Multipliers of Quantum groups from Hilbert C*-modules

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Leeds

March 2010

Multipliers

Suppose that A is an algebra: how might we embed A into a unital algebra B?

- Could use the unitisation: $A \oplus \mathbb{C}1$.
- Natural to ask that A is an *ideal* in B.
- But we don't want B to be too large: the natural condition is that A should be *essential* in B: if $I \subseteq B$ is an ideal then $A \cap I \neq \{0\}$.
- For *faithful* A, this is equivalent to: if $b \in B$ and $aba' = 0$ for all $a, a' \in A$, then $b = 0$.
- Turns out there is a maximal such B, called the *multiplier algebra* of A, written $M(A)$. Maximal in the sense that if $A \trianglelefteq B$, then $B \rightarrow M(A)$. Clearly $M(A)$ is unique.

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How to build $M(A)$

We define $M(A)$ to be the collection of maps $L, R : A \to A$ with

$$L(ab) = L(a)b, \quad R(ab) = aR(b), \quad aL(b) = R(a)b \qquad (a, b \in A).$$

- If A is faithful (which we shall assume from now on) then we only need the third condition.
- $M(A)$ is a vector space, and an algebra for the product $(L, R)(L', R') = (LL', R'R)$.
- Each $a \in A$ defines a pair $(L_a, R_a) \in M(A)$ by $L_a(b) = ab$ and $R_a(b) = ba$.
- The homomorphism $A \to M(A); a \mapsto (L_a, R_a)$ identifies A with an essential ideal in $M(A)$.
- If A is a Banach algebra, then natural to ask that L and R are bounded; but this is automatic by using the Closed Graph Theorem.

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Multipliers of C*-algebras

Let A be a C*-algebra acting non-degenerately on a Hilbert space H. Then we have that

$$M(A) = \{T \in \mathcal{B}(H) : Ta, aT \in A \ (a \in A)\}.$$

- Each such T does define a multiplier in the previous sense.
- Conversely, a bounded approximate identity argument allows you to build $T \in \mathcal{B}(H)$ given $(L, R) \in M(A)$.
- If $A = C_0(X)$ then $M(A) = C^b(X) = C(\beta X)$, so $M(A)$ is a non-commutative Stone-Čech compactification.

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Locally compact groups

Let G be a locally compact group, equipped with a left invariant Haar measure. Examples include:

- Any *discrete* group with the counting measure.
- Any *compact* group, where the Haar measure is normalised to be a probability measure.
- The real line $\mathbb{R}$ with Lebesgue measure.
- Various non-compact Lie groups give interesting examples.

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Group algebras

Turn $L^1(G)$ into a Banach algebra by using the convolution product:

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)\, dt.$$

- We can also convolve finite measures.
- Identify $M(G)$ with $C_0(G)^*$, then

$$\langle \mu * \lambda, F \rangle = \int \int F(st)\, d\mu(s)\, d\lambda(t) \qquad (\mu, \lambda \in M(G), F \in C_0(G)).$$

- [Wendel] Then we have that

$$M(L^1(G)) = M(G),$$

where for each $(L, R) \in M(L^1(G))$, there exists $\mu \in M(G)$,

$$L(a) = \mu * a, \quad R(a) = a * \mu \qquad (a \in L^1(G)).$$

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The Fourier transform

If G is abelian, then we have the dual group

$$\hat{G} = \{\chi : G \to \mathbb{T} \text{ a continuous homomorphism}\}.$$

Also we have the Fourier Transform

$$\mathcal{F} : L^1(G) \to C_0(\hat{G}) \quad \text{also} \quad L^2(G) \cong L^2(\hat{G}).$$

- The image $\mathcal{F}(L^1(G))$ is the *Fourier algebra* $A(\hat{G})$.
- As $L^1(G) = L^2(G) \cdot L^2(G)$ (pointwise product) we see that $A(\hat{G}) = L^2(G) * L^2(G) = L^2(\hat{G}) * L^2(\hat{G})$ (convolution).
- $\mathcal{F}$ extends to $M(G)$, and the image is $B(\hat{G}) \subseteq C^b(G)$, the *Fourier-Stieltjes algebra*.

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Operator algebras

The Fourier transform similarly sets up isomorphisms

$$C_0(G) \cong C_r^*(\hat{G}) \qquad L^\infty(G) \cong VN(\hat{G}).$$

Let $\lambda : G \to \mathcal{B}(L^2(G))$ be the *left-regular representation*,

$$\lambda(s) : f \mapsto g \qquad g(t) = f(s^{-1}t) \qquad (f \in L^2(G), s, t \in G).$$

Integrate this to get a homomorphism $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$.

- $C_r^*(G)$ is the closure of $\lambda(L^1(G))$.
- $C^*(G)$ is the enveloping C*-algebra of $L^1(G)$: agrees with $C_r^*(G)$ is G is abelian, compact, amenable.
- $VN(G)$ is the WOT closure of $\lambda(L^1(G))$ (or of $\lambda(G)$).
- So, $A(\hat{G})$ is the predual of $VN(\hat{G})$ and $B(\hat{G})$ is the dual of $C^*(\hat{G})$.

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The Fourier algebra

For a general G, we could hence define $A(G)$ to be:

- the predual of $VN(G)$.
- Or $A(G) = L^2(G) * L^2(G)$.
- We *hope* that these agree and that $A(G)$ is an algebra for the pointwise product.

Remember that a von Neumann algebra always has a *predual*: the space of normal functionals.

As $VN(G) \subseteq \mathcal{B}(L^2(G))$, and $\mathcal{B}(L^2(G))$ is the dual of $\mathcal{T}(L^2(G))$, the trace-class operators on $L^2(G)$, we have a quotient map

$$\mathcal{T}(L^2(G)) \twoheadrightarrow VN(G)_*.$$

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- As $\{\lambda(s) : s \in G\}$ generates $VN(G)$, for $\omega \in VN(G)_*$, if we know what $\langle \lambda(s^{-1}), \omega \rangle$ is for all s, then we know ω.
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$$\begin{aligned}\langle \lambda(s^{-1}), \omega_{\xi,\eta} \rangle &= \int_G \lambda(s^{-1})(\xi)(t)\overline{\eta(t)}\, dt = \int_G \xi(st)\overline{\eta(t)}\, dt \\ &= \int_G \xi(t)\overline{\check{\eta}}(t^{-1}s)\, dt = (\xi * \overline{\check{\eta}})(s).\end{aligned}$$

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Why an algebra? [Takesaki-Tatsumma]

There is a normal $*$-homomorphsm
$\Delta : VN(G) \to VN(G)\overline{\otimes}VN(G) = VN(G \times G)$ which satisfies

$$\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s) = \lambda(s, s).$$

- As Δ is normal, we get a (completely) contractive map $\Delta_* : A(G) \times A(G) \to A(G)$.
- Turns out that Δ_* is associative, because Δ is *coassociative*.
- This obviously induces the pointwise product on $A(G)$, as for $\omega, \sigma \in A(G)$ and $s \in G$,

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Multipliers of the Fourier algebra

- As $A(G)$ is commutative, multipliers of $A(G)$ are simply maps T on $A(G)$ with $T(ab) = T(a)b$.
- As we consider $A(G) \subseteq C_0(G)$, we find that every $T \in MA(G)$ is given by some $f \in C^b(G)$:

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- By duality, each $T \in MA(G)$ induces a map $T^* : VN(G) \to VN(G)$.
- If this is *completely bounded*– that is gives uniformly (in n) bounded maps $1 \otimes T^*$ on $\mathbb{M}_n \otimes VN(G)$– then $T \in M_{cb}A(G)$.
- [Haagerup, DeCanniere] For $f \in MA(G)$, we have that $f \in M_{cb}A(G)$ if and only if $f \otimes 1_K \in MA(G \times K)$ for all compact K (or just $K = SU(2)$).

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Properties of groups via multipliers

Lots of interesting properties of groups are related to how $A(G)$ sits in $M_{cb}A(G)$:

- $A(G)$ has a bounded approximate identity if and only if G is amenable.
- If $A(G)$ has an approximate identity, bounded in $M_{cb}A(G)$, then G is *weakly amenable*.
- For example, this is true for $SO(1, n)$ and $SU(1, n)$.
- Let Λ_G be the minimal bounded (in $M_{cb}A(G)$) for such an approximate identity.
- Then, for $G = Sp(1, n)$, then $\Lambda_G = 2n - 1$.
- [Ozawa] All hyperbolic groups are weakly amenable.

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Gilbert's view on $M_{cb}A(G)$

Theorem (Gilbert, Jolissaint)

For $f \in C^b(G)$, we have that $f \in M_{cb}A(G)$ if and only if there is a Hilbert space K and continuous bounded maps $\alpha, \beta : G \to K$ with $f(st^{-1}) = (\beta(t)|\alpha(s))$ for $s, t \in G$.

Hilbert C*-modules

Given a C*-algebra A, let X be a right module over A. Suppose that X has an A-valued inner-product:

- $(x|x) \geq 0$ (in the C*-algebra sense) and $(x|x) = 0$ if and only if $x = 0$,
- $(x|y) = (y|x)^*$,
- $(x|y \cdot a) = (x|y)a$ for $x, y \in X, a \in A$.

Then X carries a norm: $\|x\| = \|(x|x)\|^{1/2}$. If X is complete, we say that X is a Hilbert C*-module over A.

Example: $A = C_0(G)$ and $X = C_0(G, K)$ for a Hilbert space K. The module action is obvious; the inner-product is

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Abstracting $C_0(G, K)$

For any C*-algebra A and Hilbert space K, we consider the algebraic tensor product $A \odot K$, with:

$$(a \otimes \xi) \cdot b = ab \otimes \xi, \quad \big(a \otimes \xi \big| b \otimes \eta\big) = a^* b(\xi|\eta).$$

Let $A \otimes K$ be the completion.

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We're interested in $C^b(G, K)$: how can we abstract this?

Any $\alpha \in C^b(G, K)$ defines a map

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Adjointable maps

Actually, given $\alpha \in C^b(G,K)$, not only do we get a map $T : C_0(G) \to C_0(G,K)$, we get an "adjoint" $T^* : C_0(G,K) \to C_0(G)$ given by

$$T^*(x) = \big(G \to \mathbb{C}; s \mapsto (\alpha(s)|x(s))_K\big).$$

This satisfies $(T^*(x)|a) = (x|T(a))$ for $x \in C_0(G,K)$ and $a \in C_0(G)$.

- Unlike for Hilbert spaces, not all maps between Hilbert C^*-modules have adjoints.
- But, if a map is adjointable, it's automatically bounded and a module homomorphism.
- Write $\mathcal{L}(X,Y)$ for the space of maps which do have adjoints.
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Abstract duality theory: Locally Compact Quantum Groups

A *locally compact quantum group* is a von Neumann algebra M which is equipped with a normal $*$-homomorphism $\Delta : M \to M\overline{\otimes}M$ such that is coassociative: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.

- We also assume that M carries left and right invariant weights: I'll ignore these here: they are very important for the theory (but, if they exist, are unique, so in sense are intrinsic).
- As Δ is normal, we get an *associative* product on the predual M_*.
- We've seen one example: $VN(G)$ and $A(G)$.
- Another example: $L^\infty(G)$ with $\Delta : L^\infty(G) \to L^\infty(G \times G)$ given by $\Delta(F)(s,t) = F(st)$.
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C*-algebras and duality

Inside M is a C*-algebra A, and Δ restricts to a map $\Delta : A \to M(A \otimes A)$.

- For $L^\infty(G)$, we get $C_0(G)$ (mapping into $C^b(G \times G)$).
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Given M, we can form a "dual group" $(\hat{M}, \hat{\Delta})$, and we have that $(M, \Delta) \cong (\hat{\hat{M}}, \hat{\hat{\Delta}})$.

- The dual of $L^\infty(G)$ is $VN(G)$, so in some sense, this generalises Pontryagin duality.
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- In the "classical" setup, we have that $M(L^1(G)) = M(G) \to M(C_r^*(G))$. This is just the extension of $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$.
- Similarly, $M_{cb}A(G) \to C^b(G) = M(C_0(G))$.
- The duality framework gives a map $\hat{\lambda} : \hat{M}_* \to A$.
- [Daws], building heavily on work of [Kraus, Ruan]. This does indeed extend to a homomorphism $\hat{\Lambda} : M_{cb}(\hat{M}_*) \to M(A)$.
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Representing multipliers using Hilbert C*-modules

- We replace $C^b(G, K)$ by $\mathcal{L}(A, A \otimes K)$.
- Given $\alpha \in \mathcal{L}(A, A \otimes K)$, there is a way to define $\Delta * \alpha \in \mathcal{L}(A \otimes A, A \otimes A \otimes K)$. This generalises the map

$$C^b(G, K) \to C^b(G \times G, K); \quad f \mapsto \big(G \times G \to K; (s, t) \mapsto f(st)\big).$$

- We say that a pair of maps $\alpha, \beta \in \mathcal{L}(A, A \otimes K)$ is "invariant" if

$$(1 \otimes \beta^*)(\Delta * \alpha) \in M(A) \otimes 1.$$

(This is always in $\mathcal{L}(A \otimes A) \cong M(A \otimes A)$). This generalises the possibility of finding $f \in C^b(G)$ with

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Representing multipliers using Hilbert C^*-modules

- We replace $C^b(G,K)$ by $\mathcal{L}(A, A \otimes K)$.
- Given $\alpha \in \mathcal{L}(A, A \otimes K)$, there is a way to define $\Delta * \alpha \in \mathcal{L}(A \otimes A, A \otimes A \otimes K)$. This generalises the map

$$C^b(G,K) \to C^b(G \times G, K); \quad f \mapsto \big(G \times G \to K; (s,t) \mapsto f(st)\big).$$

- We say that a pair of maps $\alpha, \beta \in \mathcal{L}(A, A \otimes K)$ is "invariant" if

$$(1 \otimes \beta^*)(\Delta * \alpha) \in M(A) \otimes 1.$$

(This is always in $\mathcal{L}(A \otimes A) \cong M(A \otimes A)$). This generalises the possibility of finding $f \in C^b(G)$ with

$$f(st^{-1}) = (\beta(t)|\alpha(s)) \Leftrightarrow (\beta(t)|\alpha(st)) \text{ constant in } t.$$

Representing multipliers

There is a bijection between:

1. "Represented" completely bounded left multipliers L. That is, cb maps $L : \hat{M}_* \to \hat{M}_*$ with $L(\hat{\omega}\hat{\sigma}) = L(\hat{\omega})\hat{\sigma}$, and such that there is $x \in M(A)$ with $x\hat{\lambda}(\hat{\omega}) = \hat{\lambda}(L(\hat{\omega}))$. (Notice that this is always true for the "left half" of a cb multiplier (L, R)).
2. Invariant pairs (α, β) in $\mathcal{L}(A, A \otimes K)$ with $(1 \otimes \beta^*)(\Delta * \alpha) = x \otimes 1$.

In this case, the map $L^* : \hat{M} \to \hat{M}$ is given by

$$L^* = \tilde{\beta}^*(x \otimes 1)\tilde{\alpha} \qquad (x \in \hat{M}).$$

Here $\tilde{\alpha} : H \to H \otimes K$ is built from α, where H is the canonical Hilbert space given by M.

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