Weakly almost periodic functionals on the measure algebra

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Outline

Setting the scene

Throughout, G will be a locally compact group.
$L^1(G)$ is a Banach algebra with respect to convolution.
$M(G)$, the collection of finite measures on G, is also a Banach algebra under convolution.

Weakly almost periodic functionals

For $f \in C^b(G)$ and $s \in G$, define the left translate by

$$C^b(G) \ni L_s(f) : r \mapsto f(s^{-1}r) \qquad (r \in G).$$

We call $f \in C^b(G)$ *periodic* if the left translates

$$L_G(f) = \{L_s(f) : s \in G\}$$

span a finite-dimensional subspace of $C^b(G)$.
As $L_G(f)$ is bounded, f periodic implies that $L_G(f)$ is (relatively) compact.

Generalise: f is *almost periodic* if $L_G(f)$ is (relatively) compact.

Generalise: f is *weakly almost periodic* if $L_G(f)$ is (relatively) compact, in the *weak* topology on $C^b(G)$.

Links with compactifications

A group *compactification* of G is a pair (H, ϕ) of a compact group H and a continuous homomorphism $\phi : G \to H$, which has dense range (but may not be injective).

The Bohr (or almost periodic) compactification is the maximal group compactification of G, say $\mathfrak{b}G$.

Let $\mathrm{ap}(G) \subseteq C^b(G)$ be collection of all almost periodic functions. Then $\mathrm{ap}(G)$ is a (commutative) C*-subalgebra of $C^b(G)$, with character space $\mathfrak{b}G$. There is a natural way to lift the product from G to the character space of $\mathrm{ap}(G)$.

Replace "compact group" by "compact semitopological semigroup" (that is, separate continuity of the product) and we replace "almost periodic" by "weakly almost periodic".

For Banach algebras

For a Banach algebra $\mathcal{A}$, a functional $\mu \in \mathcal{A}^*$ is (weakly) almost periodic if the orbit

$$\{a \cdot \mu : a \in \mathcal{A}, \|a\| = 1\}$$

is relatively (weakly) compact in $\mathcal{A}$. Here $\mathcal{A}$ acts on $\mathcal{A}^*$ in the usual way. Write $\mathrm{wap}(\mathcal{A})$ or $\mathrm{ap}(\mathcal{A})$.

A bounded approximate identity argument shows that

$$\mathrm{ap}(L^1(G)) = \mathrm{ap}(G), \quad \mathrm{wap}(L^1(G)) = \mathrm{wap}(G),$$

where $C^b(G) \subseteq L^\infty(G) = L^1(G)^*$. (See Ulger, 1986, or Wong, 1969, or Lau, 1977).

$\mathrm{wap}(\mathcal{A})$ has interesting links with the Arens products on $\mathcal{A}^{**}$.

In general, little can be said about $\mathrm{wap}(\mathcal{A})$ and $\mathrm{ap}(\mathcal{A})$.

Measure algebras

What can we say about ap($M(G)$) or wap($M(G)$)?

To be more precise: To show that wap($L^1(G)$) is a subalgebra of $L^\infty(G)$ requires the result that wap($L^1(G)$) = wap(G), and then an application of Grothendieck's repeated limit criterion for weak compactness.

Representation theory

A *representation* of G is a group homomorphism $\pi : G \to \mathrm{iso}(E)$, the isometry group of a Banach space E, which is weak operator topology continuous.

A *representation* of $L^1(G)$ is a contractive Banach algebra homomorphism $\hat\pi : L^1(G) \to \mathcal{B}(E)$.

Johnson: There is a bijection between (non-degenerate) representations of G and (non-degenerate) representations of $L^1(G)$.

$$\hat\pi(f) = \int_G f(s)\pi(s)\,ds,$$

Bounded approximate identities allows you to build π from $\hat\pi$.

"Multiplying" functionals

Given $\pi : G \to \mathrm{iso}(E)$, a *coefficient functional* of π is

$$F \in C^b(G), \quad F(s) = \langle \mu, \pi(s)x\rangle \quad (s \in G),$$

where $\mu \in E^*$ and $x \in E$. Write $F = \omega_{\pi,\mu,x}$.

Given $\pi_i : G \to \mathrm{iso}(E_i)$ and $F_i = \omega_{\pi_i,\mu_i,x_i}$, we define

$$\pi = \pi_1 \otimes \pi_2 : G \to \mathrm{iso}(E_1 \otimes E_2), \quad s \mapsto \pi_1(s) \otimes \pi_2(s),$$

and then

$$(F_1F_2)(s) = \langle \mu_1 \otimes \mu_2, \pi(s)(x_1 \otimes x_2)\rangle \quad (s \in G).$$

Mantra: Multiplication of coefficient functionals is the same as tensoring representations.

This is exactly the proof that the Fourier-Stieltjes algebra is an algebra (all coefficient functionals of unitary representations).

Young, Kaiser and Interpolation

The celebrated theorem of Davis, Figiel, Johnson and Pełczyǹski tells us the weakly compact operators are precisely the operators which factor through reflexive Banach spaces.

Young adapted the proof to Banach algebras; Kaiser recast it in the language of interpolation spaces.

Theorem

$\mu \in$ wap($\mathcal{A}^*$) if and only if there exists a reflexive Banach space E, a representation $\pi : \mathcal{A} \to \mathcal{B}(E)$, and $x \in E, \mu \in E^*$ with

$$\langle \mu, a\rangle = \langle \mu, \pi(a)(x)\rangle \qquad (a \in \mathcal{A}).$$

So $F \in$ wap($L^1(G)$) if and only if F is the coefficient functional of a representation on a *reflexive* Banach space.

Reflexive tensor products

Let E and F be reflexive Banach spaces. There exists a norm on $E \otimes F$ such that:

1. $\|x \otimes y\| = \|x\|\|y\|$ for $x \in E, y \in F$;
2. Given $T \in \mathcal{B}(E)$ and $S \in \mathcal{B}(F)$, the map $T \otimes S$ is bounded, with norm $\|T\|\|S\|$;
3. the completion is reflexive.

So:

- wap($L^1(G)$) is the space of coefficient functionals on reflexive spaces;
- Multiplication is the same as tensoring;
- Reflexive spaces are stable under tensoring.

So wap($L^1(G)$) is a subalgebra of $C^b(G)$.

The measure algebra

There is a measure space X such that $M(G) = L^1(X)$ as Banach spaces.

Seemingly no way to express the convolution product on $M(G)$ in terms of X.

For example, no link between representations of $M(G)$ and a "representation" of X.

Change categories!

Look at Hopf von Neumann algebras and corepresentations.

Hopf von Neumann algebras

A (commutative) Hopf von Neumann algebra is a pair $(L^\infty(X), \Gamma)$ where $\Gamma : L^\infty(X) \to L^\infty(X \times X)$ is a unital, normal, $*$-homomorphism which is co-associative:

$$\begin{array}{ccc} L^\infty(X) & \xrightarrow{\Gamma} & L^\infty(X \times X) \\ \downarrow{\scriptstyle\Gamma} & & \downarrow{\scriptstyle \mathrm{id}\otimes\Gamma} \\ L^\infty(X \times X) & \xrightarrow{\Gamma\otimes\mathrm{id}} & L^\infty(X \times X \times X) \end{array}$$

As Γ is normal, it drops to give a contraction

$$L^1(X) \times L^1(X) \longrightarrow L^1(X \times X) \xrightarrow{\Gamma_*} L^1(X).$$

Then Γ is co-associative if and only if this product is associative.

Examples

The motivating example is $L^\infty(G)$ with the map

$$\Gamma : L^\infty(G) \to L^\infty(G \times G);$$
$$\Gamma(F)(s,t) = F(st) \qquad (F \in L^\infty(G), s, t \in G).$$

Then Γ_* induces the usual convolution product on $L^1(G)$.

As $M(G) = C_0(G)^*$, we can lift the product from $C_0(G)$ to $M(G)^* = C_0(G)^{**}$, so $M(G)^*$ becomes a commutative von Neumann algebra.

We can lift the product from $M(G)$ to a co-associative map on $M(G)^*$, turning $M(G)^*$ into a Hopf von Neumann algebra.

Representations?

A suitable generalisation of a representation is a *co-representation* of $(L^\infty(X), \Gamma)$.

A co-representation of $L^\infty(X)$ on a Hilbert space H is an element $W \in L^\infty(X)\overline{\otimes}\mathcal{B}(H)$ (von Neumann tensor product); with

$$(\Gamma \otimes \mathrm{id})W = W_{13}W_{23} \in L^\infty(X \times X)\overline{\otimes}\mathcal{B}(H).$$

Here $W_{23}(x_1 \otimes x_2 \otimes x_3) = x_1 \otimes W(x_2 \otimes x_3)$. $W_{13} = \chi W_{23} \chi$ where $\chi(x_1 \otimes x_2 \otimes x_3) = x_2 \otimes x_1 \otimes x_3$.

The von Neumann algebra $L^\infty(X)\overline{\otimes}\mathcal{B}(H)$ has predual

$$L^1(X)\widehat{\otimes}\mathcal{T}(H),$$

the *projective tensor product* of $L^1(X)$ and the trace-class operators on H.

Co-representations

$$L^\infty(X)\overline{\otimes}\mathcal{B}(H) = \left(L^1(X)\widehat{\otimes}\mathcal{T}(H)\right)^* = \mathcal{B}(L^1(X), \mathcal{B}(H)),$$

via the dual pairing

$$\langle T, f \otimes \tau \rangle = \langle T(f), \tau \rangle \qquad \begin{pmatrix} T \in \mathcal{B}(L^1(X), \mathcal{B}(H)), \\ f \in L^1(X), \tau \in \mathcal{T}(H) \end{pmatrix}$$

So $W \in L^\infty(X)\overline{\otimes}\mathcal{B}(H)$ induces $\pi : L^1(X) \to \mathcal{B}(H)$; W is a corepresentation if and only if π is a (Banach algebra) representation.

Tensoring co-representations

Given $\pi_i : L^1(X) \to \mathcal{B}(H_i)$ representations, the tensored representation

$$\pi = \pi_1 \otimes \pi_2 : L^1(X) \to \mathcal{B}(H_1 \otimes H_2),$$

is associated to

$$W_{12}^{(1)} W_{13}^{(2)} \in L^\infty(X)\overline{\otimes}\mathcal{B}(H_1)\overline{\otimes}\mathcal{B}(H_2).$$

A coefficient functional associated to π is

$$\langle F, a \rangle = \langle \mu, \pi(a)(x) \rangle = \langle (\mathrm{id} \otimes \omega_{\mu,x})W, a \rangle \quad (a \in L^1(X)),$$

where $\omega_{\mu,x} \in \mathcal{T}(H)$ is the normal functional

$$\mathcal{B}(H) \to \mathbb{C}; \quad T \mapsto \langle \mu, T(x) \rangle.$$

For reflexive spaces?

So multiplying coefficient functionals is equivalent to "multiplying" co-representations.

At least on Hilbert spaces!

So we need a co-representation theory for reflexive Banach spaces!

Weak*-tensor products

Fix a reflexive space E. We define $L^\infty(X)\overline{\otimes}\mathcal{B}(E)$ to be the weak*-closure of $L^\infty(X)\otimes\mathcal{B}(E)$ inside $\mathcal{B}(L^2(X,E))$.

Here $L^2(X,E)$ is a vector-valued L^2 space.

That is, the closure of $L^2(X)\otimes E$ for some norm.

Using the *approximation property* for $L^1(X)$, we can show that

$$\mathcal{B}(L^1(X),\mathcal{B}(E)) \cong L^\infty(X)\overline{\otimes}\mathcal{B}(E).$$

Then co-representations all still work, and are compatible with our way of tensoring reflexive spaces.

A result!

Theorem

Let $(L^\infty(X),\Gamma)$ be a commutative Hopf von Neumann algebra. The $\mathrm{wap}(L^1(X))$ is a C^*-subalgebra of $L^\infty(X)$.

Proof.

Easy to see that $\mathrm{wap}(L^1(X))$ is closed and self-adjoint.
Need to show that given $F_1, F_2 \in \mathrm{wap}(L^1(X))$, we have $F_1F_2 \in \mathrm{wap}(L^1(X))$.
F_i associated to $\pi_i : L^1(X)\to\mathcal{B}(E_i)$, associated to $W^{(i)}\in L^\infty(X)\overline{\otimes}\mathcal{B}(E_i)$.
Then can take product $W = W^{(1)}W^{(2)} \in L^\infty(X)\overline{\otimes}\mathcal{B}(E_1\otimes E_2)$, induces $\pi : L^1(X)\to\mathcal{B}(E_1\otimes E_2)$, induces F_1F_2. □

The analogous result for $\mathrm{ap}(L^1(X))$ is easy, once you think in terms of Γ (and not just look at $L^1(X)$).

But what is $\mathrm{wap}(M(G))$?

For $L^1(G)$, we have that $\mathrm{wap}(L^1(G)) = \mathrm{wap}(G) = C(K)$ where K is some compact semigroup, which we can characterise in terms of G.

We know that $\mathrm{wap}(M(G)) = C(K)$ for some K. It would be natural that Γ somehow induce a map $K\times K\to K$.

But we only expect *separate* continuity, so we cannot expect something simple, like Γ restricting to a map $C(K)\to C(K\times K)$.

Not clear that co-representations give much insight.

Weakly compact operators

We have that

$$L^\infty(X\times X) = L^\infty(X)\overline{\otimes}L^\infty(X) = \big(L^1(X)\widehat{\otimes}L^1(X)\big)^* = \mathcal{B}(L^1(X),L^\infty(X)).$$

Let $\mathcal{W}(L^1(X),L^\infty(X))$ be the collection of all weakly-compact operators $L^1(X)\to L^\infty(X)$.

Again using factorisation results, it is possible to show:

Theorem

Identify $\mathcal{B}(L^1(X),L^\infty(X))$ with $L^\infty(X\times X)$. Then $\mathcal{W}(L^1(X),L^\infty(X))$ is a subalgebra of $L^\infty(X\times X)$.

This immediately implies that $\mathrm{wap}(L^1(X))$ is a subalgebra!

Semitopological semigroups

Recall that a topological semigroup K is *semitopological* if the product is separately continuous.

Theorem

Let $(L^\infty(X),\Gamma)$ be a commutative Hopf von Neumann algebra. Let K be the character space of $\mathrm{wap}(L^1(X))$. Then Γ naturally induces a semigroup product on K turning K into a compact semitopological semigroup.

For the measure algebra

We can apply this to $\mathrm{wap}(M(G))\cong C(K)$.

We now know that K is, naturally, a compact semitopological semigroup.

But what can we say about K? It would be good to have an abstract characterisation of K in terms of G.

Non-commutative issues

I initially thought about these problems for *non-commutative* Hopf von Neumann algebras, specifically for locally compact quantum groups.

Let (M, Γ) be a Hopf von Neumann algebra; let M_* be the predual of M; let E be a reflexive (operator) space.

1. What is a good replacement for $L^2(X, E)$? Maybe Pisier's notion of vector-valued non-commutative L^p spaces? But does M act nicely on these?
2. Lacking the approximation property, can we show that $\mathcal{CB}(M_*, \mathcal{CB}(E))$ is equal to $M_* \overline{\otimes} \mathcal{CB}(E)$? (True if E is a Hilbert space).
3. How to tensor two reflexive operator spaces?