Multipliers and the Fourier algebra

Matthew Daws

Leeds

July 2009

Outline

Centralisers

For an algebra $\mathcal{A}$, let $M(\mathcal{A})$ be the space of *double centralisers*, that is, pairs of linear maps (L,R) of $\mathcal{A}\to\mathcal{A}$ with

$$\begin{cases} L(ab)=L(a)b, \quad R(ab)=aR(b), \\ \qquad aL(b)=R(a)b \end{cases} \qquad (a,b\in\mathcal{A}).$$

We always assume that $\mathcal{A}$ is faithful, meaning that if $a\in\mathcal{A}$ with $bac=0$ for any $b,c\in\mathcal{A}$, then $a=0$.
When $\mathcal{A}$ is a Banach algebra, we naturally ask that L and R are linear and bounded. However…
A Closed Graph argument shows that if (L,R) is a pair of maps $\mathcal{A}\to\mathcal{A}$ with

$$aL(b)=R(a)b \qquad (a,b\in\mathcal{A}),$$

then already $(L,R)\in M(\mathcal{A})$.

Centralisers continued

Then $M(\mathcal{A})$ becomes a Banach algebra for the product

$$(L,R)(L',R')=(LL',R'R).$$

We can identify $\mathcal{A}$ as a subalgebra of $M(\mathcal{A})$ by

$$a\mapsto(L_a,R_a), \qquad L_a(b)=ab,\ R_a(b)=ba \qquad (a,b\in\mathcal{A}).$$

$M(\mathcal{A})$ is a well understood and useful tool in the C*-algebra world.
If $\mathcal{A}$ is a Banach algebra with a bounded approximate identity, then most of what we expect from the C*-world works for $M(\mathcal{A})$.

Centralisers / Multipliers?

If $\mathcal{A}$ is commutative, then $L = R$, and so

$$M(\mathcal{A}) = \{T : \mathcal{A} \to \mathcal{A} : T(ab) = aT(b)\ (a, b \in \mathcal{A})\}.$$

Let X be a locally compact space, and suppose that $\mathcal{A}$ is a commutative algebra of functions on X which separates the points of X. Then each $T \in M(\mathcal{A})$ gives rise to a function $f : X \to \mathbb{C}$ with $T(a) = fa$ for $a \in \mathcal{A}$. Thus

$$M(\mathcal{A}) = \{f : X \to \mathbb{C} : fa \in \mathcal{A}\ (a \in \mathcal{A})\}.$$

Hence $M(\mathcal{A})$ is the algebra of functions on X which "multiply $\mathcal{A}$ into $\mathcal{A}$".
I shall henceforth not distinguish between multipliers and centralisers.

Dual Banach algebras and multipliers

A *dual Banach algebra* is a Banach algebra $\mathcal{A}$ which is (isomorphic to) the dual of some Banach space $\mathcal{A}_*$, such that the product on $\mathcal{A}$ is separately weak*-continuous.
The multiplier algebra of a C*-algebra is rarely a dual Banach algebra:

$$M(c_0) = \ell^\infty = (\ell^1)^*, \qquad M(C_0(K)) = C^b(K) \cong C(\beta K).$$

For a locally compact group G,

$$M(L^1(G)) = M(G) = C_0(G)^*,$$

where for each $(L, R) \in M(L^1(G))$, there exists $\mu \in M(G)$,

$$L(a) = \mu * a, \quad R(a) = a * \mu \qquad (a \in L^1(G)).$$

The Fourier Algebra

For a locally compact group G let λ be the left regular representation

$$\big(\lambda(s)\xi\big)(t) = \xi(s^{-1}t) \qquad (s, t \in G, \xi \in L^2(G)).$$

This induces a homomorphism $\lambda : L^1(G) \to \mathcal{B}(L^2(G))$.
Let $C^*_\lambda(G)$ and $VN(G)$ be the norm and σ-weak closures of $\lambda(L^1(G))$, respectively. So $VN(G) = C^*_\lambda(G)''$.
Let $A(G)$ be the predual of $VN(G)$. As $VN(G)$ is in standard position on $L^2(G)$, for each $\omega \in A(G)$, there exist $\xi, \eta \in L^2(G)$ with

$$\omega = \omega_{\xi,\eta} \qquad \langle x, \omega\rangle = \big(x(\xi)|\eta\big) \qquad (x \in VN(G)).$$

Fourier Algebra: The product

As $\{\lambda(s) : s \in G\}$ also generates $VN(G)$, we see that $\{\langle\lambda(s), \omega\rangle : s \in G\}$ determines $\omega \in A(G)$. For $\omega_{\xi,\eta} \in A(G)$ and $s \in G$,

$$\langle\lambda(s), \omega_{\xi,\eta}\rangle = \int \xi(s^{-1}t)\overline{\eta(t)}\ dt = \overline{\eta} * \check{\xi}(s).$$

Here $\check{\xi}(s) = \xi(s^{-1})$.
It's not hard to see that $\overline{\eta} * \check{\xi} \in C_0(G)$, and so we have an injection

$$\Phi : A(G) \to C_0(G).$$

Then $\Phi(A(G))$ is a subalgebra of $C_0(G)$, and $A(G)$ is a Banach algebra, [Eymard].

Multipliers

So we can form $MA(G)$:

- $MA(G) = \{f \in C^b(G) : fa \in A(G) \ (a \in A(G))\}$;
- $MA(G) = B(G)$, the Fourier-Stieltjes algebra, if and only if G is amenable [Losert].

As the predual of a von Neumann algebra, $A(G)$ is an operator space. Actually a completely contractive Banach algebra. Hence natural to consider the *completely bounded multipliers*, written $M_{cb}(A(G))$.
[De Canniere, Haagerup]: For $f \in MA(G)$, TFAE:

- $f \in M_{cb}A(G)$;
- $f \otimes 1_K \in MA(G \times K)$ for all compact groups K;
- $f \otimes 1_K \in MA(G \times K)$ for $K = SU(2)$.

$MA(G)$ and $M_{cb}A(G)$ are dual

[De Canniere, Haagerup]: Let Q be the completion of $L^1(G)$ under the norm

$$\|f\|_Q = \sup\left\{\left|\int f(s)a(s)\,ds\right| : a \in MA(G), \|a\| \leq 1\right\}.$$

Then $Q^* = MA(G)$.
Let Q_0 be the completion of $L^1(G)$ under the norm

$$\|f\|_{Q_0} = \sup\left\{\left|\int f(s)a(s)\,ds\right| : a \in M_{cb}A(G), \|a\| \leq 1\right\}.$$

Then $Q_0^* = M_{cb}A(G)$.
Easy to check that $MA(G)$ and $M_{cb}A(G)$ hence become dual Banach algebras.

An abstract approach

Notice that $A(G) = \overline{\text{lin}}\{ab : a, b \in A(G)\}$. Let

$$\iota : A(G) \to B(G)$$

be the isometric inclusion map; so $\iota(A(G))$ is an *essential* ideal.
We hence get a map $\theta : B(G) \to MA(G)$,

$$\theta(a) : A(G) \to A(G); b \mapsto ab \qquad (a \in B(G), b \in A(G)).$$

As $A(G)$ is faithful, $\iota(A(G))$ being essential is equivalent to θ being injective.

An abstract approach, cont.

$$\iota : A(G) \to B(G), \quad \theta : B(G) \to MA(G).$$

Theorem

$MA(G)$ admits a weak-topology such that $MA(G)$ becomes a dual Banach algebra.*

Proof.

Consider $X_0 = (A(G)\widehat{\otimes}C^*(G)) \oplus_1 (A(G)\widehat{\otimes}C^*(G))$ which has dual space $\mathcal{B}(A(G), B(G)) \oplus_\infty \mathcal{B}(A(G), B(G))$. Let $X \subseteq X_0$ be the closed linear span of

$$(b \otimes x \cdot \iota(a)) \oplus (-a \otimes \iota(b) \cdot x),$$

for $a, b \in A(G), x \in C^*(G)$. A calculation shows that

$$X^\perp = \{(\iota L, \iota R) : (L, R) \in MA(G)\}.$$

$\square$

Comments

Theorem

The weak-topology on $MA(G)$ is unique such that:*

- *$MA(G)$ is a dual Banach algebra;*
- *We have that $b_\alpha \to b$ weak* in $B(G)$ if and only if $\theta(b_\alpha) \to \theta(b)$ weak* in $MA(G)$.*

So this gives the same predual as DeCanniere and Haagerup. Also, we could have used $B^*_\lambda(G) = C^*_\lambda(G)^*$ instead of $B(G)$.
If we want $M_{cb}A(G)$ then simply work in the category of Operator Spaces!
All this works, with essentially no change, for locally compact quantum groups.
Inspired by [Selivanov].

Representations

Results of [Daws] in the Banach space case, and [Uygul] in the Operator space case give:

Theorem

Let $\mathcal{A}$ be a (completely contractive) dual Banach algebra. Then there exists a reflexive Operator / Banach space E and a (completely) isometric, weak-weak*-continuous homomorphism $\pi : \mathcal{A} \to \mathcal{B}(E)$.*

If we know more about $\mathcal{A}$ (say, $\mathcal{A} = M(L^1(G))$ or $MA(G)$) can we choose E in a "nice" way?

An idea of Young

Fix a group G. Let $(p_n) \subseteq (1,\infty)$ be a sequence tending to 1, and let

$$E = \ell^2 - \bigoplus_n L^{p_n}(G).$$

- $L^1(G)$ acts by convolution on each $L^{p_n}(G)$, and hence on E.
- Similarly $M(G)$ acts by convolution on E, extending the action of $L^1(G)$.
- Actually, the homomorphism $\pi : M(G) \to \mathcal{B}(E)$ is an *isometry*, and is weak*-weak* continuous.
- The image of $M(G)$ in $\mathcal{B}(E)$ is the *idealiser* of $\pi(L^1(G))$:

$$\pi(M(G)) = \left\{ T \in \mathcal{B}(E) : \begin{matrix} T\pi(a), \pi(a)T \in \pi(L^1(G)) \\ (a \in L^1(G)) \end{matrix} \right\}.$$

The details

Let $\mu \in M(G)$, $p \in (1,\infty)$, $\xi \in L^p(G)$ and $\eta \in L^{p'}(G)$, so

$$\begin{aligned} \langle \pi(\mu)\xi, \eta \rangle &= \int d\mu(s) \xi(s^{-1}t)\eta(t)\, dt \\ &= \int\int \xi(s^{-1}t)\eta(t)\, dt\, d\mu(s) = \langle \mu, \eta * \check{\xi} \rangle. \end{aligned}$$

Here $\eta * \check{\xi} \in A_p(G)$, the Figa-Talamanca–Herz algebra.

- For $\xi, \eta \in C_{00}(G)$, we have $\lim_{p\to 1} \|\eta * \check{\xi}\|_{A_p} = \|\eta * \check{\xi}\|_\infty$.
- As $C_{00}(G)^2$ dense in $C_0(G)$, this shows that $\pi : M(G) \to \mathcal{B}(E)$ is an isometry (as $p_n \to 1$).
- This also shows that π is weak*-weak*-continuous: the preadjoint is the direct sum of the maps

$$L^{p_n}(G) \widehat{\otimes} L^{p_n'}(G) \to A_{p_n}(G) \to C_0(G).$$

Abstracting the construction

It is well-known that $L^p(G)$ can be realised as the *complex interpolation* space, of parameter $1/p$, between $L^\infty(G)$ and $L^1(G)$.
I won't explain this in detail but observe that:

- We regard $L^\infty = L^\infty(G)$ and $L^1 = L^1(G)$ as spaces of functions on G, so it makes sense to talk about $L^\infty \cap L^1$ and $L^\infty + L^1$.
- We have inclusions $L^\infty \cap L^1 \subseteq L^p \subseteq L^\infty + L^1$;
- (Riesz-Thorin) If $T : L^\infty + L^1 \to L^\infty + L^1$ is linear, and restricts to give map $L^1 \to L^1$ and $L^\infty \to L^\infty$, then

$$\|T : L^p \to L^p\| \leq \|T : L^\infty \to L^\infty\|^{1-1/p} \|T : L^1 \to L^1\|^{1/p}.$$

Convolution action

For $\mu \in M(G)$, we have a convolution action of μ on $L^1(G)$ and $L^\infty(G)$.
Interpolating gives the convolution action on $L^p(G)$.
However, from an abstract point of view, this is actually a little odd:

- $M(G)$ acts entirely naturally on $L^1(G)$ as $M(L^1(G)) = M(G)$.
- $L^\infty(G)$ is the dual space of $L^1(G)$.
- So we have the dual (technically, adjoint) action of $M(G)$ on $L^\infty(G)$.
- This *is not* the usual convolution action of $M(G)$ on $L^\infty(G)$.

So, if we are to generalise this, we need a new idea.

Analogous ideas for $A(G)$

So for the Fourier algebra, we might proceed as follows:

- Find some way to embed $A(G)$ and $VN(G)$ into a Hausdorff topological space;
- so we can form $VN(G) \cap A(G)$ and $VN(G) + A(G)$.
- Use the complex interpolation method with parameter $1/p$.
- Find some module action of $MA(G)$ on $VN(G)$ which agrees with the standard action of $MA(G)$ on $A(G)$ in $VN(G) \cap A(G)$.
- Then do the same again at the Operator Space level!

Bizarrely the last point suggests a novel way to get the module actions.

Non-commutative L^p spaces

Using the complex interpolation method applied to von Neumann algebras is a well established way to construct *non-commutative* L^p spaces, say $L^p(VN(G))$.

- If G is discrete, then $VN(G)$ admits a finite trace: $\varphi : x \mapsto (x\delta_e|\delta_e)$ for $x \in VN(G)$. Then $L^p(VN(G))$ is the completion of $VN(G)$ under the norm $\|x\|_p = \varphi(|x|^p)^{1/p}$, where $|x| = (x^*x)^{1/2}$.
- In general, $VN(G)$ only admits a weight, which satisfies $\varphi(\lambda(f * g)) = (f * g)(e)$ for, say, $f, g \in C_{00}(G)$.
- If G is compact, then

$$VN(G) \cong \prod_i \mathbb{M}_{n_i}, \qquad L^p(VN(G)) \cong \ell^p - \bigoplus_i S^p_{n_i},$$

where S^p_n is $\mathbb{M}_n$ equipped with the pth Schatten-class norm.

Operator Space Structures

For further details on the complex interpolation approach to non-commutative L^p spaces, see [Kosaki], [Terp] and [Izumi].
Eventually we want a *natural* Operator Space structure on $L^p(VN(G))$:

- Under favourable circumstances, we except that non-commutative L^2 is a Hilbert space;
- A Hilbert space is *self-dual*;
- The unique Operator Space structure on a Hilbert space with this property is Pisier's Operator Hilbert Space;
- To recover this, we need to interpolate between a von Neumann algebra M and M_*^{op}, see [Pisier].
- Here M_*^{op} is the predual of M equipped with the *opposite* structure,

$$\|(\omega_{ij})\|_{M_*^{\mathrm{op}}} = \|(\omega_{ji})\|_{M_*}. \qquad \big((\omega_{ij}) \in \mathbb{M}_n(M_*)\big).$$

Standard position

The trick, following [Junge, Ruan, Xu], is to use that $VN(G)$ is in *standard position* on $L^2(G)$. There is a map

$$J : L^2(G) \to L^2(G); \quad J\xi(s) = \overline{\xi(s^{-1})}\nabla(s)^{-1/2}.$$

Then

$$JVN(G)J = VN(G)' = VN_r(G), \quad JVN_r(G)J = VN(G).$$

Here $VN_r(G)$ is the right group von Neumann algebra, generated by the right regular representation.
Then we have a normal completely isometric isomorphism

$$\phi : VN(G) \to VN_r(G)^{\mathrm{op}}; \quad x \mapsto Jx^*J.$$

The pre-adjoint is

$$\phi_* : A_r(G)^{\mathrm{op}} \to A(G); \quad \omega_{\xi,\eta} \mapsto \omega_{J\eta,J\xi}.$$

The spaces

So if we privilege $A(G)$, it makes sense to interpolate between $VN_r(G)$ and $A_r(G)^{\mathrm{op}} \cong A(G)$.
If we follow Terp's interpolation method through, then in $A(G) \cap VN_r(G)$, we find that

$$a = \rho(\nabla^{-1/2} a) \qquad \big(a \in A(G) \cap C_{00}(G)^2\big).$$

Here ρ is the right regular representation.
Obviously $A(G)$ acts on itself by pointwise multiplication; the above suggests that $A(G)$ should act on $VN_r(G)$ in such a way that

$$a \cdot \rho(f) = \rho(af) \qquad (a \in A(G), f \in L^1(G)).$$

The module action

Let K be the linear version of J,

$$K\xi(s) = \xi(s^{-1})\nabla(s)^{-1/2} \qquad (\xi \in L^2(G), s \in G).$$

Then we can form a normal completely isometric isomorphism,

$$\hat{\phi} : VN(G) \to VN_r(G); \quad x \mapsto KxK,$$

and similarly this drops to a complete isometry

$$\hat{\phi}_* : A_r(G) \to A(G); \quad \omega_{\xi,\eta} \mapsto \omega_{K\xi,K\eta}.$$

Now, naturally, $VN_r(G)$ is a completely contractive $A_r(G)$ module, and so using $(\hat{\phi}_*)^{-1}$, we see that $VN_r(G)$ is a completely contractive $A(G)$ module, which satisfies

$$a \cdot \rho(f) = \rho(af) \qquad (a \in A(G), f \in L^1(G)).$$

The spaces

So we interpolate between $VN_r(G)$ and $A(G)$, leading to $L^p(\hat{G})$ say. If G is abelian, this *is* the L^p space of the dual group $\hat{G}$.
We interpolate the module actions, so $L^p(\hat{G})$ becomes a (completely contractive) $A(G)$ module. A similar argument establishes that $MA(G)$ and $M_{cb}A(G)$ act on $L^p(\hat{G})$, extending the action of $A(G)$.
[Izumi] implies that there is a natural dual pairing between $L^p(\hat{G})$ and $L^{p'}(\hat{G})$, where $p^{-1} + p'^{-1} = 1$.
By taking the adjoint, we get a map

$$L^p(\hat{G}) \widehat{\otimes} L^{p'}(\hat{G}) \to MA(G)^*,$$

which actually takes values in Q, showing that the action of $MA(G)$ on $L^p(\hat{G})$ is weak*-weak* continuous. The same idea applies to $M_{cb}A(G)$.

Links with work of Forrest, Lee and Samei

These authors interpolate between $VN(G)$ and $A(G)^{\mathrm{op}}$, leading to $OL^p(VN(G))$, say. Then they define

$$L^p(VN(G)) = \begin{cases} OL^p(VN(G))^{\mathrm{op}} & : 1 < p \leq 2, \\ OL^p(VN(G)) & : 2 \leq p < \infty. \end{cases}$$

An $A(G)$-module action is defined by working with $L^2(G) \cong L^2(VN(G))$, and using row and column Hilbert spaces, and interpolation.

Theorem

For $1 < p \leq 2$, $L^p(VN(G)) \cong L^p(\hat{G})$, as $A(G)$ modules, by interpolating the map

$$\phi : VN(G) \to VN_r(G)^{\mathrm{op}}; \quad x \mapsto Jx^*J.$$

For $2 \leq p < \infty$, $L^p(VN(G)) \cong L^p(\hat{G})$, this time using $\hat{\phi}$.

The theorem

Let (p_n) be a sequence in $(1, \infty)$ tending to 1. Let

$$E = \ell^2 - \bigoplus_n L^{p_n}(\hat{G}).$$

Let $\pi : MA(G) \to \mathcal{B}(E)$ be the diagonal action.

Theorem

The homomorphism π is an isometric, weak-weak*-continuous isomorphism onto its range, which is equal to the idealiser of $\pi(A(G))$ in $\mathcal{B}(E)$,*

$$\pi(MA(G)) = \left\{ T \in \mathcal{B}(E) : \begin{matrix} T\pi(a), \pi(a)T \in \pi(A(G)) \\ (a \in A(G)) \end{matrix} \right\}.$$

Completely bounded case

Give each $L^{p_n}(\hat{G})$ its canonical operator space structure. Then

$$\ell^\infty - \bigoplus_n L^{p_n}(\hat{G}), \quad \ell^\infty - \bigoplus_n L^{p_n}(\hat{G})^*,$$

both carry natural operator space structures. Then

$$\ell^1 - \bigoplus_n L^{p_n}(\hat{G}) \subseteq \Big(\ell^\infty - \bigoplus_n L^{p_n}(\hat{G})^* \Big)^*,$$

and hence has an operator space structure. Then

$$E = \ell^2 - \bigoplus_n L^{p_n}(\hat{G})$$

is a complex interpolation between the ℓ^∞ and ℓ^1 direct sums, and so has an operator space structure, see [Pisier] and [Xu].

Completely bounded case (cont.)

$$E = \ell^2 - \bigoplus_n L^{p_n}(\hat{G})$$

Let $\pi : M_{cb}A(G) \to \mathcal{CB}(E)$ be the diagonal map.

Theorem

The homomorphism π is a completely isometric, weak*-weak*-continuous isomorphism onto its range, which is equal to the idealiser of $\pi(A(G))$ in $\mathcal{CB}(E)$,

$$\pi(M_{cb}A(G)) = \left\{ T \in \mathcal{CB}(E) : \begin{array}{c} T\pi(a), \pi(a)T \in \pi(A(G)) \\ (a \in A(G)) \end{array} \right\}.$$

Notice that E, and the $A(G)$ action, is the same in either case. The idealiser in $\mathcal{B}(E)$ is $MA(G)$, while the idealiser in $\mathcal{CB}(E)$ is $M_{cb}A(G)$.

Figa-Talamanca–Herz algebras

The adjoint of the action of $A(G)$ on $L^p(\hat{G})$ gives a map

$$\pi_*^p : L^p(\hat{G}) \widehat{\otimes} L^{p'}(\hat{G}) \to A(G)^* = VN(G),$$

which takes values in $C_\lambda^*(G)$.
Let $A_p(\hat{G})$ be the image of π_*^p in $C_\lambda^*(G)$, equipped with the norm as a quotient of $L^p(\hat{G}) \widehat{\otimes} L^{p'}(\hat{G})$.

Theorem

$A_2(\hat{G}) = \lambda(L^1(G))$ in $C_\lambda^*(G)$, with the $A_2(\hat{G})$ norm being equal to the $L^1(G)$ norm.

Again, if G is abelian, then $A_2(\hat{G}) = A(\hat{G}) = L^1(G)$.

Is $A_p(\hat{G})$ an algebra for $p \neq 2$?