Noncommutative Graphs

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Channels

A channel sends an input message (element of a finite set A) to an output message (element of a finite set B) perhaps with *noise* so that there is a probability that $a \in A$ is mapped to different $b \in B$.

- Input "o" might be sent to "o" or "0" or "a".

$$p(b|a) = \text{probability that } b \text{ is received given that } a \text{ was sent}$$

Define a (simple, undirected) graph structure on A by

$$(a_1, a_2) \text{ an edge when } p(b|a_1)p(b|a_2) > 0 \text{ for some } b.$$

This is the *confusability graph* of the channel.
If we want to communicate with *zero error* then we seek a maximal *independent set* in A.

Quantum Mechanics

- A *state* is a unit vector $|\psi\rangle$ in a (finite dim) Hilbert space H.
- More generally, a *density* is a positive, trace one operator $\rho \in \mathcal{B}(H)$.
- A rank-one density is always of the form $|\psi\rangle\langle\psi|$ for some state ψ.
- (Use Trace duality, so $\omega \in \mathcal{B}(H)^*$ is associated uniquely to $A \in \mathcal{B}(H)$ with $\omega(T) = \mathrm{tr}(AT)$. Then densities are exactly the *states* on $\mathcal{B}(H)$.)

A *(quantum) channel* is a trace-preserving, completely positive (CPTP) map $\mathcal{B}(H_A) \to \mathcal{B}(H_B)$:

- positive and trace-preserving so it maps densities to densities;
- completely positive so you can tensor with another system and still have positivity.

Stinespring and Kraus

The Stinespring Representation Theorem tells us that any CP map $\mathcal{E} : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ has the form

$$\mathcal{E}(x) = V^*\pi(x)\,V \qquad (x \in \mathcal{B}(H_A)),$$

where $V : H_B \to K$, and $\pi : \mathcal{B}(H_A) \to \mathcal{B}(K)$ is a $*$-representation.

- Any such π is of the form $\pi(x) = x \otimes 1$ where $K \cong H_A \otimes K'$.
- Take an o.n. basis (e_i) for K' so $V(\xi) = \sum_i K_i^*(\xi) \otimes e_i$ for some operators $K_i : H_A \to H_B$.

We arrive at the *Kraus form*:

$$\mathcal{E}(x) = \sum_i K_i x K_i^* \qquad (x \in \mathcal{B}(H_A)).$$

Trace-preserving when $\sum_i K_i^* K_i = 1$.

Quantum zero-error

We turn $\mathcal{B}(H)$ into a Hilbert space using the trace: $(T|S) = \mathrm{tr}(T^*S)$, so densities ρ, σ are *orthogonal* when

$$0 = \mathrm{tr}(\rho\sigma) = \mathrm{tr}(\sigma^{1/2}\rho^{1/2}\rho^{1/2}\sigma^{1/2}) \quad \Leftrightarrow \quad \rho^{1/2}\sigma^{1/2} = 0.$$

Let $\mathcal{E}(x) = \sum_i K_i x K_i^*$ be a quantum channel. We can distinguish densities exactly when $\mathcal{E}(\rho) \perp \mathcal{E}(\sigma)$. As $\mathcal{E}$ is positive, this is equivalent to

$$\mathcal{E}(|\psi\rangle\langle\psi|) \perp \mathcal{E}(|\phi\rangle\langle\phi|) \qquad (\psi \in \mathrm{Im}\,\rho, \phi \in \mathrm{Im}\,\sigma).$$

Thus

$$\begin{aligned} 0 = \mathrm{tr}\left(\mathcal{E}(|\psi\rangle\langle\psi|)\mathcal{E}(|\phi\rangle\langle\phi|)\right) &= \sum_{i,j} \mathrm{tr}\left(K_i|\psi\rangle\langle\psi|K_i^*K_j|\phi\rangle\langle\phi|K_j^*\right) \\ &= \sum_{i,j} |\langle\psi|K_i^*K_j|\phi\rangle|^2 \end{aligned}$$

is equivalent to $\langle\psi|K_i^*K_j|\phi\rangle = 0$ for each i, j.

To operator systems

So ψ, ϕ are distinguishable when

$$\langle \psi | T | \phi \rangle = 0 \quad \text{for each} \quad T \in \text{lin}\{K_i^* K_j\}.$$

Set $\mathcal{S} = \text{lin}\{K_i^* K_j\}$ which has properties:

- $\mathcal{S}$ is a linear subspace;
- $T \in \mathcal{S}$ if and only if $T^* \in \mathcal{S}$;
- $1 \in \mathcal{S}$ (as $\sum_i K_i^* K_i = 1$ as $\mathcal{E}$ is CPTP).

That is, $\mathcal{S}$ is an *operator system*, which depends only on $\mathcal{E}$ and not the choice of (K_i).

Theorem (Duan)

For any operator system $\mathcal{S} \subseteq \mathcal{B}(H_A)$ there is some quantum channel $\mathcal{E} : \mathcal{B}(H_A) \to \mathcal{B}(H_B)$ giving rise to $\mathcal{S}$.

In the classical case

Given a classical channel from A to B with probabilities $p(b|a)$, define Kraus operators

$$K_{ab} = p(b|a)^{1/2} |b\rangle\langle a| : H_A \to H_B.$$

Here $(\langle a|)$ is the canonical basis of $H_A = \ell^2(A) \cong \mathbb{C}^{|A|}$.

$$\sum_{ab} K_{ab} |c\rangle\langle c| K_{ab}^* = \sum_{ab} p(b|a) |b\rangle\langle a|c\rangle\langle c|a\rangle\langle b| = \sum_{b} p(b|c) |b\rangle\langle b|.$$

So the pure state $|c\rangle\langle c|$ is mapped to the combination of pure states which can be received, given that message c is sent.

$$\begin{aligned} \mathcal{S} &= \operatorname{lin}\{K_{ab}^* K_{cd}\} = \operatorname{lin}\{p(b|a)^{1/2} p(d|c)^{1/2} |a\rangle\langle b|d\rangle\langle c|\} \\ &= \operatorname{lin}\{|a\rangle\langle c| : a \sim c\} \end{aligned}$$

Thus $\mathcal{S}$ is directly linked to the confusability graph of the channel.

Quantum relations

Simultaneously, and motivated more by "noncommutative geometry", Weaver studied:

Definition

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. A *quantum relation* on M is a weak*-closed subspace $S \subseteq \mathcal{B}(H)$ with $M'SM' \subseteq S$. The relation is:

1. *reflexive* if $M' \subseteq S$;
2. *symmetric* if $S^* = S$ where $S^* = \{x^* : x \in S\}$;
3. *transitive* if $S^2 \subseteq S$ where $S^2 = \overline{\text{lin}}^{w^*}\{xy : x, y \in S\}$.

When $M = \ell^\infty(X) \subseteq \mathcal{B}(\ell^2(X))$ there is a bijection between the usual meaning of "relation" on X and quantum relations on M, given by

$$S = \overline{\text{lin}}^{w^*}\{e_{x,y} : x \sim y\}.$$

Quantum graphs

As a graph on a (finite) vertex set V is simply a relation, and

- undirected graph corresponds to a symmetric relation;
- a reflexive relation corresponds to having a "loop" at every vertex.

Definition (Weaver)

A *quantum graph* on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is a reflexive, symmetric quantum relation. That is, a unital, self-adjoint, weak*-closed subspace $S \subseteq \mathcal{B}(H)$, which is an M'-bimodule ($M'SM' \subseteq S$).

If $M = \mathcal{B}(H)$ with H finite-dimensional, then as $M' = \mathbb{C}$, a quantum graph is just an operator system: that is, exactly what we had before! [Duan, Severini, Winter; Stahlke]

Adjacency matrices

Given a graph $G = (V, E)$ consider the $\{0, 1\}$-valued matrix A with

$$A_{i,j} = \begin{cases} 1 & : (i, j) \in E, \\ 0 & : \text{otherwise}, \end{cases}$$

the *adjacency matrix* of G.

- A is idempotent for the *Schur product*;
- G is undirected if and only if A is self-adjoint;
- A has 1s down the diagonal when G has a loop at every vertex.

We can think of A as an operator on $\ell^2(V)$. This is the GNS space for the C^*-algebra $\ell^\infty(V)$ for the state induced by the uniform measure.

General C^*-algebras

Let B be a finite-dimensional C^*-algebra, and let φ be a faithful state on B, with GNS space $L^2(B)$. Thus B bijects with $L^2(B)$ as a vector space, and so we get:

- The multiplication on B induces a map $m : L^2(B) \otimes L^2(B) \to L^2(B)$;
- The unit in B induces a map $\eta : \mathbb{C} \to L^2(B)$.

We get an analogue of the Schur product:

$$x \bullet y = m(x \otimes y)m^* \qquad (x, y \in \mathcal{B}(L^2(B))).$$

Quantum adjacency matrix

Definition (Many authors)

A *quantum adjacency matrix* is a self-adjoint $A \in \mathcal{B}(L^2(B))$ with:

- $m(A \otimes A)m^* = A$ (so Schur product idempotent);
- $(1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1) = A$;
- $m(A \otimes 1)m^* = \mathrm{id}$ (a "loop at every vertex");

The middle axiom is a little mysterious: it roughly corresponds to "undirected".

I want to sketch why this definition is equivalent to the previous notion of a "quantum graph".

Subspaces to projections

Fix a finite-dimensional C^*-algebra (von Neumann algebra) M. A "quantum graph" is either:

- A subspace of $\mathcal{B}(H)$ (where $M \subseteq \mathcal{B}(H)$) with some properties; or
- An operator on $L^2(M)$ with some properties.

How do we move between these?

$S \subseteq \mathcal{B}(H)$ is a bimodule over M'. As H is finite-dimensional, $\mathcal{B}(H)$ is a Hilbert space for

$$(x|y) = \mathrm{tr}(x^* y).$$

Then $M \otimes M^{\mathrm{op}}$ is represented on $\mathcal{B}(H)$ via

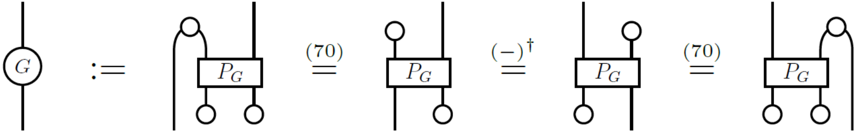
$$\pi : M \otimes M^{\mathrm{op}} \to \mathcal{B}(\mathcal{B}(H)); \quad \pi(x \otimes y) : T \mapsto xTy.$$

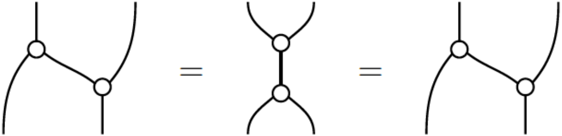
- The commutant of $\pi(M \otimes M^{\mathrm{op}})$ is naturally $M' \otimes (M')^{\mathrm{op}}$.
- So an M'-bimodule of $\mathcal{B}(H)$ corresponds to an $M' \otimes (M')^{\mathrm{op}}$-invariant subspace of the Hilbert space $\mathcal{B}(H)$;
- Which corresponds to a *projection* in $M \otimes M^{\mathrm{op}}$.

Operators to algebras

So how can we relate:

- Operators $A \in \mathcal{B}(L^2(M))$;
- Projections in $M \otimes M^{\text{op}}$?





[Musto, Reutter, Verdon]

Operators to algebras 2

Recall the GNS construction for a *tracial* state ψ on M:

$$\Lambda : M \to L^2(M); \quad (\Lambda(x)|\Lambda(y)) = \psi(x^*y).$$

As $L^2(M)$ is finite-dimensional, every operator on $L^2(M)$ is a linear combination of rank-one operators of the form

$$\theta_{\Lambda(a),\Lambda(b)} : \xi \mapsto (\Lambda(a)|\xi)\Lambda(b) \qquad (\xi \in L^2(M)).$$

Define a bijection

$$\Psi : \mathcal{B}(L^2(M)) \to M \otimes M^{\mathrm{op}}; \quad \theta_{\Lambda(a),\Lambda(b)} = b \otimes a^*,$$

and extend by linearity!

Operators to algebras 3

$$\Psi : \mathcal{B}(L^2(M)) \to M \otimes M^{\mathrm{op}}; \quad \theta_{\Lambda(a),\Lambda(b)} = b \otimes a^*,$$

- Ψ is a homomorphism for the "Schur product" $A_1 \bullet A_2 = m(A_1 \otimes A_2)m^*$;
- $A \mapsto (1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^*\eta \otimes 1)$ corresponds to the anti-homomorphism $\sigma : a \otimes b \mapsto b \otimes a$;
- $A \mapsto A^*$ corresponds to $e \mapsto \sigma(e)^*$.

Conclude: A quantum adjacency matrix corresponds to a projection e with $\sigma(e) = e$. BUT: There is no clean one-to-one correspondence between the axioms.

Non-tracial case

If the functional ψ on M is not tracial, then this correspondence fails. However:

Theorem (D.)

There is a bijection between:

- *"Schur idempotent", self-adjoint operators A on $L^2(M)$;*
- *$e \in M \otimes M^{\mathrm{op}}$ with $e^2 = e$ and $e = \sigma(e)^*$;*
- *self-adjoint M'-bimodules $S \subseteq \mathcal{B}(H)$ such that there is another self-adjoint M'-bimodule S_0 with $S \oplus S_0 = \mathcal{B}(H)$*

KMS States

Any faithful state ψ is KMS: there is an automorphism σ' of M with

$$\psi(ab) = \psi(b\sigma'(a)) \qquad (a, b \in M).$$

Indeed, there is $Q \in M$ positive and invertible with

$$\psi(a) = \mathrm{tr}(Qa) \qquad \sigma'(a) = QaQ^{-1}.$$

Theorem (D.)

Twisting our bijection Ψ using σ' allows us to establish a bijection between:

- *Quantum adjacency operators $A \in \mathcal{B}(L^2(M))$;*
- *projections $e \in M \otimes M^{\mathrm{op}}$ with $e = \sigma(e)$ and $(\sigma' \otimes \sigma')(e) = e$;*
- *self-adjoint M'-bimodules $S \subseteq \mathcal{B}(H)$ with $QSQ^{-1} = S$.*

So this is *more restrictive* than the tracial case.

Further developments

- This whole business about "a loop at every vertex" can be handled naturally.
- There is an asymmetry in the axiom

$$(1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^* \eta \otimes 1) = A$$

$$\text{or??} \quad (\eta^* m \otimes 1)(1 \otimes A \otimes 1)(1 \otimes m^* \eta) = A$$

  But these are actually equivalent.
- There are various notions of "homomorphism" or "pushforward / pullback" along a CP map. To a greater or lesser extent, these interact with the different "pictures".
- People have studied things like "colourings" of quantum graphs. E.g. a graph can be k-coloured if there is a homomorphism $G \to K_k$. So just let G be quantum.

Isomorphisms

An *isomorphism* of a quantum adjacency operator $A \in \mathcal{B}(L^2(M))$ is an automorphism θ of M which preserves the state ψ, and which commutes with A. This means:

- Think of A as a map on M, so simply $A \circ \theta = \theta \circ A$; or
- θ preserves ψ, so induces a unitary operator

$$\hat{\theta} : L^2(M) \to L^2(M); \quad \Lambda(a) \mapsto \Lambda(\theta(a)).$$

  Then require that $\hat{\theta} A \hat{\theta}^* = A$.

What can we say about an M'-bimodule $S \subseteq \mathcal{B}(H)$?

- Not every automorphism of M lifts to $\mathcal{B}(H)$;
- Seems we get dependence on H here;

Does all work if $H = L^2(M)$: then an automorphism of S is an isomorphism of $\mathcal{B}(H)$, which restricts to a ψ-persevering aut of M, and which restricts to a bijection on S.

Quantum Isomorphisms

(Extremely briefly. . .) A *quantum isomorphism* is a coaction of a compact quantum group (A, Δ) on M, say $\alpha : M \to M \otimes A$ which *commutes* with the quantum adjacency operator A_G:

$$\alpha A_G = (A_G \otimes \mathrm{id})\alpha.$$

Here A_G is thought of as a linear map on M.
Any such coaction is associated to a unitary (co)representation $U \in \mathcal{B}(L^2(M)) \otimes A$, because we assume that α leaves ψ invariant. (Copy the construction of the fundamental unitary from the coproduct.) Then equivalently $(A_G \otimes 1)U = U(A_G \otimes 1)$.
Lots of previous interest in quantum isomorphisms of classical graphs.
Also an equivalent definition from [Musto, Reutter, Verdon] using 2-categories.

Quantum Isomorphisms of operator bimodules

From the coaction α form the corep $U \in \mathcal{B}(L^2(M)) \otimes A$. Then there is a coaction of (A, Δ) on $\mathcal{B}(L^2(M))$:

$$\alpha_U : T \mapsto U(T \otimes 1)U^* \qquad (T \in \mathcal{B}(L^2(M))).$$

Might this leave $S \subseteq \mathcal{B}(L^2(M))$ invariant if and only if U commutes with A_G?

- No, as the "trivial quantum graph" is $S = M'$, which should always be invariant, but α_U leaves M invariant, not M'.
- Instead, we can use the *modular conjugation* and *antipode* to form a "commutant" coaction α'_U; or equivalently, look at α_U but work with

$$S' := \{JTJ : T \in S\}.$$

Theorem (D.)

α leaves A_G invariant if and only if α_U leaves S' invariant.