is the usual representation of $L^1(G)$ on $L^2(G)$ by convolution. The image is σ-weakly dense in $VN(G)$, and norm dense in $C_r^*(G)$.

The map

$$A(G) \to L^\infty(G); \quad \omega \mapsto (\iota \otimes \omega)(W),$$

is the usual embedding of $A(G)$ into $L^\infty(G)$ (the Gelfand map, if you wish). The image is σ-weakly dense in $L^\infty(G)$, and norm dense in $C_0(G)$.

The group inverse is represented by the antipode $S : L^\infty(G) \to L^\infty(G); S(F)(t) = F(t^{-1})$. We have that

$$S\big((\iota \otimes \omega)(W)\big) = (\iota \otimes \omega)(W^*).$$

More non-commutative framework

- M a von Neumann algebra;
- Δ a normal injective $*$-homomorphism $M \to M\overline{\otimes}M$ with $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$;
- a "left-invariant" weight φ, with $(\iota \otimes \varphi)\Delta(\cdot) = \varphi(\cdot)1$ (in some loose sense).
- a "right-invariant" weight ψ, with $(\psi \otimes \iota)\Delta(\cdot) = \psi(\cdot)1$.

Let H be the GNS space for φ. There is a unitary W on $H \otimes H$ with

$$\Delta(x) = W^*(1 \otimes x)W, \quad M = \overline{\text{lin}\{(\iota \otimes \omega)(W)\}}^{\sigma\text{-weak}}.$$

Again form $\hat{W} = \sigma W^* \sigma$. Then

$$\hat{M} = \overline{\text{lin}\{(\iota \otimes \omega)(\hat{W})\}}^{\sigma\text{-weak}}$$

is a von Neumann algebra, we can define $\hat{\Delta}(\cdot) = \hat{W}^*(1 \otimes \cdot)\hat{W}$ which is "coassociative". It is possible to define weights $\hat{\varphi}$ and $\hat{\psi}$.

Antipode not bounded

Can again define

$$S\big((\iota\otimes\omega)(W)\big) = (\iota\otimes\omega)(W^*).$$

However, in general S will be an unbounded, σ-weakly-closed operator.

- We can factor S as $S = R\circ\tau_{-i/2}$.
- R is the "unitary antipode", a normal anti-$*$-homomorphism $M\to M$ which is an anti-homomorphism on M_*
- $S\circ *\circ S\circ * = \iota$.
- $\tau_{-i/2}$ is the analytic generator of a one-parameter automorphism group (τ_t) of M. Each τ_t also induces a homomorphism on M_*.
- Via Tomita-Takesaki theory, the weight $\hat\varphi$ has a modular operator $\hat\nabla$. Then $\tau_t(\cdot) = \hat\nabla^{it}(\cdot)\hat\nabla^{-it}$. (τ_t) is the scaling group.

A few names

An incomplete list...

- This viewpoint on $L^\infty(G)$ and $VN(G)$ comes from Takesaki, Tatsuuma.
- Using W in a more general setting comes from Baaj, Skandalis.
- Work of Woronowicz on the compact case.
- Enock & Schwartz, Kac & Vainerman developed "Kac algebras" (essentially when $S = R$).
- Masuda, Nakagami, Woronowicz gave a more complicated (but equivalent) set of axioms
- Current axioms are from Kustermans, Vaes.
- Prioritising W leads to the notion of a "manageable multiplicative unitary" from Woronowicz (and Sołtan).
- Various more algebraic approaches from van Daele.

Representation theory

A corepresentation of (M, Δ) is a unitary $U \in M\overline{\otimes}\mathcal{B}(K)$ with $(\Delta \otimes \iota)(U) = U_{13}U_{23}$.
If $M = L^\infty(G)$ and π is a unitary representation of G on K, then let

$$U = \big(\pi(t)\big)_{t\in G} \in L^\infty(G, \mathcal{B}(K)) \cong L^\infty(G)\overline{\otimes}\mathcal{B}(K).$$

That $(\Delta \otimes \iota)(U) = U_{13}U_{23}$ is equivalent to $\pi(st) = \pi(s)\pi(t)$.
The relation $\pi(s)^* = \pi(s^{-1})$ becomes reflected in the general fact that

$$(\iota \otimes \omega)(U) \in D(S), \quad S\big((\iota \otimes \omega)(U)\big) = (\iota \otimes \omega)(U^*).$$

W is a corepresentation– the left regular corepresentation on H.

Reduced and universal C*-algebras

Taking the norm closure of $\{(\iota \otimes \omega)(W)\}$ gives a C*-algebra A. Then Δ gives a "morphism" $A \to A \otimes A$ (a non-denegenerate $*$-homomorphism $A \to M(A \otimes A)$). The weights restrict to densely defined KMS weights.
There is a parallel C*-algebraic theory, though the axioms are more subtle.
There is a "maximal" corepresentation $\mathcal{W}$ (formed from a suitable direct sum argument). Then

$$\hat{A}_u = \text{closure}\{(\omega \otimes \iota)(\mathcal{W})\}$$

is a C*-algebra, which also admits a coproduct and invariant weights (thought these might fail to be faithful).
Any corepresentation U is of the form $U = (\iota \otimes \phi)(\mathcal{W})$ where $\phi : \hat{A}_u \to \mathcal{B}(K)$ is a unique non-degenerate $*$-representation.
This parallels the formation of $C^*(G)$ vs $C^*_r(G)$.

Multipliers

I'm interested in the algebra M_*, but this is only unital when (M,Δ) is said to be *discrete*.

- So you can study the multipliers of M_*.
- Turn A^* into a Banach algebra by using Δ (analogue of the measures on a group).
- Then M_* is an essential ideal in A^*.
- Indeed, the same is true for A_u^*.
- In the commutative case, $A = A_u$ and you get all the multipliers of $M_* = L^1(G)$ as measures.
- In the cocommutative case, A_u^* is the Fourier-Stieltjes algebra $B(G)$, but you get all multipliers of $M_* = A(G)$ if and only if G is amenable (Bożejko, Losert, Nebbia).

Completely positive case

Suppose that a is a completely positive multiplier of $A(G)$.

- To be precise, multiplication by a induces a map $A(G) \to A(G)$.
- So the adjoint is a map on $VN(G)$, and we ask that this is completely positive in the usual way.
- (Gilbert) There is a continuous map $\alpha : G \to K$ with $a(t^{-1}s) = (\alpha(t)|\alpha(s))_K$.
- (de Canniere, Haagerup) Now immediate that a is positive definite (and conversely).
- Notice that a is then also a positive member of $B(G)$, that is, a positive functional on $C^*(G)$.
- So if G amenable if and only if the span of the completely positive multipliers equals the space of all (completely bounded) multipliers.

Result in quantum case

Let $L_* : \hat{M}_* \to \hat{M}_*$ be a completely bounded (left) multiplier. So:

- $L_*(\omega \star \tau) = L_*(\omega) \star \tau$;
- the adjoint $L = (L_*)^* : \hat{M} \to \hat{M}$ is completely bounded.

Theorem (Junge, Neufang, Ruan)

There is a unique $x \in M$ such that, if we embed $\hat{M}_*$ into M via $\omega \mapsto (\omega \otimes \iota)(\hat{W})$, then L_* is given by left multiplication by x.

Theorem (D.)

$x \in M(A)$ and $x^* \in D(S)$ with $S(x^*)$ also inducing a left multiplier.

Picture: abstract multiplier of $A(G)$ corresponds to multiplication by a (continuous) function on G.

Completely positive case

Theorem (D.)

Let L_* be a left multiplier, associated to $x \in M(A)$. The following are equivalent:

- *$L = (L_*)^*$ is completely positive ($L : \hat{M} \to \hat{M}$).*
- *There is a positive functional $\mu \in \hat{A}_u^*$ with $L_*(\omega) = \mu \star \omega$ (recall: M_* ideal in $\hat{A}_u^*$), and $x = (\iota \otimes \mu)(\mathcal{W}^*)$.*
- *There is a unitary corepresentation U of (M, Δ) on K, and a positive $\mu \in \mathcal{B}(K)_*$ with $x = (\iota \otimes \mu)(U^*)$, and with*

$$L(\hat{x}) = (\iota \otimes \mu)\big(U(\hat{x} \otimes 1)U^*\big) \qquad (\hat{x} \in \hat{M}).$$

Link with Haagerup tensor product

So $L(\hat{x}) = (\iota \otimes \mu)(U(\hat{x} \otimes 1)U^*)$. By adjusting the space U acts on, we may assume that μ is a vector state ω_ξ, and then taking (e_i) an orthonormal basis of K, define

$$a_i = (\iota \otimes \omega_{\xi,e_i})(U^*) \in M \implies \sum_i a_i^* \hat{x} a_i = L(\hat{x}).$$

The extended (or weak*) Haagerup tensor product (Effros-Ruan, Blecher-Smith, Haagerup (unpublished)) of M with itself is the space

$$\Big\{ u \in M \overline{\otimes} M : u = \sum_i x_i \otimes y_i \text{ with } \sum_i x_i x_i^*, \sum_i y_i^* y_i < \infty \Big\}.$$

So

$$\sum_i a_i^* \otimes a_i \in M \overset{eh}{\otimes} M.$$

Sketch proof that CP multiplier $\Rightarrow$ corep

Actually, if we start with a CP left multiplier, then [JNR] (and a little bookkeeping) shows that for some $\sum_{i\in I} a_i^* \otimes a_i \in M \overset{eh}{\otimes} M$ we have $L(\hat{x}) = \sum_i a_i^* \hat{x} a_i$.

- By applying the [JNR] construction twice, you find that

$$\sum_i a_i^* \otimes a_i \otimes 1 = \sum_i \Delta(a_i^*)_{13} \Delta(a_i)_{23}.$$

- This is enough to construct an isometry U^* on $H \otimes \ell^2(I)$ and $\xi \in \ell^2(I)$ with

$$(\Delta \otimes \iota)(U^*) = U_{23}^* U_{13}^*, \quad (\iota \otimes \omega_{\xi,e_i})(U^*) = a_i.$$

- So U is a corepresentation; only remains to show that U is unitary. This follows by using that we can find the (a_i) from the Stinespring representation, and so we have some sort of minimality condition, and then using $\hat{M} M'$ is linearly, σ-weakly dense in $\mathcal{B}(H)$.

"Positive definite" elements

Recall that a function f on G is positive definite if

$$\big(f(st^{-1})\big)_{s,t\in G\times G} = (\iota\otimes S)\Delta(f)$$

is a positive kernel on $G\times G$.
$\mathcal{B}(H)\overset{eh}{\otimes}\mathcal{B}(H)$ is isomorphic to the space of completely bounded normal maps on $\mathcal{B}(H)$,

$$x\otimes y\mapsto \big(a\mapsto xay\big) \qquad (a\in\mathcal{B}(H)).$$

So can talk of "complete positivity".

Theorem (D. & Salmi)

$x\in M$ is a completely positive multiplier if and only if $(\iota\otimes S)\Delta(x)\in\mathcal{B}(H)\overset{eh}{\otimes}\mathcal{B}(H)$ and is completely positive.

To right multipliers

Introduce an involution J_K on K by $J_K(e_i)=e_i$ (and anti-linearity).
Define $\tau(\cdot)=J_K(\cdot)^*J_K$, an anti-$*$-automorphism on $\mathcal{B}(K)$. Then set

$$U^c=(R\otimes\tau)(U) \implies (\Delta\otimes\iota)(U^c)=(R\otimes R\otimes\tau)(\sigma\otimes\iota)(U_{13}U_{23})$$
$$=(\sigma\otimes\iota)(U^c_{23}U^c_{13})=U^c_{13}U^c_{23}.$$

- So U^c is a unitary corepresentation.
- So $L'(\hat{x})=(\iota\otimes\omega)(U^c(\hat{x}\otimes 1)(U^c)^*)$ is a left multiplier.
- The point is that $r=\hat{R}\circ L'\circ\hat{R}$ is then the adjoint of a completely positive right multiplier of $\hat{M}_*$, with (L,r) a double multiplier.
- Not surprising from the viewpoint that we're multiplying the (two-sided!) ideal $\hat{M}_*$ by elements of $\hat{A}_u^*$. But. . .

For Kac algebras

Have L associated to $a_i = (\iota \otimes \omega_{\xi,e_i})(U^*)$, and now L' associated to $b_i = (\iota \otimes \omega_{\xi,e_i})((U^c)^*)$. Supposing that $S = R$,

$$\begin{aligned} a_i^* &= (\iota \otimes \omega_{\xi,e_i})(U^*)^* = (\iota \otimes \omega_{e_i,\xi})(U) = R\big((\iota \otimes \omega_{e_i,\xi})(U^*)\big) \\ &= (\iota \otimes \omega_{J_K\xi, J_K e_i})((U^c)^*) = (\iota \otimes \omega_{\xi,e_i})((U^c)^*) = b_i. \end{aligned}$$

(Assume $J_K\xi = \xi$). So curiously,

$$\sum_i a_i a_i^* = \sum_i b_i^* b_i = L'(1) = 1.$$

So on all of $\mathcal{B}(H)$,

$$x \mapsto \sum_i a_i^* x a_i,$$

is a unital completely positive map, and a trace-preserving completely positive map.

In Quantum Information Theorey, such maps are called "bistochastic quantum channels". There is a small amount of literature...

From a Haagerup tensor product perspective

There is an asymmetry in the extended Haagerup tensor product, so the swap map σ is unbounded on $M \overset{eh}{\otimes} M$.

Yet we find that $u = \sum_i a_i^* \otimes a_i$ is such that both u and $\sigma(u)$ are in $M \overset{eh}{\otimes} M$.

Theorem (Pisier & Shlyakhtenko, Haagerup & Musat)

Let $u \in \mathcal{B}(K)\overline{\otimes}\mathcal{B}(K)$ be such that both u and $\sigma(u)$ are in $\mathcal{B}(K) \overset{eh}{\otimes} \mathcal{B}(K)$. Then the map

$$\mathcal{B}(K)_* \odot \mathcal{B}(K)_* \to \mathbb{C}; \quad \omega_1 \otimes \omega_2 \mapsto \langle u, \omega_1 \otimes \omega_2 \rangle$$

is bounded for the minimal operator space tensor norm.

Curiosity; and a question

If also $L = L'$ (the $\mu \in \hat{A}_u^*$ satisfies $\mu = \mu \circ \hat{R}_u$) then in $M \overset{eh}{\otimes} M$,

$$u = \sum_i a_i^* \otimes a_i = \sum_i b_i^* \otimes b_i = \sum_i a_i \otimes a_i^*,$$

so we even have that $\sigma(u) = u$.

But in general, $u \in \mathcal{B}(K) \overset{eh}{\otimes} \mathcal{B}(K), \sigma(u) = u$ does **not** imply that $u = \sum_i x_i \otimes y_i$ with $\sum_i x_i^* x_i, \sum_i y_i^* y_i < \infty$.

- What extra conditions on u would give this representation?
- If $u = \sum c_i^* \otimes d_i$ and $x \mapsto \sum_i c_i^* x d_i$ is completely *positive*, that's enough!
- Application: Completely bounded maps $A(G) \to VN(G)$ which factor through a column or row Hilbert space.