# Shift-invariant preduals of $\ell^1(\mathbb{Z})$

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# Banach spaces and duality

A first course in Banach spaces (not Hilbert spaces!) will introduce the Banach spaces $\ell^1 = \ell^1(\mathbb{N})$, and $c_0 = c_0(\mathbb{N})$:

$$\ell^1 = \Big\{ (a_n) : \|(a_n)\|_1 = \sum_n |a_n| < \infty \Big\}$$

$$c_0 = \Big\{ (a_n) : \lim_n a_n = 0 \Big\} \quad \text{with} \quad \|(a_n)\|_\infty = \sup_n |a_n|.$$

Remember that the dual space $E^*$ is the collection of bounded linear maps $E \to \mathbb{C}$. Then $c_0^* = \ell^1$. To be precise, for each $f \in c_0^*$ there exists $(f_n) \in \ell^1$ such that

$$f((a_n)) = \sum_n f_n a_n \qquad ((a_n) \in c_0),$$

and with $\|f\| = \|(f_n)\|_1$.

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# Other preduals of $\ell^1$

Let $K$ be a compact Hausdorff space; let $C(K)$ be the Banach space of continuous functions on $K$ with the supremum norm; let $M(K)$ be the space of regular Borel measures on $K$, with the total variation norm.

Then each member of $C(K)^*$ arising from integrating against a member of $M(K)$. So we can write $C(K)^* = M(K)$.

Now suppose that $K$ is countable– we can enumerate $K$ as $K = \{k_n : n \in \mathbb{N}\}$ say. Then any $\mu \in M(K)$ is countably additive, and so for $f \in C(K)$,

$$\int_K f \, d\mu = \sum_n f(k_n)\mu(\{k_n\}).$$

Hence we have an isometric isomorphism $\theta : \ell^1 \to C(K)^*$ which sends $a = (a_n) \in \ell^1$ to the functional $\theta_a \in C(K)^*$ given by

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# The weak$^*$-topology

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$$\theta_a(f) = \sum_n f(k_n) a_n \qquad (f \in C(K)).$$

To simplify notation, we shall write $\langle \cdot, \cdot \rangle$ for the dual pairing $\ell^1 \times C(K) \to \mathbb{C}$, so $\langle a, f \rangle = \theta_a(f)$.

So the isomorphism $\ell^1 \cong C(K)^*$ induces a weak$^*$-topology on $\ell^1$. For example, as $K$ is compact, we have non-trivial limiting sequences– say $(k_{n_i}) \to k_n$ as $i \to \infty$.

Write $\delta_k$ for the "point-mass" in $\ell^1$ at $k$– that is, the sequence which is 0 except for a 1 in the $k$th place. Thus for $f \in C(K)$,

$$\lim_i \langle \delta_{k_{n_i}}, f \rangle = \lim_i f(k_{n_i}) = f(k_n) = \langle \delta_{k_n}, f \rangle,$$

and so $\delta_{k_{n_i}} \to \delta_{k_n}$ weak$^*$. Of course, this does not hold for the "usual" weak$^*$-topology induced by $c_0^* = \ell^1$.

# The weak*-topology

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# Think more abstractly about preduals

Given a Banach space $E$, a *predual* for $E$ is a Banach space $F$ together with an isomorphism (not assumed isometric) $\theta : E \to F^*$.

- Note that the map $\theta$ is very important.
- It seems reasonable to say that two preduals "are the same" if they induce the same weak*-topology on $E$.
- As usual, we identify $F$ with a closed subspace of its bidual $F^{**}$, and so we can talk about the image of $F$ under the adjoint map $\theta^* : F^{**} \to E^*$. Call this $F_0$.
- Then $F_0 \subseteq E^*$ is a closed subspace such that:
  - $F_0$ separates the points of $E$;
  - every functional $\mu \in F_0^*$ is given by some element of $E$.
- We call such a subspace $F_0 \subseteq E^*$ a *concrete predual*.
- It's not hard to see that two concrete preduals $F_0, F_1$ induce the same weak*-topology on $E$ if and only if $F_0 = F_1$.

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# An "exotic" predual of $\ell^1(\mathbb{Z})$

I now want to describe, in detail, the construction of a concrete predual for $\ell^1(\mathbb{Z})$ which has an unusual property– namely, the bilateral shift $S : \ell^1(\mathbb{Z}) \to \ell^1(\mathbb{Z})$ will be weak*-continuous.
This is joint work with Richard Haydon, Thomas Schlumprecht, and Stuart White, see arXiv:1101.5696v1 [math.FA].

For an integer $n \geq 1$, let $b(n)$ be the number of ones in the binary expansion of $n$, so $b(1) = b(2) = b(4) = 1$, $b(3) = b(5) = b(6) = 2$, $b(7) = 3$ and so on. Set $b(0) = 0$, and for $n < 0$, set $b(n) = \infty$.
Let $x_0 = (2^{-b(n)})_{n\in\mathbb{Z}} \in \ell^\infty(\mathbb{Z})$, so

$$x_0 = \left( \cdots, 0, 0, 1, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{2}, \cdots \right).$$

Let $F$ be the closed, shift-invariant subspace of $\ell^\infty(\mathbb{Z})$ generated by $x_0$. So $F$ is the closed linear span of $\{S^k(x_0) : k \in \mathbb{Z}\}$.
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# Separates points

- Given $x \in \ell^\infty(\mathbb{Z})$, we view $x$ as a function $x : \mathbb{Z} \to \mathbb{C}$. Extend this to a function $x : \mathbb{Q} \to \mathbb{C}$ by setting $x(q) = 0$ for $q \in \mathbb{Q} \setminus \mathbb{Z}$.
- Define $\tau : \ell^\infty(\mathbb{Z}) \to \ell^\infty(\mathbb{Z})$ by $\tau(x)(n) = x(n/2)$.
- We can check that $\tau S = S^2 \tau$.
- We claim that $\tau^k(x_0) \in F$ for every $k \geq 1$.
  - First prove the identity

$$(1 - \tfrac{1}{2}S)(x_0)(n) = \textstyle\sum_{j\geq 1} 2^{-j}\tau^j(x_0)(n) \qquad (n \in \mathbb{Z}).$$

  - Then show that

$$(1 - \tfrac{1}{2}S)(x_0) = (1 - \tfrac{1}{4}S^2)\tau(x_0).$$

  - Then invert to get

$$\tau(x_0) = \sum_{j\geq 0} 4^{-j} S^{-2j}(1 - \tfrac{1}{2}S)(x_0) \in F.$$

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- Given $x \in \ell^\infty(\mathbb{Z})$, we view $x$ as a function $x : \mathbb{Z} \to \mathbb{C}$. Extend this to a function $x : \mathbb{Q} \to \mathbb{C}$ by setting $x(q) = 0$ for $q \in \mathbb{Q} \setminus \mathbb{Z}$.
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- So for any $a = (a_n) \in \ell^1(\mathbb{Z})$, we see that

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# Functionals on $F$

It is rather harder to show that every bounded linear functional $F \to \mathbb{C}$ is induced by a member of $\ell^1(\mathbb{Z})$.

- For example, a "typical" functional on $\ell^\infty(\mathbb{Z})$ which is not given by an element $\ell^1(\mathbb{Z})$ is the functional

$$\mu : x \mapsto \lim_{n\to\infty} x(2^n).$$

(To make this converge on all of $\ell^\infty(\mathbb{Z})$, limit down an ultrafilter).

- Let's restrict $\mu$ to $F$. It's enough to compute $\mu$ on $S^k(x_0)$ for $k \in \mathbb{Z}$

$$\mu(S^k(x_0)) = \lim_n x_0(2^n - k) = \lim_n 2^{-b(2^n-k)} = 2^{-1-b(-k)}.$$

- But then note that

$$\langle S^k(x_0), \tfrac{1}{2}\delta_0\rangle = \frac{1}{2}S^k(x_0)(0) = \frac{1}{2}x_0(-k) = \frac{1}{2}2^{-b(-k)}.$$

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# Shift-invariant preduals

So we conclude that $F$ is a predual for $\ell^1(\mathbb{Z})$. By construction, it is shift-invariant, so it follows that $S$ is weak*-continuous.

- By the calculation on the previous slide, we see that

$$\text{weak}^* - \lim_n \delta_{2^n} = \frac{1}{2}\delta_0.$$

  This gives but one example of a non-trivial weak*-limit point.
- In particular, $F$ does not give the same weak*-topology as $c_0(\mathbb{Z})$.
- There was nothing special about using 2– this could have been any $\lambda \in \mathbb{C}$ with $|\lambda| > 1$. So we get an uncountable number of mutually non-isomorphic preduals.
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# What Banach space is $F$?

If I were to give the actually proof that every functional on $F$ is given by $\ell^1(\mathbb{Z})$, then I would use that $\ell^\infty(\mathbb{Z}) = C(\beta\mathbb{Z})$. We'd then check that there was a partition of $\beta\mathbb{Z} \setminus \mathbb{Z}$, say

$$\{X_t^{(k)} : t \in \mathbb{Z}, k \geq 1\} \cup \{X^{(\infty)}\},$$

such that $x \in F$ if and only if

$$x(\omega) = \begin{cases} 2^{-k}x(t) & : \omega \in X_t^{(k)}, \\ 0 & : \omega \in X^{(\infty)}. \end{cases}$$

- It follows that $F$ is a "G-space", in the sense of Benyamini (Israel J. Math (1973)). Thus $F$ is, as a Banach space, isomorphic to some $C(K)$ space.
- So we can calculate the Szlenk index to work out which $C(K)$ space $F$ is isomorphic to.
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# Algebraic constraints

- Recall that (in the weak*-topology) $\delta_{2^n} \to \frac{1}{2}\delta_0$.
- By shift-invariance, we must also have that $\delta_{2^n+1} \to \frac{1}{2}\delta_1$, and so forth.
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- By shift-invariance, we must also have that $\delta_{2^n+1} \to \frac{1}{2}\delta_1$, and so forth.
- But then, consider

$$\lim_n \lim_m \delta_{2^n+2^m} = \lim_n \frac{1}{2}\delta_{2^n} = \frac{1}{4}\delta_0.$$

- This is all fine, as $\{2^n : n \in \mathbb{N}\}$ is sufficiently "sparse".
- But suppose, instead, we had that

$$\lim_n \delta_{2n} = \frac{1}{2}\delta_0.$$

- Then we'd have that

$$\lim_n \delta_{2n} = \lim_n \delta_{2(n+1)} = \frac{1}{2}\delta_2,$$

which would be a contradiction!

# A general theorem

We shall say that $J \subseteq \mathbb{Z}$ is "additively sparse" if given $t \in \mathbb{Z}$ and $r, s \in \mathbb{N}$, we can find $N$ such that, if

$$j_1 + \cdots + j_r = l_1 + \cdots + l_s + t,$$

with $(j_i), (l_i) \subseteq J$ and $N < |j_1| < \cdots < |j_r|, N < |l_1| < \cdots < |l_s|$, then necessarily $r = s, t = 0$ and $j_i = l_i$ for each $i$.

Define a multiplication on $\ell^1(\mathbb{Z})$ by $\delta_n \delta_m = \delta_{n+m}$.

**Theorem**

*Let $J \subseteq \mathbb{Z}$ be additively sparse, and let $J = J^{(1)} \cup \cdots \cup J^{(r)}$ be a partition. For each i, let $a_i \in \ell^1(\mathbb{Z})$ be a power-bounded element with $\|a_i^n\|_\infty \to 0$. Then there is a shift-invariant $\ell^1(\mathbb{Z})$ predual E such that $\delta_n \to a_i$ weak* as $n \to \infty$ through $J^{(i)}$.*

The example given before has $J = \{2^n : n \in \mathbb{N}\}$, $r = 1$ and $a_1 = \frac{1}{2}\delta_0$.

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# Further examples

- Let $J = \{2^n\}$ and $r = 1$. Let $a_1 = \frac{1}{2}(\delta_0 + \delta_1)$. Thus

$$a_1^2 = \frac{1}{4}(\delta_0 + 2\delta_1 + \delta_2), \quad a_2^3 = \frac{1}{8}(\delta_0 + 3\delta_1 + 3\delta_2 + \delta_3).$$

  You can check that $\|a_1^n\|_1 = 1$ for any $n$, but that we do have $\|a_1^n\|_\infty \to 0$.
- Some fiddling with the Szlenk index shows that the resulting predual, in this case, is not isomorphic to $c_0$.
- Now set $a_1 = 5^{-1/2}(\delta_0 + \delta_1 - \delta_2)$. Then $\|a_1\| = 3/\sqrt{5} > 1$, but (if you know where to look!) this is a power-bounded element. Taking the Fourier transform shows that $\|a_1^n\|_\infty \to 0$.
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Hold on: I said that $C(K)$ spaces, for countable $K$, provide simple examples of preduals of $\ell^1$.

- So could we cleverly choose a (compact, Hausdorff) topology on $\mathbb{Z}$ such that $C(\mathbb{Z})$ provided a shift-invariant predual?
- Well, $\mathbb{Z}$ would then be countable and (locally) compact, so it would be a Baire Space, and hence would have some $k \in \mathbb{Z}$ with $\{k\}$ being open.
- The identification of $C(\mathbb{Z})$ as a closed subspace of $\ell^\infty(\mathbb{Z})$ is simply the identification of functions. So $C(\mathbb{Z})$ will be shift-invariant if and only if the shift on $\mathbb{Z}$ is continuous.
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# Original motivation

A Banach algebra is a Banach space with an algebra product which is contractive: $\|ab\| \leq \|a\|\|b\|$.

- For example, $\ell^1(\mathbb{Z})$ with the convolution product.
- More generally, let $G$ be a discrete group, and consider $\ell^1(G)$ with the convolution product.
- This example has a predual: $c_0(G)$. Furthermore, the algebra product is (separately) weak*-continuous. That is, if $a_i \to a$ weak*, then also $a_i b \to ab$ weak*, and similarly $ba_i \to ba$.
- We say that $\ell^1(G)$ is a *dual Banach algebra* (with respect to $c_0(G)$).
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# Unique preduals

A *von Neumann algebra* is a C$^*$-algebra $A$ which is the *isometric* dual of some Banach space $E$.

- Sakai's Theorem then says that $A$ becomes a dual Banach algebra with respect to $E$.
- Furthermore, if also $A = F^*$ isometrically, then $E$ and $F$ are isometrically isomorphic, and *induce the same weak$^*$-topology on $A$*.
- Another way to state this is: if $A$ and $B$ are von Neumann algebras and $\theta : A \to B$ is an isometric isomorphism, then necessarily $\theta$ is weak$^*$-continuous.
- Pełcynski showed that $L^\infty[0,1]$ and $\ell^\infty$ are, as Banach space, isomorphic. But of course, $L^1[0,1]$ and $\ell^1$ are not isomorphic.

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**Theorem (D., Le Pham, White)**

*Let $A$ be a von Neumann algebra, and let $B$ be a dual Banach algebra. If $\theta : A \to B$ is an isomorphism (not necessarily isometric) which is also an algebra homomorphism, then necessarily $\theta$ is weak$^*$-continuous.*

**Theorem (D.)**

*Let $E$ be a reflexive Banach space with the approximation property, and denote by $\mathcal{B}(E)$ the algebra of bounded operators on $E$. Let $B$ be a dual Banach algebra. If $\theta : \mathcal{B}(E) \to B$ is an isomorphism (not necessarily isometric) which is also an algebra homomorphism, then necessarily $\theta$ is weak$^*$-continuous.*

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**Theorem (D., Le Pham, White)**

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Of course, the main task of this talk has been to show:

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# For semigroups

Together with Le Pham and White, we showed that for semigroups, the situation is very different.

**Theorem**

*With $S = \mathbb{Z} \times \mathbb{Z}_+$, consider the Banach algebra $\ell^1(S)$. There is a continuum of preduals of $\ell^1(S)$ which all turn $\ell^1(S)$ into a dual Banach algebra, and which are all subalgebras of $\ell^\infty(S)$.*

**Theorem**

*Let $S = \mathbb{N}$ equipped with the semigroup product $\max$. Then $\ell^1(S)$ is a dual Banach algebra with respect to $c_0(S)$. If $B$ is a dual Banach algebra and $\theta : \ell^1(S) \to B$ is an isomorphism which is an algebra homomorphism, then necessarily $\theta$ is weak$^*$-continuous.*

# For semigroups

Together with Le Pham and White, we showed that for semigroups, the situation is very different.

### Theorem

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### Theorem

*Let $S = \mathbb{N}$ equipped with the semigroup product $\max$. Then $\ell^1(S)$ is a dual Banach algebra with respect to $c_0(S)$. If $B$ is a dual Banach algebra and $\theta : \ell^1(S) \to B$ is an isomorphism which is an algebra homomorphism, then necessarily $\theta$ is weak*-continuous.*