Around the Approximation Property for (Quantum) Groups

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Glasgow, April 2023

Fejér's Theorem

Let's recall this from classical Fourier Analysis. Identify $\mathbb{T} = [-\pi, \pi)$ which has Haar measure $\frac{ds}{2\pi}$. For a "nice" function f on $\mathbb{T}$ define

$$c_k = \int_{-\pi}^{\pi} f(s) e^{-iks} \frac{ds}{2\pi}, \qquad s_n(f, x) = \sum_{k=-n}^{n} c_k e^{ikx}.$$

Theorem

For $f \in C(\mathbb{T})$, the Cesàro sums

$$\sigma_n(f, x) = \frac{1}{n} \sum_{k=0}^{n-1} s_k(f, x)$$

converge uniformly to $f(x)$.

Think about this in a "quantum" framework

For me, the Fourier transform is between Hilbert spaces:

$$\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z}); \quad f \mapsto (c_k) = \Big(\int_{-\pi}^{\pi} f(s) e^{-iks} \, \frac{ds}{2\pi} \Big).$$

We end up with a *unitary* $\mathcal{F}$.

- Let $C(\mathbb{T})$ be the algebra of continuous functions on $\mathbb{T}$.
- Then $C(\mathbb{T})$ naturally acts on $L^2(\mathbb{T})$ by multiplication, and so becomes a *concrete* C^*-algebra.

Consider $\ell^2(\mathbb{Z})$ with canonical orthonormal basis $(\delta_k)_{k\in\mathbb{Z}}$. For each $n \in \mathbb{Z}$ let λ_n be the *translation operator* given by

$$\lambda_n : \delta_k \mapsto \delta_{k+n}.$$

- $\lambda_n \circ \lambda_m = \lambda_{n+m}$ and $\lambda_n^* = \lambda_{-n}$, so $\mathbb{Z} \ni n \mapsto \lambda_n$ is a unitary group representation.
- Denote by $C_r^*(\mathbb{Z})$ the closed linear span of the λ_n. This is the *(reduced) group C^*-algebra*.

Think about this in a "quantum" framework (cont.)

Have $C(\mathbb{T})$ acting on $L^2(\mathbb{T})$, and $C^*_r(\mathbb{Z})$ acting on $\ell^2(\mathbb{Z})$.
Have $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$.

We then obtain

$$\mathcal{F}_0 : C(\mathbb{T}) \to C^*_r(\mathbb{Z}); \quad f \mapsto \mathcal{F} f \mathcal{F}^{-1}.$$

Why does this make sense?

- Calculation shows that $\mathcal{F}_0^{-1} : \lambda_n \mapsto (e^{ins})_{s\in\mathbb{T}}$;
- As $\mathcal{F}_0$ is conjugation by unitaries, it is an isomorphism between C^*-algebras.
- So by density and continuity it must give an isomorphism between $C(\mathbb{T})$ and $C^*_r(\mathbb{Z})$.

Continued 1: Normal functionals

For a C^*-algebra $A \subseteq \mathcal{B}(H)$, given $\xi, \eta \in H$, let $\omega_{\xi,\eta} \in A^*$ be the (normal) functional

$$A \ni a \mapsto (a\xi|\eta) \in \mathbb{C}.$$

We can think of $L^1(\mathbb{T})$ as those functionals on $C(\mathbb{T})$ of this form. Indeed, given $\xi, \eta \in L^2(\mathbb{T})$ and $f \in C(\mathbb{T})$,

$$\langle f, \omega_{\xi,\eta} \rangle = (f\xi|\eta) = \int_{-\pi}^{\pi} f(s)\xi(s)\overline{\eta(s)}\ \frac{ds}{2\pi} = \langle f, \xi\overline{\eta} \rangle.$$

So $\omega_{\xi,\eta}$ agrees with $\xi\overline{\eta} \in L^1(\mathbb{T})$ on $C(\mathbb{T})$.

Similarly, define the *Fourier Algebra* $A(\mathbb{Z})$ to be the collection of such normal functionals on $C_r^*(\mathbb{Z})$. (That this is a *closed subspace* is true, but not obvious).

Continued 2: Function Spaces

Given $\omega = \omega_{\xi,\eta} \in A(\mathbb{Z})$, we can identify this with a function on $\mathbb{Z}$ by

$$\omega \leftrightarrow (\omega(n))_{n\in\mathbb{Z}} = (\langle \lambda_{-n}, \omega \rangle)_{n\in\mathbb{Z}}.$$

As $C_r^*(\mathbb{Z})$ is the span of $\{\lambda_n : n \in \mathbb{Z}\}$, the values $\{\omega(n) : n \in \mathbb{Z}\}$ determines ω. Use of "$-n$" seems odd, but makes things work (and occurs in the general quantum theory).

Recall $\mathcal{F}_0 : C(\mathbb{T}) \to C_r^*(\mathbb{Z})$. The Banach space adjoint is $\mathcal{F}_0^* : C_r^*(\mathbb{Z})^* \to C(\mathbb{T})^*$. Restricting this to $A(\mathbb{Z})$ gives

$$\mathcal{F}_1 = \mathcal{F}_0^* : A(\mathbb{Z}) \to L^1(\mathbb{T}); \omega_{\xi,\eta} \mapsto \omega_{\mathcal{F}^*(\xi),\mathcal{F}^*(\eta)}.$$

This is a bijection, and the inverse $L^1(\mathbb{T}) \to A(\mathbb{Z})$ is just the usual Fourier transform (thought of as acting between function spaces).

Continued 3: Algebras

$L^1(\mathbb{T})$ is an algebra under convolution, and $A(\mathbb{Z})$ is an algebra of functions with the pointwise product.

- $\mathcal{F}_1 : A(\mathbb{Z}) \to L^1(\mathbb{T})$ is a homomorphism.
- $\mathcal{F}_0 : C(\mathbb{T}) \to C_r^*(\mathbb{Z})$ is a homomorphism.

Given any (Banach) algebra A, the dual space becomes an A-bimodule. $A(\mathbb{Z})$ acts on its dual space, and this restricts to turn $C_r^*(\mathbb{Z})$ into an $A(\mathbb{Z})$-module. Similarly for $L^1(\mathbb{T})$ acting on $C(\mathbb{T})$.

$$\omega \cdot \lambda_n = \omega(-n)\lambda_n, \qquad f \cdot F = F \star \check{f} \qquad \begin{pmatrix} F \in C(\mathbb{T}), f \in L^1(\mathbb{T}) \\ \lambda_n \in C_r^*(\mathbb{Z}), \omega \in A(\mathbb{Z}) \end{pmatrix}.$$

Here $\check{f}(s) = f(-s)$.

- $\mathcal{F}_0$ is a module homomorphism.

Back to Fejér

For $F \in C(\mathbb{T})$ we have

$$\sigma_n(F, \cdot) = F \star F_n = \check{F}_n \cdot F,$$

where $F_n \in L^1(\mathbb{T})$ is the Fejér kernel; we have $\check{F}_n = F_n$.
Push this through $\mathcal{F}_0$ to obtain $\omega_n = \mathcal{F}_1(\check{F}_n)$ with

$$\begin{aligned}\omega_n \cdot a = \mathcal{F}_1(\check{F}_n) \cdot \mathcal{F}_0(\mathcal{F}_0^{-1}(a)) &= \mathcal{F}_0(\check{F}_n \cdot \mathcal{F}_0^{-1}(a)) \\ \xrightarrow{n\to\infty} \mathcal{F}_0(\mathcal{F}_0^{-1}(a)) &= a \qquad (a \in C_r^*(\mathbb{Z})).\end{aligned}$$

Indeed, ω_n, as a function on $\mathbb{Z}$, is simply the "triangle", piecewise linear with $\omega_n(0) = 1$ and $\omega_n(n) = \omega_n(-n) = 0$.

> *We obtain a sequence of (normalised, positive definite) functions in $A(\mathbb{Z})$ which acts on $C_r^*(\mathbb{Z})$ as an "approximate identity".*

Generalise?

Amenability

Let G be a discrete (locally compact) group. Form $\ell^2(G)$, and form the translation operators $\{\lambda_g : g \in G\}$, given by

$$\lambda_g : \delta_s \mapsto \delta_{gs} \qquad (g, s \in G).$$

The norm closed linear span is $C_r^*(G)$, and the bicommutant is $VN(G)$ the *group von Neumann algebra*.

The predual of $VN(G)$ is $A(G)$, the *Fourier Algebra*, considered as an algebra of functions in the same way, $\omega \leftrightarrow (\omega_g)_{g\in G} = (\langle \lambda_{g^{-1}}, \omega \rangle)_{g\in G}$.

Turn $C_r^*(G)$ and $VN(G)$ into $A(G)$-modules for the dual action.

Amenability (cont.)

Theorem

The following are equivalent:

- *$A(G)$ contains a net of normalised positive definite functions (i.e. normal states on $VN(G)$) which form an approximate identity for $C_r^*(G)$, or a weak*-approximate identity for $VN(G)$;*
- *$A(G)$ contains some bounded approximate identity (bai);*
- *G is amenable.*

If you think of G being amenable as the existance of a Følner net (F_i) of subsets of G, then $\xi_i = \chi_{F_i} / \sqrt{|F_i|}$ is a net of unit vectors in $\ell^2(G)$, and (ω_{ξ_i,ξ_i}) is a net of normalised, positive definite functions in $A(G)$ forming a bai.

A-T-menability or the Haagerup property

Question

Can we expand the space of functions away from $A(G)$ to obtain a larger class of groups than those which are amenable?

Instead of using the predual of $VN(G)$, could we use the dual of $C_r^*(G)$? No: if this has a bai then it has a unit, and G is amenable. Could we use the dual of the full group C^*-alegbra $C^*(G)$? No: this is always unital. But all functions in $A(G)$ vanish at infinity.

Definition

G has the *Haagerup Property* if there is a net of normalised positive-definite functions which vanish at infinity, and converge to 1 uniformly on compacta.

E.g. Groups acting properly on (locally finite) trees; free products of amenable groups.

Completely bounded multipliers

A key property of $A(G)$ functions is that they "multiply" (or act on) $C_r^*(G)$ and $VN(G)$.

Definition

A *multiplier* of $A(G)$ is a function f on G such that $f\omega \in A(G)$ for each $\omega \in A(G)$.

Such an f is automatically continuous. By the Closed Graph Theorem, the resulting map $A(G) \to A(G); \omega \mapsto f\omega$ is continuous.
Such an f acts on $VN(G)$ and, by restriction, on $C_r^*(G)$.

Definition

A multiplier f is *completely bounded* if the resulting map on $VN(G)$, say M_f, (equivalently $C_r^*(G)$) is completely bounded.

$$M_f \otimes \mathrm{id} : VN(G) \otimes \mathbb{M}_n \to VN(G) \otimes \mathbb{M}_n.$$

Weak amenability

Of course, each $\omega \in A(G)$ is itself a (cb-)multiplier.

Theorem (Losert)

The following are equivalent:

- *G is amenable*
- *the map from $A(G)$ into the algebra of multipliers of $A(G)$ is bounded below;*
- *the map from $A(G)$ into the algebra of cb-multipliers of $A(G)$ is bounded below.*

Definition

G is *weakly amenable* if there is a net (ω_i) in $A(G)$, bounded in the $\|\cdot\|_{cb}$ norm, forming an approximate identity for $C_r^*(G)$.

E.g. (Haagerup) $\mathbb{F}_2$.

The approximation property

The space of cb-multipliers, $M_{cb}A(G)$, is a dual space (and a dual Banach algebra).

- Each $f \in L^1(G)$ defines a bounded functional on $M_{cb}A(G)$ (by integration of functions).
- The closure of such functionals in $M_{cb}A(G)^*$, say $Q_{cb}A(G)$, is a predual for $M_{cb}A(G)$.

Definition

G has the *approximation property (AP)* when there is a net (ω_i) in $A(G)$ which converges to 1 weak* in $M_{cb}A(G)$.

If such a net is bounded in $M_{cb}A(G)$ then G is already weakly amenable.

Examples

The class of groups with the AP is closed under extensions, while the class of weakly amenable groups is not (not even closed under semi-direct products).

- Let $\Lambda_{cb}(G)$ be the infimum of $M > 0$ such that $A(G)$ contains a net (ω_i) converging to 1 on compacta, with $\|\omega_i\|_{cb} \leq M$.
- So G is weakly amenable exactly when $\Lambda_{cb}(G) < \infty$.
- [Cowling–Haagerup] $\Lambda_{cb}(G_1 \times G_2) = \Lambda_{cb}(G_1)\Lambda_{cb}(G_2)$.
- Wreath products give examples with $\Lambda_{cb}(G) > 1$.
- Taking an infinite product gives a non-weakly amenable group which has the AP.
- In fact, much is known now about Lie groups and lattices therein.
- [Lafforgue–de la Salle] $SL_3(\mathbb{Z})$ does not have the AP.

Applications: finite-rank approximations

For those familiar with the notion of *nuclearity* the following should look slightly familiar.

Definition

A C^*-algebra A has the *operator approximation property* (OAP) if there is a net of continuous finite-rank operators (φ_i) which converges to 1_A in the point-stable topology: $(\varphi_i \otimes \mathrm{id})(u) \to u$ in norm, for each $u \in A \otimes \mathcal{K}(\ell^2)$.

Theorem (Haagerup–Kraus)

For a discrete group G the following are equivalent:

- *G has the AP;*
- *$C_r^*(G)$ has the OAP.*

Similar definitions/results hold for von Neumann algberas, and $VN(G)$.

L^p variants

We can replace $L^2(G)$ by $L^p(G)$ when definining the Fourier algebra and $VN(G)$. The operators $(\lambda_s)_{s\in G}$ act on $L^p(G)$ (by left-invariance of the Haar measure). The weak*-linear span in $\mathcal{B}(L^p(G))$ is $PM_p(G)$, the algebra of *p-pseudo measures*. Its predual is $A_p(G)$ the *Figa-Talamanca–Herz algebra*.

We can also look at right-translation variants, leading to $PM_p^r(G)$. Let the commutant of this be $CV_p(G)$, the algebra of *p-convolvers*. We always have that $CV_p(G) \supseteq PM_p(G)$.

Question

Is it true that $CV_p(G) = PM_p(G)$?

L^p variants, continued

Question

Is it true that $CV_p(G) = PM_p(G)$?

Yes, if $p = 2$.

Theorem (Cowling; see D.–Spronk)

If G has the AP then $CV_p(G) = PM_p(G)$

The idea of the proof is that the net (ω_i) in $A(G)$ approximating the identity can be made to act on $CV_p(G)$ in a way which weak*-approximates the identity and which maps $CV_p(G)$ into $PM_p(G)$.

Locally compact quantum groups

We introduce these objects by way of two examples.

For a (locally compact) group G consider $L^\infty(G)$. We identify the von Neumann algebra tensor product $L^\infty(G)\bar{\otimes}L^\infty(G)$ with $L^\infty(G\times G)$. We can then "dualise" the group product to define a normal injective $*$-homomorphism by, for $F\in L^\infty(G), g,h\in G$,

$$\Delta : L^\infty(G)\to L^\infty(G\times G); \qquad \Delta(F)(g,h) = F(gh).$$

Product associative $\implies$ Δ is *coassociative*: $(\Delta\otimes\mathrm{id})\Delta = (\mathrm{id}\otimes\Delta)\Delta$.

Let $\varphi : L^\infty(G)^+\to[0,\infty]$ be the left "Haar weight"

$$\varphi(F) = \int_G F(g)\ dg.$$

Then, for $f\in L^1(G)^+$ and $F\in L^\infty(G)^+$ we have

$$\varphi\big((f\otimes\mathrm{id})\Delta(F)\big) = \int_G dh \int_G dg\ f(g)F(gh) = \int_G\int_G f(g)F(gh)\ dh\ dg$$
$$= \int_G\int_G f(g)F(h)\ dh\ dg = \varphi(F)\langle 1,f\rangle.$$

Co-commutative case

Alternatively, form $VN(G)$, which is generated by the translation operators λ_g. There exists a normal injective $*$-homomorphism

$$\widehat{\Delta} : VN(G) \to VN(G) \bar{\otimes} VN(G) \cong VN(G \times G); \quad \lambda_g \mapsto \lambda_g \otimes \lambda_g.$$

If $\sigma : VN(G) \bar{\otimes} VN(G) \to VN(G) \bar{\otimes} VN(G)$ is the tensor swap map, then $\widehat{\Delta} = \sigma \circ \widehat{\Delta}$: this is the *co-commutative* condition.
Similarly, "one can show" that there is a weight $\widehat{\varphi} : VN(G)^+ \to [0, \infty]$ with

$$\widehat{\varphi}\big((\omega \otimes \mathrm{id})\widehat{\Delta}(x)\big) = \widehat{\varphi}(x)\omega(1) \qquad (x \in VN(G)^+, \omega \in A(G)^+).$$

Indeed, $\widehat{\varphi}(\lambda(f)) = f(e)$ for suitably nice $f \in L^1(G)$.

Locally compact quantum groups

Abstract object $\mathbb{G}$ with:

- von Neumann algebra $L^\infty(\mathbb{G})$;
- equipped with a coproduct $\Delta : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\bar{\otimes}L^\infty(\mathbb{G})$ which is coassociative: $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$;
- which has weights φ, ψ which are left/right invariant, e.g.

$$\varphi\big((\omega \otimes \mathrm{id})\Delta(x)\big) = \varphi(x)\omega(1) \qquad (x \in \mathcal{M}_\varphi^+, \omega \in L^1(\mathbb{G})^+).$$

From this, one gets:

- $L^1(\mathbb{G})$ becomes a Banach algebra, product induced by Δ;
- GNS for φ gives $L^2(\mathbb{G})$ with $L^\infty(\mathbb{G})$ in standard position;
- a multiplicative unitary W, so $W_{12}W_{13}W_{23} = W_{23}W_{12}$;

Multiplicative unitaries

Let's think more about this W. It is a unitary W on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ which "encodes" Δ and $L^\infty(\mathbb{G})$.

- We use *leg numbering* notation: on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$ we let $W_{12} = W \otimes 1$, so W acting on "legs 1 and 2";
- W_{13} is analogously W acting on legs 1 and 3.

E.g. for $L^\infty(G)$ for a group G, we find that W is the unitary on $L^2(G \times G)$ given by

$$(W\xi)(g,h) = \xi(g, g^{-1}h) \qquad (\xi \in L^2(G \times G), g, h \in G).$$

In general, W gives us Δ by

$$\Delta(x) = W^*(1 \otimes x)W \qquad (x \in L^\infty(\mathbb{G})).$$

W remembers $L^\infty(\mathbb{G})$ as

$$L^\infty(\mathbb{G}) = \{(\mathrm{id} \otimes \omega)(W) : \omega \in L^1(\mathbb{G})\}''.$$

Duality

$$\lambda : L^1(\mathbb{G}) \to \mathcal{B}(L^2(\mathbb{G})); \quad \omega \mapsto (\omega \otimes \mathrm{id})(W)$$

is a homomorphism. The closure of its image is a C^*-algebra $C_0(\widehat{\mathbb{G}})$.

- There indeed exists $\widehat{\mathbb{G}}$ a LCQG; $L^\infty(\widehat{\mathbb{G}})$ is the WOT closure.
- There is $\widehat{\varphi}$ so that $L^2(\widehat{\mathbb{G}}) = L^2(\mathbb{G})$ canonically.
- $W \in L^\infty(\mathbb{G}) \bar{\otimes} L^\infty(\widehat{\mathbb{G}})$ and $\widehat{W} = \sigma(W^*)$ where σ is the swap map.

For G a locally compact group, if we set $L^\infty(\mathbb{G}) = L^\infty(G)$, then we indeed find that $L^\infty(\widehat{\mathbb{G}}) = VN(G)$ and $C_0(\widehat{\mathbb{G}}) = C_r^*(G)$, with $\widehat{\Delta}$ as before.

Indeed, that $\widehat{\Delta}$ exists (we only defined it on λ_g) follows from using the formula

$$\widehat{\Delta}(x) = \widehat{W}^*(1 \otimes x)\widehat{W} \quad \text{where} \quad \widehat{W} = \sigma(W^*).$$

Duality continued: Fourier algebra

Again with $\mathbb{G} = G$ a genuine group, the map $\lambda : L^1(G) \to C_r^*(G) \subseteq \mathcal{B}(L^2(G))$ is the usual left-regular representation. We also have

$$\widehat{\lambda} : L^1(\widehat{\mathbb{G}}) = A(G) \to C_r^*(\widehat{\mathbb{G}}) = C_0(\mathbb{G}) = C_0(G)$$

which agrees with our map before. This "explains" our use of g^{-1}.

For general $\mathbb{G}$... We define $A(\mathbb{G}) = \widehat{\lambda}(L^1(\widehat{\mathbb{G}}))$ with the norm from $L^1(\widehat{\mathbb{G}})$, but thought of as a subalgebra of $C_0(\mathbb{G})$.

[Stop?]

Centralisers and Multipliers

We can think of a multiplier of $A(G)$ as a map $T : A(G) \to A(G)$ with $T(\omega_1\omega_2) = T(\omega_1)\omega_2$, that is, a module homomorphism.

Definition

A *left centraliser* of $L^1(\widehat{\mathbb{G}})$ is a right module homomorphism, $L(\widehat{\omega}_1 \star \widehat{\omega}_2) = L(\widehat{\omega}_1) \star \widehat{\omega}_2$.

Definition

A *left multiplier* of $A(\mathbb{G})$ is $a \in L^\infty(\mathbb{G})$ with $a\widehat{\lambda}(\widehat{\omega}) \in \widehat{\lambda}(L^1(\widehat{\mathbb{G}})) = A(\mathbb{G})$ for each $\widehat{\omega} \in L^1(\widehat{\mathbb{G}})$.

As $\widehat{\lambda}$ is injective, a left multiplier a induces a (unique) left centraliser L with $a\widehat{\lambda}(\widehat{\omega}) = \widehat{\lambda}(L(\widehat{\omega}))$.
We say that L (and thus a) is *completely bounded* if the adjoint $L^* : L^\infty(\widehat{\mathbb{G}}) \to L^\infty(\widehat{\mathbb{G}})$ is completely bounded.

Centralisers are multipliers

Theorem (Junge–Neufang–Ruan; D.)

For any cb left centraliser L there exists $a \in M(C_0(\mathbb{G})) \subseteq L^\infty(\mathbb{G})$ an associated multiplier.

We write $M_{cb}(A(\mathbb{G}))$ for the collection of all multipliers, equipped with the norm (operator space structure) arising as centralisers, that is, maps on $L^1(\widehat{\mathbb{G}})$.

Following the classical situation, $M_{cb}(A(\mathbb{G}))$ is a dual space: let $Q_{cb}(A(\mathbb{G}))$ be the closure of the image of $L^1(\mathbb{G})$ in $M_{cb}(A(\mathbb{G}))^*$ where $\omega \in L^1(\mathbb{G})$ is paired against $a \in M_{cb}(A(\mathbb{G})) \subseteq L^\infty(\mathbb{G}) = L^1(\mathbb{G})^*$ in the canonical way.

Definition (D.-Krajczok–Voigt)

$\mathbb{G}$ has the AP if there is a net in $A(\mathbb{G})$ which converges to 1 weak* in $M_{cb}(A(\mathbb{G}))$.

(We used "left"; there is a "right" analogue; this gives the same notion.)

Other notions of convergence

Each $a \in M_{cb}(A(\mathbb{G}))$ is associated to a centraliser $L : L^1(\widehat{\mathbb{G}}) \to L^1(\widehat{\mathbb{G}})$ and hence to a map $L^* = \Theta(a) \in \mathcal{CB}(L^\infty(\widehat{\mathbb{G}}))$.

Definition (Crann; Kraus–Ruan)

$\mathbb{G}$ has the (strong) AP when there is a net (a_i) in $A(\mathbb{G})$ such $(\Theta(a_i) \otimes \mathrm{id})(x) \to x$ weak* for each $x \in L^\infty(\widehat{\mathbb{G}}) \bar{\otimes} \mathcal{B}(\ell^2)$ (that is, *stable point-weak** convergence to id).

Proposition (DKV)

AP and strong AP are equivalent.

Proof.

Only (AP) $\implies$ (strong AP) needs a proof. Follows from a careful study of $Q_{cb}(A(\mathbb{G}))$ and adapting some classical work of Kraus–Haagerup: as sometimes happens you end up proving a little bit more in the abstract setting of LCQGs. □

Discrete case

Proposition (Kraus–Ruan)

For discrete $\mathbb{G}$, consider the following:

1. $\mathbb{G}$ *has AP;*
2. $C(\widehat{\mathbb{G}})$ *has the OAP;*
3. $L^\infty(\widehat{\mathbb{G}})$ *has the* w^*OAP

Then (1)⇒(2) and (1)⇒(3) and when $\mathbb{G}$ *is unimodular, all are equivalent.*

$\mathbb{G}$ is *unimodular* when the left and right Haar weights coincide.

Relative w^*OAP

Let M be a general von Neumann algebra. Let M have the w^*OAP: $\varphi_i \to \mathrm{id}$ stable point-w^*.

Let φ be a weight on M with GNS space $L^2(\varphi)$, definition ideal

$$\mathfrak{n}_\varphi = \{x \in M : \varphi(x^*x) < \infty\},$$

and GNS map $\Lambda : \mathfrak{n}_\varphi \to L^2(\varphi)$. Define that φ_i has an L^2-*implementation* when $\varphi_i(\mathfrak{n}_\varphi) \subseteq \mathfrak{n}_\varphi$, and there is $T_i \in \mathcal{B}(L^2(\varphi))$ with $T_i\Lambda(x) = \Lambda(\varphi_i(x))$ for $x \in \mathfrak{n}_\varphi$.

Definition

Let $N \subseteq \mathcal{B}(L^2(\varphi))$ be a von Neumann algebra. M has the w^*OAP *relative to* N when each $T_i \in N$.

Relative w^*OAP and AP

Theorem (DKV)

For a discrete quantum group $\mathbb{G}$ the following are equivalent:

1. $\mathbb{G}$ *has* AP;
2. $L^\infty(\widehat{\mathbb{G}})$ *has* w^*OAP *relative to* $\ell^\infty(\mathbb{G})$;
3. $L^\infty(\widehat{\mathbb{G}})$ *has* w^*OAP *relative to* $\ell^\infty(\mathbb{G})'$;

Permanence properties

Theorem (DKV)

Let $\mathbb{G}$ have the AP, and let $\mathbb{H}$ be a closed quantum subgroup of $\mathbb{G}$. Then $\mathbb{H}$ has the AP.

Proof.

Almost by definition, $\mathbb{H} \leq \mathbb{G}$ means that there is a quotient map $A(\mathbb{G}) \to A(\mathbb{H})$ (classically this is the Herz Restriction Theorem). $\square$

Free products

Theorem (DKV)

Let $\mathbb{G}_1, \mathbb{G}_2$ be discrete quantum groups with the AP. Then $\mathbb{G}_1 \star \mathbb{G}_2$ has the AP.

Is there a reference in the classical case?

Proof.

With $\mathbb{G} = \mathbb{G}_1 \star \mathbb{G}_2$, by definition, $C(\widehat{\mathbb{G}}) = C(\widehat{\mathbb{G}}_1) \star C(\widehat{\mathbb{G}}_2)$. We use operator algebraic methods to deal with this C^*-algebraic free product, especially results of [Ricard–Xu]. Then check that their ideas arise (or can be made to arise) from operations on cb-multipliers which are weak*-continuous. $\square$

Double crossed product

Let $\mathbb{G}_1, \mathbb{G}_2$ be locally compact quantum groups. Following [Baaj–Vaes], a *matching* is an injective normal $*$-homomorphism (which is automatically a $*$-isomorphism) $m : L^\infty(\mathbb{G}_1)\bar{\otimes}L^\infty(\mathbb{G}_2) \to L^\infty(\mathbb{G}_1)\bar{\otimes}L^\infty(\mathbb{G}_2)$ with

$$(\Delta_1 \otimes \mathrm{id})m = m_{23}m_{13}(\Delta_1 \otimes \mathrm{id}), \qquad (\mathrm{id} \otimes \Delta_2)m = m_{13}m_{12}(\mathrm{id} \otimes \Delta_2).$$

From this, we can construct the *double crossed product* $\mathbb{G}_m$ with

$$L^\infty(\mathbb{G}_m) = L^\infty(\mathbb{G}_1)\bar{\otimes}L^\infty(\mathbb{G}_2), \quad \Delta_m = (\mathrm{id} \otimes \sigma m \otimes \mathrm{id})(\Delta_1^{\mathrm{op}} \otimes \Delta_2).$$

(Notice that the product is a very special case of this.)

Quantum double: results

Proposition (DKV)

If $\mathbb{G}_m$ has the AP then so do $\mathbb{G}_1$ and $\mathbb{G}_2$.

Proof.

$\mathbb{G}_1^{\mathrm{op}}$ and $\mathbb{G}_2$ are closed quantum subgroups of $\mathbb{G}_m$. □

Theorem (DKV)

If $\widehat{\mathbb{G}}_1$ and $\widehat{\mathbb{G}}_2$ have the AP, then so does $\widehat{\mathbb{G}}_m$.

Proof.

The idea is to translate the approximating nets from $A(\widehat{\mathbb{G}}_1)$ and $A(\widehat{\mathbb{G}}_2)$ to $A(\widehat{\mathbb{G}}_m)$. At a key point, this doesn't seem to quite work, but the issue can be side-stepped by using a construction of [Junge–Neufang–Ruan] to extend a centraliser $\Theta(a) \in \mathcal{CB}(L^\infty(\widehat{\mathbb{G}}))$ to all of $\mathcal{B}(L^2(\mathbb{G}))$. □

Products

Corollary

For locally compact quantum groups $\mathbb{G}_1, \mathbb{G}_2$ *the following are equivalent:*

1. $\mathbb{G}_1, \mathbb{G}_2$ *both have AP;*
2. $\mathbb{G}_1 \times \mathbb{G}_2$ *has AP.*

The end

We would like to know more about when $\mathbb{G}_m$ has (or does not have) the AP.

Further things one could mention:

- Central AP.
- Links with representation categories.

Thanks for your attention!