Locally compact quantum groups
2. C^*-algebras and compact quantum groups

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Obligatory non-commutative topology 2

Theorem (Gelfand)

Let A be a commutative C^*-algebra, and let Φ_A be the collection of characters on A, given the relative weak*-topology. Then Φ_A is a locally compact Hausdorff space, and the map

$$\mathcal{G} : A \to C_0(\Phi_A); \quad \mathcal{G}(a)(\varphi) = \varphi(a),$$

is an isometric isomorphism.

But how do we capture the notion of a continuous map between Φ_A and Φ_B?

- $*$-homomorphisms $A \to B$ correspond to *proper* continuous maps $\Phi_B \to (\Phi_A)_\infty$, the one-point compactification of Φ_A.

Multiplier algebras

Let A be a C*-algebra.

- Regard A as acting non-degenerately (so $\operatorname{lin}\{a(\xi) : a \in A, \xi \in H\}$ is dense in H) on H. Then

$$M(A) = \{T \in \mathcal{B}(H) : Ta, aT \in A \ (a \in A)\}.$$

- Regard A as a subalgebra of its bidual A^{**}; then

$$M(A) = \{x \in A^{**} : xa, ax \in A \ (a \in A)\}.$$

- These are isomorphic (and independent of H).

An abstract way to think of $M(A)$ is as the pairs of maps (L, R) from A to A with $aL(b) = R(a)b$. A little closed graph argument shows that L and R are bounded, and that

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \qquad (a, b \in A).$$

The involution in this picture is $(L, R)^* = (R^*, L^*)$ where $R^*(a) = R(a^*)^*$, $L^*(a) = L(a^*)^*$. You can move between these pictures by a bounded approximate identity argument.

Multiplier algebras 2

- $M(A)$ is the largest C*-algebra containing A as an *essential* ideal: if $x \in M(A)$ and $axb = 0$ for all $a, b \in A$, then $x = 0$.
- So $M(A)$ is the largest (sensible) unitisation of A.

Applied to $C_0(X)$, unitisations correspond to compactifications of X.

- Indeed, $M(C_0(X))$ is isomorphic to $C^b(X)$ the algebra of all bounded continuous functions on X.
- The character space of $C^b(X)$ is βX, the Stone-Čech compactification.

Morphisms

A *morphism* $A \to B$ between C^*-algebras is a *non-degenerate* $*$-homomorphism $\theta : A \to M(B)$.

- θ is *non-degenerate* if $\{\theta(a)b : a \in A, b \in B\}$ is linearly dense in B.

The *strict topology* on $M(B)$ is:

$$x_\alpha \to x \quad \Leftrightarrow \quad x_\alpha b \to xb,\ bx_\alpha \to bx \quad (b \in B).$$

Non-degeneracy is equivalent to:

- For any (or all) bounded approximate identity (e_α) in A, the net $(\theta(e_\alpha))$ converges strictly to $1 \in M(B)$;
- θ is the restriction of a strictly continuous $*$-homomorphism $\tilde\theta : M(A) \to M(B)$.

We can construct the extension: $\tilde\theta(x)\theta(a)b = \theta(xa)b$ and so forth.

Application

Theorem

Let X, Y be locally compact spaces.

- *Given a continuous map $\phi : Y \to X$, the map $\theta : C_0(X) \to C^b(Y); f \mapsto f \circ \phi$ is a morphism.*
- *Any morphism $C_0(X) \to C_0(Y)$ is induced in this way.*

So we have some machinery: but it captures exactly what we want!

Compact quantum groups

Let G be a compact semigroup (associative, continuous product).

- Define $\Delta : C(G) \to C(G \times G); \Delta(f)(s,t) = f(st)$ which is a unital $*$-homomorphism;
- again this is coassociative $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;
- Every coassociative $\Delta : C(G) \to C(G \times G)$ arises in this way (from some product on G).

How do we capture the notion of a group?

- Write down the identity and inverse, as maps on $C(G)$?
- Inelegant; doesn't generalise.

Theorem

A compact semigroup G is a group if and only if satisfies cancellation:

$$st = sr \implies t = r, \qquad ts = rs \implies t = r.$$

If you're bored: prove this.

Cancellation as density

Theorem

G satisfies cancellation if and only if

$$\operatorname{lin}\{(a \otimes 1)\Delta(b) : a, b \in C(G)\}, \qquad \operatorname{lin}\{(1 \otimes a)\Delta(b) : a, b \in C(G)\}$$

are dense in $C(G \times G) = C(G) \otimes C(G)$.

Sketch proof.

- Commutative, so these are $*$-subalgebras, so can apply Stone-Weierstrauss: dense if and only if they separate points;
- $(a \otimes 1)\Delta(b)(s,t) = a(s)b(st)$;
- so $st = sr$ if and only if $f(s,t) = f(s,r)$ for all f in the 1st set;
- so separates points if and only if cancellation.

□

Compact quantum groups

Definition (Woronowicz)

A compact quantum group is a unital C^*-algebra A with a coassociative unital $*$-homomorphism $\Delta : A \to A \otimes A$ with

$$\{(a \otimes 1)\Delta(b) : a, b \in A\}, \qquad \{(1 \otimes a)\Delta(b) : a, b \in A\}$$

linearly dense in $A \otimes A$.

So if A is commutative, we exactly capture the notion of a compact group.

Let Γ be a discrete group, and $A = C_r^*(\Gamma)$ the reduced group C^*-algebra, say generated by $\{\lambda(s) : s \in \Gamma\}$.

- Exactly as in the last lecture, can construct a coproduct $\Delta : \lambda(s) \mapsto \lambda(s) \otimes \lambda(s)$.
- Cancellation is easy to verify: $(\lambda(st^{-1}) \otimes 1)\Delta(\lambda(t)) = \lambda(s) \otimes \lambda(t)$.
- Every cocommutative ($\Delta = \sigma\Delta$) compact quantum group is of this form.

Construction of Haar state

- From now on, (A, Δ) is a compact quantum group.
- Turn A^* into a (completely contractive) Banach algebra:

$$\langle \mu \star \lambda, a \rangle = \langle \mu \otimes \lambda, \Delta(a) \rangle \qquad (\mu, \lambda \in A^*, a \in A).$$

Theorem

There is a unique state φ with $(\varphi \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varphi)\Delta(a) = \langle \varphi, a \rangle 1$.

Very sketch proof.

- Equivalent to $\varphi \star \mu = \mu \star \varphi = \langle \mu, 1 \rangle \varphi$ for all $\mu \in A^*$.
- If want this for one state μ then $\varphi = \lim \frac{1}{n}(\mu + \mu^2 + \cdots + \mu^n)$.

See van Daele, PAMS 1995. □

For $a \in C(G)$:

$$(\mathrm{id} \otimes \varphi)\Delta(a)(t) = \int_G a(ts) \ d\varphi(s), \quad \langle \varphi, a \rangle 1(t) = \int_G a(s) \ d\varphi(s).$$

Regular representation

Let $\mathbb{G}$ be the "object" which is our compact quantum group.

- Let $L^2(\mathbb{G})$ be the GNS space for the Haar state φ. Let π_φ, ξ_φ be the representation and the cyclic vector.

Let $\pi : A \to \mathcal{B}(K)$ be some auxiliary non-degenerate $*$-representation.

Theorem

There is a unitary $U \in \mathcal{B}(K \otimes L^2(\mathbb{G}))$ with

$$U^*(\xi \otimes \pi_\varphi(a)\xi_\varphi) = (\pi \otimes \pi_\varphi)\big(\Delta(a)\big)(\xi \otimes \xi_\varphi).$$

(All this theory is due to Woronowicz; some presentation motivated by Maes, van Daele, Timmermann.)

Position, implementation, representations

- We have that U is a multiplier of $\pi(A) \otimes \mathcal{B}_0(L^2(\mathbb{G}))$.
- $\mathcal{B}_0(L^2(\mathbb{G}))$ is the compact operators on $L^2(\mathbb{G})$.
- Also $(\pi \otimes \pi_\varphi)\Delta(a) = U^*(1 \otimes \pi_\varphi(a))U$.

A SOT continuous unitary representation π of a compact group G gives a map

$$G \to \mathcal{B}(H) = M(\mathcal{B}_0(H)); \quad s \mapsto \pi(s).$$

This is continuous for the strict topology; given $f \in C_0(G, \mathcal{B}_0(H))$ the map

$$G \to \mathcal{B}_0(H); \quad s \mapsto \pi(s)f(s)$$

is continuous. So

$$(\pi(s))_{s\in G} \in M\big(C_0(G) \otimes \mathcal{B}_0(H)\big).$$

Given $V \in M\big(C_0(G) \otimes \mathcal{B}_0(H)\big)$ how do we recognise that it's a representation?

Representations continued

$$C^b_{str}(G, \mathcal{B}_0(H)) \cong M\big(C_0(G) \otimes \mathcal{B}_0(H)\big)$$

$$(\pi(s)) \leftrightarrow V \qquad \big(s \mapsto f(s)\pi(s)\xi\big) \leftrightarrow V(f \otimes \xi) \quad (f \in C_0(G), \xi \in H).$$

- $\pi(s)$ unitary for all s corresponds to V being a unitary operator.
- a *representation* means:

$$(\Delta \otimes \mathrm{id})V \leftrightarrow (\pi(st))_{(s,t)\in G\times G} = (\pi(s)\pi(t))_{(s,t)\in G\times G} \leftrightarrow V_{13}V_{23}.$$

- This is "leg-numbering notation": $V_{23} = 1 \otimes V$ acts on the 2nd/3rd components; $V_{13} = \sigma_{12}V_{23}\sigma_{12}$.

Definition

A *corepresentation* of (A, Δ) is $V \in M(A \otimes \mathcal{B}_0(H))$ with $(\Delta \otimes \mathrm{id})(V) = V_{13}V_{23}$.

Left regular representation

Theorem

If $\pi : A \to \mathcal{B}(K)$ is faithful, then $U \in M(\pi(A) \otimes \mathcal{B}_0(L^2(\mathbb{G})))$ is a corepresentation.

- π faithful, so $M(\pi(A) \otimes \mathcal{B}_0(L^2(\mathbb{G}))) \cong M(A \otimes \mathcal{B}_0(L^2(\mathbb{G})))$.

Theorem

For $a, b \in A$ set $\xi = \pi_\varphi(a)\xi_\varphi, \eta = \pi_\varphi(b)\xi_\varphi$. Then

$$(\mathrm{id} \otimes \omega_{\xi,\eta})(U) = (\mathrm{id} \otimes \varphi)(\Delta(b^*)(1 \otimes a))$$
$$(\mathrm{id} \otimes \omega_{\xi,\eta})(U^*) = (\mathrm{id} \otimes \varphi)((1 \otimes b^*)\Delta(a))$$

(Here I supress the π).

- By cancellation, such slices are hence dense in A.

Finite dimensional corepresentations

- If H finite dimensional then pick a basis, $H \cong \mathbb{C}^n$.
- $\mathcal{B}_0(H) \cong \mathbb{M}_n$ and $M(A \otimes \mathcal{B}_0(H)) = A \otimes \mathcal{B}_0(H) \cong \mathbb{M}_n(A)$.
- A unitary $V = (V_{ij})$ is a corepresentation if and only if

$$\Delta(V_{ij}) = \sum_{k=1}^{n} V_{ik} \otimes V_{kj}.$$

- A subspace $K \subseteq H$ is *invariant* for V if

$$V(1 \otimes p) = (1 \otimes p)V(1 \otimes p)$$

  for $p : H \to K$ the orthogonal projection.
- Given $V \in M(A \otimes \mathcal{B}_0(H_V))$ and $W \in M(A \otimes \mathcal{B}_0(H_W))$ an operator $T : H_V \to H_W$ is an *intertwiner* if $W(1 \otimes T) = (1 \otimes T)V$.
- Hence have notions of being *irreducible*, a *subcorepresentation*, *(unitary) equivalence* and so forth.

Schur's lemma

Theorem (Schur's Lemma)

Let x intertwine corepresentations W, V. The kernel, and the closure of the image, of x are invariant subspaces of W, respectively, V. If

- *W and V are irreducible; or*
- *W and V are finite-dimensional of the same dimension and one is irreducible,*

then $x = 0$ if W, V are not equivalent; if $x \neq 0$ then x is invertible. Then span of such invertibles is one-dimensional.

Averaging with the Haar state

Theorem

Let W, V be corepresentations, and let $x \in \mathcal{B}(H_W, H_V)$. Then

$$y = (\varphi \otimes \mathrm{id})(V^*(1 \otimes x)W) \in \mathcal{B}(H_W, H_V)$$

satisfies $V^*(1 \otimes y)W = 1 \otimes y$. If x compact, so is y.

Proof.

Using $(\varphi \otimes \mathrm{id})\Delta(\cdot) = \varphi(\cdot)1$,

$$(\varphi \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(V^*(1 \otimes x)W) = 1 \otimes (\varphi \otimes \mathrm{id})(V^*(1 \otimes x)W) = 1 \otimes y$$
$$(\Delta \otimes \mathrm{id})(V^*(1 \otimes x)W) = V_{23}^* V_{13}^*(1 \otimes 1 \otimes x)W_{13}W_{23}$$
$$(\varphi \otimes \mathrm{id} \otimes \mathrm{id})\big(V_{23}^* V_{13}^*(1 \otimes 1 \otimes x)W_{13}W_{23}\big) = V^*(1 \otimes y)W.$$

If V is unitary then $(1 \otimes y)W = V(1 \otimes y)$ so we have an intertwiner. □

Applications 1

Theorem

An irreducible unitary corepresentation is finite-dimensional.

Proof.

Let V be the corepresentation.

- Pick a compact $x \in \mathcal{B}_0(H_V)$ and average to a compact intertwiner

$$y = (\varphi \otimes \mathrm{id})(V^*(1 \otimes x)V) \in \mathcal{B}(H_V)$$

- By Schur, $y = 0$ or $y \in \mathbb{C}1$.
- y is compact, so if $y = t1$ for $t \neq 0$ we're done.
- Let x vary through a net of finite-dimensional orthogonal projections to see that y must be non-zero for some choice.

□

Applications 2

Theorem

Any unitary corepresentation V decomposes as the direct sum of irreducibles.

Sketch proof.

- If V is unitary then if K is an invariant subspace for V so is $K^\perp$.
- So the collection of intertwiners from V to itself is a C^*-algebra B say.
- The previous averaging argument shows that we can find a bounded approximate identity in B consisting of *compact* operators.
- So B is the direct sum of matrix algebras.
- So V decomposes as finite-dimensional corepresentations.
- Can obviously decompose finite-dimensional corepresentations into irreducibles.

□

Applications 3

Theorem

Let V be an irreducible unitary corepresentation of (A, Δ). Then V is equivalent to a subrepresentation of U.

Proof.

- Pick any $x \in \mathcal{B}(L^2(\mathbb{G}), H_V)$ and average to an intertwiner

$$y = (\varphi \otimes \mathrm{id})(V^*(1 \otimes x)U).$$

- If y is non-zero, use Schur to conclude y is onto.
- As V, U are unitary, it follows that y^* is also an intertwiner, injective by Schur, so gives required equivalence.

□

Continued proof

$$y = (\varphi \otimes \mathrm{id})(V^*(1 \otimes x)U).$$

- Maybe $y = 0$ for all x, so test on rank-one maps $x = \theta_{\xi, a\xi_\varphi}$, giving

$$\begin{aligned} 0 = (yb\xi_\varphi | \eta) &= \langle \varphi \otimes \omega_{b\xi_\varphi, \eta}, V^*(1 \otimes \theta_{\xi, a\xi_\varphi})U \rangle \\ &= \varphi\big((\mathrm{id} \otimes \omega_{\xi,\eta})(V^*)(\mathrm{id} \otimes \omega_{b\xi_\varphi, a\xi_\varphi})(U)\big) \\ &= \varphi\big((\mathrm{id} \otimes \omega_{\xi,\eta})(V^*)(\mathrm{id} \otimes \varphi)(\Delta(a^*)(1 \otimes b))\big) \end{aligned}$$

- Think of $V = (V_{ij}) \in \mathbb{M}_n(A)$.
- By cancellation, and taking ξ, η to be basis vectors, conclude that $0 = \varphi(V_{ij}^* a)$ for all $a \in A$.
- But V is unitary, so taking $a = V_{ij}$ gives

$$0 = \sum_i \varphi(V_{ij}^* V_{ij}) = \varphi(1) = 1.$$

Algebra of "matrix elements"

Definition

Let $A_0 \subseteq A$ be the linear span of matrix elements V_{ij} arising from all finite-dimensional (irreducible) unitary corepresentations $V = (V_{ij})$.

- U decomposes as a direct sum of (all the) irreducible (finite-dimensional) corepresentations.
- So also $L^2(\mathbb{G})$ decomposes as (finite-dimensional) invariant subspaces.
- Given $\xi, \eta \in L^2(\mathbb{G})$, approximate by vectors with "finite-support".
- So can approximate $(\mathrm{id} \otimes \omega_{\xi,\eta})(U)$ by linear combination of matrix elements.
- So A_0 dense in A.
- A_0 is an algebra: tensor product of corepresentations ($V \textcircled{\scriptsize T} W = V_{12} W_{13}$).
- Is A_0 a $*$-algebra?