We now turn to the general case, which is joint work in progress with White, Schlumprecht and Haydon.

Suppose that $E \subseteq \ell^\infty(\mathbb{Z})$, and let $\mathcal{A}$ be the unital C*-algebra generated by E, inside $\ell^\infty(\mathbb{Z})$. As $\mathcal{A}$ is commutative, $\mathcal{A} \cong C(\Omega)$, and so $\mathcal{A}' \cong M(\Omega)$.

As E, and hence $\mathcal{A}$, separates the points of $\ell^1(\mathbb{Z})$, we get an injection $\mathbb{Z} \to \Omega$. We can then extend the product on $\mathbb{Z}$ to Ω, making Ω into a compact semitopological semigroup. We can also extend this injection to a map $\ell^1(\mathbb{Z}) \to M(\Omega)$, which is bounded below as E is a predual.

Of course, we are now putting no extra conditions on E, and so in general Ω will not be $\mathbb{Z}$. The maximal compact semitopological semigroup which contains $\mathbb{Z}$ densely is $\mathbb{Z}^{\mathsf{wap}}$; this is not in general separable, and so at least Ω cannot be as large as $\mathbb{Z}^{\mathsf{wap}}$. It would be convienient if Ω were always countable.

Theorem. There is a projection $P : M(\Omega) \to \ell^1(\mathbb{Z})$ which is an algebra homomorphism, such that

$$E = {}^\perp(\ker P) := \{f \in C(\Omega) : \langle a, f\rangle = 0 \ (P(a) = 0)\}.$$

Furthermore, $\ker P$ is weak*-closed.

The projection P can be constructed as follows. For each $\mu \in M(\Omega) = \mathcal{A}'$, the restriction of μ to E induces a unique member of $\ell^1(\mathbb{Z})$, as E is a predual for $\mathbb{Z}$. We let this be $P(\mu)$. That P is an algebra homomorphism follows because E makes the product on $\ell^1(\mathbb{Z})$ separately weak*-continuous. It easily follows that $E = {}^\perp \ker P$ and that $\ker P$ is weak*-closed.

The key idea in constructing preduals of $\ell^1(\mathbb{Z})$ is to find a suitable converse of this theorem. It turns out that it is much easier to work in the situation when Ω is countable, essentially because then $M(\Omega) = \ell^1(\Omega)$.

So let Ω be a countable compact semitopological semigroup, and suppose that Ω contains $\mathbb{Z}$ as a subgroup. We can hence regard $\ell^1(\mathbb{Z})$ as a subalgebra of $\ell^1(\Omega)$. Suppose that we have a projection $P : \ell^1(\Omega) \to \ell^1(\mathbb{Z})$ which is an algebra homomorphism.

Theorem. Let ${}^\perp(\ker P)$ induce a space of functionals on $\ell^1(\mathbb{Z})$, say $E \subseteq \ell^\infty(\mathbb{Z})$. Then E is a predual if and only if $\ker P$ is weak*-closed.

The E constructed here is rather hard to get a handle on. However, the weak*-topology it induces can be found using P.

Proposition. Let (a_α) be a bounded net in $\ell^1(\mathbb{Z})$ which tends weak* to $b \in C(\Omega)' = \ell^1(\Omega)$. Then (a_α) tends to $a = P(b)$ in the weak*-topology on $\ell^1(\mathbb{Z})$ induced by E.

Our task now is to construct suitable semigroups Ω and projections P.

We start with the simplest case: let Ω be the free abelian semigroup generated by $\mathbb{Z}$ and a single extra generator z. We shall also add a semigroup *zero*, denoted by ∞, as we ultimately want Ω to be compact.

By this, we mean that $s + \infty = \infty$ for any $s \in \Omega$. When we give Ω a topology, ∞ will be the point added "at infinity", as in the usual one-point compactification of a topological space.

Hence every member of Ω is of the form

$$kz + n \qquad (n \in \mathbb{Z}, k \geq 0).$$

The group product is simply $(kz+n)+(lz+m) = (k+l)z+(n+m)$.

Any projection P is uniquely determined by $P(\delta_z)$, as P is an algebra homomorphism:

$$P(\delta_{kz+n}) = P(\delta_z)^k \delta_n, \quad P(\delta_\infty) = 0.$$

Theorem. Suppose that Ω is given some topology such that Ω becomes a compact semitopological semigroup. Suppose, furthermore, that for each $K > 0$ there exists $k > K$ and an open neighbourhood U of kz such that $U \subseteq \{lz + s : s \in \mathbb{Z}, l \leq k\}$. When $\|P(\delta_z)\| < 1$, we have that $X = {}^\perp(\ker P)$ is a predual for $\ell^1(\mathbb{Z})$.

This is a long, technical proof. The basic idea is one of successive approximation to show that ${}^\perp(\ker P)$ is weak*-closed.

The condition that $\|P(\delta_z)\| < 1$ can be weakened to

$$\lim_k \|P(\delta_z)^k\| = 0.$$

We have a proof, however, that for $P(\delta_z) = \delta_0$, the resulting E is not a predual.

The proof of this takes a huge detor via Banach space theory, and in particular the Szlenk index. We have currently not been able to find a more (Banach) algebraic proof.

It is completely unclear if we could have $P(\delta_z) = \frac{1}{2}(\delta_0 + \delta_1)$ though.

We have an argument, far from rigourous, that we *can* have $P(\delta_z) = \frac{1}{2}(\delta_0 + \delta_1)$.

So finally we wish to construct a suitable topology on Ω.

Let $J \subseteq \mathbb{Z}$ be a very "sparse" set. We can make this rigourous, but the current working definition is a little tedious. We define a clopen neighbourhood of kz to be

$$\{kz\} \cup \bigcup_{l=1}^{k} \{(k-l)z + s : s = j_1 + \cdots + j_l, n < |j_1| < \cdots < |j_l|\}.$$

This additive structure is chosen to make the product on Ω separately continuous. We have no choice for the clopen neighbourhoods of $kz + t$ as the map $x \mapsto x + t$, and its inverse, must be continuous. As these sets are clopen, we also have a ready supply of open neigbourhoods of ∞.

We can fairly easily check that we have defined a base for a topology on Ω making Ω a semitopological semigroup which is compact. Our meaning of "sparse" is chosen so that the topology is Hausdorff.

Theorem. Let $J \subseteq \mathbb{Z}$ be an "sparse" set, and let $a \in \ell^1(\mathbb{Z})$ with $\|a\| < 1$. Then there exists a predual for $\ell^1(\mathbb{Z})$ such that $\delta_n \to a$ as n tends through J, in the weak*-topology.

For example, we can let $J = \{2^n : n \in \mathbb{N}\}$. For this choice, the natural involution on $\ell^1(\mathbb{Z})$ given by

$$\delta_n^* = \delta_{-n} \qquad (n \in \mathbb{Z})$$

is not weak*-continuous.

However, for $J = \{\pm n! : n \in \mathbb{N}\}$, as $J = -J$, it is not hard to show that the involution becomes weak*-continuous.

For the first choice, with $P(\delta_z) = \lambda\delta_0$ for some $\lambda \in \mathbb{C}$ with $|\lambda| < 1$, we can throw a lot of Banach space machinery into an argument to show that the constructed predual is isomorphic to c_0, purely as a Banach space. Furthermore, the predual is generated, as an $\ell^1(\mathbb{Z})$-bimodule, by a single element.

One can express the space as a "G-space", in the sense of Benyamini, Samuel et al. Thus the predual is a $C(K)$ space for some K. We can compute the Szlenk index in this specific case, however, which tells us that $C(K) \cong c_0$. That the predual has a single generator follows from an even more specific argument. It is tempting to conjecture that this single generator property follows from the fact that Ω is, in a sense, a single element extension of $\mathbb{Z}$.