Quantum automorphisms of quantum graphs

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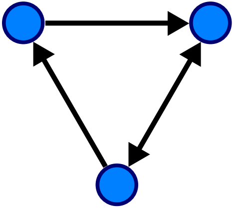
Graphs

A graph consists of a (finite) set of *vertices* V and a collection of *edges* $E \subseteq V \times V$.



$V = \{A, B, C\}$ say, and $E = \{(A, B), (B, C), (C, B), (C, A)\}$.

A graph is *undirected* if $(x, y) \in E \Leftrightarrow (y, x) \in E$. We allow *self-loops*, so $(x, x) \in E$.

Notice that a graph $G = (V, E)$ is exactly a *relation* on the set V. An undirected graph gives a symmetric relation; having a loop on each vertex gives a reflexive relation.

Adjacency matrices

A standard way to associate an "algebraic" object to a graph $G = (V, E)$ is the *adjacency matrix*. Let $V = \{1, 2, \cdots, n\}$ and define

$$A_{ij} = \begin{cases} 1 & : (i, j) \in E, \\ 0 & : \text{otherwise.} \end{cases}$$

- A is idempotent for the *Schur product*;
- G is undirected if and only if A is self-adjoint;
- A has 1s down the diagonal corresponds to G having a loop at every vertex.

We can think of A as an operator on $\ell^2(V)$. This is the GNS space for the C^*-algebra $\ell^\infty(V)$ for the state induced by the uniform measure.

Operator subspaces

Let $G = (V, E)$ be a graph, again with $V = \{1, 2, \cdots, n\}$, and consider the subspace of matrices $\mathcal{S}$ spanned by the matrix units

$$\{e_{ij} : (i, j) \in E\}.$$

- $\mathcal{S}$ is an *operator bimodule* over $\ell^\infty(V)$. That is, $x \in \mathcal{S}, a, b \in \ell^\infty(V) \implies axb \in \mathcal{S}$;
- Any bimodule over $\ell^\infty(V)$ must be spanned by matrix units, and so come from some graph.
- G is undirected if and only if $\mathcal{S}$ is self-adjoint;
- G has a loop at every vertex if and only if $1 \in \mathcal{S}$.

Recall that a self-adjoint, unital subspace of operators is an *operator system*.

Automorphisms

An *automorphism* of a graph $G = (V, E)$ is a bijection $\theta : V \to V$ which satisfies that $(i, j) \in E \implies (\theta(i), \theta(j)) \in E$. ($V$ is finite!) Set $V = \{1, \cdots, n\}$ for ease, so the adjacency matrix A is in $\mathbb{M}_n$.

Lemma

Let $P_\theta \in \mathbb{M}_n$ be permutation matrix associated with a bijection θ. Then θ is an automorphism of G if and only if $P_\theta A = A P_\theta$.

Compact Quantum groups

Definition (Woronowicz)

A *compact quantum group* is a unital C^*-algebra A together with a unital $*$-homomorphism, the *coproduct*, $\Delta : A \to A \otimes A$, which is coassociative, $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$, and such that:

$$\{(a \otimes 1)\Delta(b) : a, b \in A\}, \quad \{(1 \otimes a)\Delta(b) : a, b \in A\}$$

both have dense linear span in $A \otimes A$.

Theorem

Let (A, Δ) be a compact quantum group with A commutative. There is a compact group G with $A = C(G)$ and $\Delta : C(G) \to C(G) \otimes C(G) = C(G \times G)$ given by

$$\Delta(f)(s,t) = f(st) \qquad (f \in C(G), s, t \in G).$$

Quantum group (co)actions

An (right) action of a group G on a space/set X is a map

$$X \times G \to X.$$

So we get a $*$-homomorphism

$$\alpha : C(X) \to C(X) \otimes C(G),$$

- $(\mathrm{id} \otimes \Delta)\alpha = (\alpha \otimes \mathrm{id})\alpha$ corresponds to $x \cdot st = (x \cdot s) \cdot t$;
- $\mathrm{lin}\{\alpha(b)(1 \otimes a) : a \in C(G), b \in C(X)\}$ is dense in $C(X) \otimes C(G)$ corresponds to $x \cdot e = x$.

Definition (Podleś)

A (right) coaction of a compact quantum group (A, Δ) on a C^*-algebra B is a unital $*$-homomorphism $\alpha : B \to B \otimes A$ with these two conditions.

Coactions on ℓ^∞_n

Fix a compact quantum group (A, Δ).

- The algebra ℓ^∞_n is spanned by projections $(e_i)_{i=1}^n$.
- So $\alpha : \ell^\infty_n \to \ell^\infty_n \otimes A$ is determined by (u_{ij}) in A with

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes u_{ji}.$$

- α is a $*$-homomorphism $\Leftrightarrow$ each u_{ji} a projection and $u_{ji}u_{jk} = \delta_{ik}u_{ji}$;
- α is unital $\Leftrightarrow$ $\sum_i u_{ji} = 1$;
- α satisfies the coaction equation $\Leftrightarrow$ $\Delta(u_{ji}) = \sum_k u_{jk} \otimes u_{ki}$;
- α satisfies the Podleś density condition $\Leftrightarrow$ $\sum_i u_{ji} = 1$.
- General Theory $\implies$ $\sum_j u_{ji} = 1$. So (u_{ij}) is a *magic unitary*.

Quantum symmetry group of the space of n points

For $\ell_n^\infty = C(\{1, 2, \cdots, n\})$,

$$\alpha(e_i) = \sum_{j=1}^{n} e_j \otimes u_{ji},$$

with $u = (u_{ij})$ a magic unitary.

Theorem (Wang)

Let S_n^+ be the "universal" C^*-algebra generated by a magic unitary. Then S_n^+ is the "largest" compact quantum group which acts on $\mathbb{C}^n$ is a "non-degenerate" way.

We think of S_n^+ as the "quantum symmetry group" of $\{1, 2, \cdots, n\}$.

(Co)actions on graphs

Recall that a permutation θ gives an automorphism of G when

$$P_\theta A_G = A_G P_\theta.$$

Here A_G is the adjacency matrix of G, which we can think of as also a linear map $\ell_n^\infty \to \ell_n^\infty$.

So $\mathrm{Aut}(G)$ acts in a way which preserves A_G:

$$\alpha : \ell_n^\infty \to \ell_n^\infty \otimes C(\mathrm{Aut}(G)); \quad \alpha A_G = (A_G \otimes \mathrm{id})\alpha.$$

Definition (Banica)

The *quantum automorphism group* of G is the maximal compact quantum group $\mathrm{QAut}(G)$ with a coaction satisfying

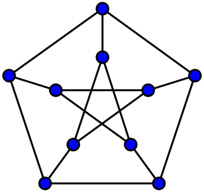
$$\alpha : \ell_n^\infty \to \ell_n^\infty \otimes \mathrm{QAut}(G); \quad \alpha A_G = (A_G \otimes \mathrm{id})\alpha.$$

Equivalently, the underlying magic unitary $U = (u_{ij})$ has to commute with the adjacency matrix A_G. This allows us to construct $\mathrm{QAut}(G)$ as a quotient of S_n^+.

Examples

We say that a graph *has quantum symmetry* if $\mathrm{Aut}(G) \neq \mathrm{QAut}(G)$.

- By now, we have many examples.
- For example, the Petersen graph has no quantum symmetry [Schmidt].



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- Recently, [Roberson, Schmidt] have constructed G with $\mathrm{Aut}(G) \neq \mathrm{QAut}(G)$ and yet $\mathrm{QAut}(G)$ is finite.

(Co)actions on operator bimodules

What is an "automorphism" of $\mathcal{S} \subseteq \mathcal{B}(\ell^2(V))$?

- Start with a bijection $\theta : V \to V$, hence giving $P_\theta \in \mathcal{B}(\ell^2(V))$.
- Then get an action on $\mathcal{B}(\ell^2(V))$ as $\hat{\theta} : x \mapsto P_\theta x P_\theta^*$ (as $P_\theta^* = P_\theta^{-1}$).
- When is $\mathcal{S}$ left invariant: $P_\theta \mathcal{S} P_\theta^* = \mathcal{S}$?

$$P_\theta e_{ij} P_\theta^* = e_{\theta(i),\theta(j)}$$

So $P_\theta \mathcal{S} P_\theta^* = \mathcal{S}$ exactly when $(i,j) \in E \Leftrightarrow (\theta(i), \theta(j)) \in E$, that is θ is an automorphism of G.

How to phrase this in terms of coactions?

Unitary implementations

Given a coaction $\alpha : \ell^\infty(V) \to \ell^\infty(V) \otimes A$ of (A, Δ) on $\ell^\infty(V)$, we saw before that α gives rise to a magic unitary $u = (u_{ij})_{i,j \in V}$,

$$\alpha(e_i) = \sum_{j \in V} e_j \otimes u_{ji} \qquad (i \in V).$$

Lemma

Let $\ell^\infty(V) \subseteq \mathcal{B}(\ell^2(V))$. Then

$$\alpha(x) = u(x \otimes 1)u^* \qquad (x \in \ell^\infty(V)).$$

Coactions on operator bimodules

$$\alpha(x) = u(x \otimes 1)u^* \qquad (x \in \ell^\infty(V) \subseteq \mathcal{B}(\ell^2(V))).$$

It hence make sense...

Definition

α is a coaction on $\mathcal{S} \subseteq \mathcal{B}(\ell^2(V))$ exactly when $u(x \otimes 1)u^* \in \mathcal{S} \otimes A$ for each $x \in \mathcal{S}$.

One can check (non-trivially) that we then get the following.

Theorem (Eifler)

If a graph G is associated to the $\ell^\infty(V)$-operator bimodule $\mathcal{S}$, then a coaction of (A, Δ) on $\ell^\infty(V)$ gives a coaction on G if and only if it gives a coaction on $\mathcal{S}$.

Non-commutative graphs

Both approaches to graphs can be adapted to a general, finite-dimensional C^*-algebra B, replacing $\ell^\infty(V)$.

- For adjacency matrices, we need a Hilbert space to act on...
- Fix a faithful state ψ on B and let $L^2(B) = L^2(B, \psi)$ be the GNS space. (We will mostly assume ψ is a trace.)
- As B is finite-dimensional, B and $L^2(B)$ are linearly isomorphic.

Let $m : B \otimes B \to B$ be the multiplication map, so we get $m^* : L^2(B) \to L^2(B) \otimes L^2(B)$. An analogue of the Schur Product is

$$A_1 \bullet A_2 = m(A_1 \otimes A_2)m^* \qquad (A_1, A_2 \in \mathcal{B}(L^2(B))).$$

(For $B = \ell^\infty(\{1, \cdots, n\})$ this gives the Schur Product on $\mathbb{M}_n \cong \mathcal{B}(\ell^2_n)$.)

- As B is unital, we also obtain the "unit map" $\eta : \mathbb{C} \to L^2(B)$.

Quantum adjacency matrix

Definition (Many authors)

A *quantum adjacency matrix* is a self-adjoint $A_G \in \mathcal{B}(L^2(B))$ with:

- $m(A_G \otimes A_G)m^* = A_G$ (so Schur product idempotent);
- $(1 \otimes \eta^* m)(1 \otimes A_G \otimes 1)(m^* \eta \otimes 1) = A_G$;
- $m(A_G \otimes 1)m^* = \mathrm{id}$ (a "loop at every vertex");

The middle axiom is a little mysterious: it roughly corresponds to "undirected".

Quantum relations

Motivated by "noncommutative geometry", Weaver studied:

Definition (Weaver)

Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra. A *quantum relation* on M is a weak*-closed subspace $\mathcal{S} \subseteq \mathcal{B}(H)$ with $M'\mathcal{S}M' \subseteq \mathcal{S}$. The relation is:

1. *reflexive* if $M' \subseteq \mathcal{S}$;
2. *symmetric* if $\mathcal{S}^* = \mathcal{S}$ where $\mathcal{S}^* = \{x^* : x \in \mathcal{S}\}$;
3. *transitive* if $\mathcal{S}^2 \subseteq \mathcal{S}$ where $\mathcal{S}^2 = \overline{\operatorname{lin}}^{w^*}\{xy : x, y \in \mathcal{S}\}$.

When $M = \ell^\infty(X) \subseteq \mathcal{B}(\ell^2(X))$ there is a bijection between the usual meaning of "relation" on X and quantum relations on M, given by

$$\mathcal{S} = \overline{\operatorname{lin}}^{w^*}\{e_{x,y} : x \sim y\}.$$

Quantum graphs

Definition (Weaver)

A *quantum graph* on a von Neumann algebra $M \subseteq \mathcal{B}(H)$ is a reflexive, symmetric quantum relation. That is, a unital, self-adjoint, weak*-closed subspace $\mathcal{S} \subseteq \mathcal{B}(H)$, which is an M'-bimodule ($M'\mathcal{S}M' \subseteq \mathcal{S}$).

- This definition depends upon H, but in fact is really independent of the choice.
- When $M = B$ is finite-dimensional, we could (and will!) let $H = L^2(B)$, as before.
- If $M = \mathcal{B}(H)$ with H finite-dimensional, then as $M' = \mathbb{C}$, and so a quantum graph is just an operator system.
- Independently, this was defined and studied by [Duan, Severini, Winter; Stahlke] and others in relation to quantum channels.

Equivalent?

These notions seem different: an operator A_G, and a subspace $\mathcal{S}$. They are in fact equivalent: let us see why.

$\mathcal{S} \subseteq \mathcal{B}(H)$ is a bimodule over B'. As H is finite-dimensional, $\mathcal{B}(H)$ is a Hilbert space for

$$(x|y) = \mathrm{tr}(x^* y).$$

Then $B \otimes B^{\mathrm{op}}$ is represented on $\mathcal{B}(H)$ via

$$\pi : B \otimes B^{\mathrm{op}} \to \mathcal{B}(\mathcal{B}(H)); \quad \pi(x \otimes y) : T \mapsto xTy.$$

- The commutant of $\pi(B \otimes B^{\mathrm{op}})$ is naturally $B' \otimes (B')^{\mathrm{op}}$.
- So an B'-bimodule of $\mathcal{B}(H)$ corresponds to an $B' \otimes (B')^{\mathrm{op}}$-invariant subspace of the Hilbert space $\mathcal{B}(H)$;
- Which corresponds to a *projection* in $B \otimes B^{\mathrm{op}}$.

Operators to algebras

Recall the GNS construction for a *tracial* state ψ on B:

$$\Lambda : B \to L^2(B); \quad (\Lambda(x)|\Lambda(y)) = \psi(x^* y).$$

As $L^2(B)$ is finite-dimensional, every operator on $L^2(B)$ is a linear combination of rank-one operators of the form

$$\theta_{\Lambda(a),\Lambda(b)} : \xi \mapsto (\Lambda(a)|\xi)\Lambda(b) \qquad (\xi \in L^2(B)).$$

Define a bijection

$$\Psi : \mathcal{B}(L^2(B)) \to B \otimes B^{\mathrm{op}}; \quad \theta_{\Lambda(a),\Lambda(b)} = b \otimes a^*,$$

and extend by linearity!

Operators to algebras cont.

$$\Psi : \mathcal{B}(L^2(B)) \to B \otimes B^{\mathrm{op}}; \quad \theta_{\Lambda(a),\Lambda(b)} = b \otimes a^*,$$

- Ψ is a homomorphism for the "Schur product" $A_1 \bullet A_2 = m(A_1 \otimes A_2)m^*$;
- $A \mapsto (1 \otimes \eta^* m)(1 \otimes A \otimes 1)(m^*\eta \otimes 1)$ corresponds to the anti-homomorphism $\sigma : a \otimes b \mapsto b \otimes a$;
- $A \mapsto A^*$ corresponds to $e \mapsto \sigma(e)^*$.

Conclude: A quantum adjacency matrix corresponds to a projection $e \in B \otimes B^{\mathrm{op}}$ with $\sigma(e) = e$. BUT: There is no clean one-to-one correspondence between the axioms.
(This result is due to [Musto, Reutter, Verdon] but the proof here is mine; one can also work with non-tracial states.)

Coactions on C^*-algebras

A coaction of (A, Δ) on B is, as before,

$$\alpha : B \to B \otimes A; \quad (\mathrm{id} \otimes \Delta)\alpha = (\alpha \otimes \mathrm{id})\alpha,$$

and satisfying the Podleś density condition.

Theorem (Wang)

There is no maximal compact quantum group coacting on B.
If ψ is a faithful state on B, there is a maximal compact quantum group coacting on B and preserving ψ: $(\psi \otimes \mathrm{id})\alpha(x) = \psi(x)1$ for $x \in B$. Write $\mathrm{QAut}(B, \psi)$ for this.

Coactions on quantum adjacency matrices

There is now a clear definition:

Definition (Brannan et al.)

Let A_G be a quantum adjacency matrix on (B,ψ). We say that (A,Δ) coacts on A_G when $\alpha : B \to B \otimes A$ is a coaction, which preserves ψ, and with $(A_G \otimes \mathrm{id})\alpha = \alpha A_G$.

- Here we regard A_G as a linear map on B.
- That α preserves ψ allows us to define a unitary $U \in \mathcal{B}(L^2(B)) \otimes A$ which implements α, as $\alpha(x) = U(x \otimes 1)U^*$. Indeed, one way to prove Wang's theorem is to start with such a U and impose certain conditions on it (compare Compact Quantum Matrix Groups).
- Then, equivalently, we require that U and $A_G \otimes 1$ commute.

Coactions on operator bimodules

A coaction α which preserves ψ gives a unitary U (which is a *corepresentation*) and it is then easy to see that

$$\alpha_U : \mathcal{B}(L^2(B)) \to \mathcal{B}(L^2(B)) \otimes A; x \mapsto U(x \otimes 1)U^*$$

is a coaction (which extends α).

Might this leave $\mathcal{S} \subseteq \mathcal{B}(L^2(M))$ invariant if and only if U commutes with A_G?

- No, as the "trivial quantum graph" is $\mathcal{S} = B'$, which should always be invariant, but α_U leaves B invariant, not B'.
- Instead, we can use the *modular conjugation* J and *antipode* to form a "commutant" coaction α'_U; or equivalently, look at α_U but work with

$$\mathcal{S}' := \{JTJ : T \in \mathcal{S}\}.$$

Theorem (D.)

α leaves A_G invariant if and only if α_U leaves $\mathcal{S}'$ invariant.