Outline

Group representations and algebras

Fourier algebras and operator spaces

Generalising for Figa-Talamanca–Herz algebras

Locally compact groups

A group G which is a locally compact space in such a way that the maps

$$G \times G \to G; (s, t) \mapsto st, \quad G \to G; s \mapsto s^{-1},$$

are continuous is called a *locally compact group*.

Examples:

- Any discrete group: for example, $\mathbb{Z}$ or $SL(2, \mathbb{Z})$;
- Various abelian groups: $\mathbb{R}$ or $\mathbb{T}$;
- Lie groups, such as $SL(3, \mathbb{R})$, $SO(3)$ or $GL(n, \mathbb{R})$.

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Haar measure

- The key property of locally compact groups which separates them from other topological groups is the Haar measure: a left invariant regular measure.
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$L^1(G)$ algebra

Given a Haar measure, we can define the *convolution* product on the Banach space $L^1(G)$ by

$$(f \star g)(s) = \int f(t)g(t^{-1}s)\, dt \qquad (f, g \in L^1(G), s \in G).$$

On $L^1(\mathbb{R})$, this is just the usual notion of convolution.

- $L^1(G)$ and $L^1(H)$ are *isometrically* isomorphic algebras if and only if G and H are isomorphic.
- However, for example, $L^1(C_4)$ and $L^1(C_2 \times C_2)$ are isomorphic, but not isometrically.

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Group representations

Let E be a *reflexive* Banach space. Write $\operatorname{inv}\mathcal{B}(E)$ for the invertible linear maps on E.

- For us, a *group representation* shall be a group homomorphism $\pi : G \to \operatorname{inv}\mathcal{B}(E)$ such that $\pi(s)$ is an isometry for each $s \in G$.
- We insist that for each $x \in E$, the map $G \to E; s \mapsto \pi(s)(x)$ is continuous.
- We can form an algebra homomorphism $\hat{\pi} : L^1(G) \to \mathcal{B}(E)$ by integration,

$$\hat{\pi}(f)(x) = \int_G f(s)\pi(s)(x)\, ds \qquad (f \in L^1(G), x \in E).$$

- We can form a categorical sense of *equivalence*: two representations $\pi : G \to \operatorname{inv}\mathcal{B}(E)$ and $\theta : G \to \operatorname{inv}\mathcal{B}(F)$ are equivalent when there is an isomorphism $T : E \to F$ with $T\pi(s) = \theta(s)T$ for each $s \in G$.

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$A(\pi)$ spaces

- This is a poor sense of equivalence though: for example, the trivial representations on non-isomorphic Banach spaces are not equivalent!
- Instead, we consider the bilinear map

$$\Pi : E' \times E \to C(G);\ (\mu, x) \mapsto \big(s \mapsto \langle \mu, \pi(s)(x) \rangle\big).$$

- This becomes linear by using a tensor product,

$$\Pi : E' \widehat{\otimes} E \to C(G).$$

- We define $A(\pi)$ to be the *co-image* of Π. That is, $A(\pi)$ as a vector space is the image of Π inside $C(G)$, but we give $A(\pi)$ the norm that comes from identifying this image with $E' \widehat{\otimes} E / \ker \Pi$.

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$A(\pi)$ spaces (cont.)

More concretely, $A(\pi)$ is those continuous functions $f : G \to \mathbb{C}$ such that there exist sequences $(\mu_n) \subseteq E'$ and $(x_n) \subseteq E$ with $\sum \|\mu_n\| \|x_n\| < \infty$ and

$$f(s) = \sum_{n=1}^{\infty} \langle \mu_n, \pi(s)(x_n) \rangle \qquad (s \in G).$$

We give $A(\pi)$ the norm

$$\|f\|_{A(\pi)} = \inf \left\{ \sum \|\mu_n\| \|x_n\| \right\}.$$

Then, for example, a representation π is (almost) trivial if and only if $A(\pi) = \mathbb{C}$. So studying $A(\pi)$ gives a better notion of equivalence.

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$A(\pi)$ as an algebra

- For some representations π, $A(\pi)$ is even a subalgebra of $C(G)$.
- Let $1 < p < \infty$, and let $\lambda_p : G \to \text{inv}\,\mathcal{B}(L^p(G))$ be the *left-regular representation* given by translation:

$$\lambda_p(s)(f) = g,\ g(t) = f(s^{-1}t) \qquad (f \in L^p(G), s, t \in G).$$

- Then $A_p(G) := A(\lambda_p)$ is a (Banach) algebra: called a Figa-Talamanca–Herz algebra.
- The proof that $A_p(G)$ is an algebra relies on “Fell's absorption principle”. That is, tensoring with the left-regular representation gives you nothing new, as long as the other Banach space is a “p-space”.

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Dual of $A(\pi)$

- As $A(\pi)$ is a quotient of $E'\widehat{\otimes}E$, we have that the dual of $A(\pi)$ is a subspace of the dual of $E'\widehat{\otimes}E$. As E is reflexive, we have that $(E'\widehat{\otimes}E)' = \mathcal{B}(E)$, for the duality

$$\langle T, \mu \otimes x\rangle = \langle \mu, T(x)\rangle \qquad (T \in \mathcal{B}(E), \mu \otimes x \in E'\widehat{\otimes}E).$$

- The dual of $A_p(G)$ is an algebra, denoted by $PM_p(G)$. It is the weak-operator closed algebra generated by the group of operators $\{\lambda_p(s) : s \in G\} \subseteq \mathcal{B}(L^p(G))$.
- When $p = 2$, $L^2(G)\widehat{\otimes}L^2(G)$ is just the trace-class operators on $L^2(G)$, and $PM_2(G) = VN(G)$ is the *group von Neumann algebra* of G. Then $A_2(G) = A(G)$ is the *Fourier algebra* of G, studied first by Eymard.
- Every $f \in A(G)$ is given as

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for some $x, y \in L^2(G)$. Notice we don't need a sum here.

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Why Fourier?

- Recall the Fourier transform, $\mathcal{F} : L^1(\mathbb{R}) \to C_0(\mathbb{R})$,
$$\mathcal{F}(f)(s) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(s) e^{-ist}\, dt.$$
- Then $A(\mathbb{R})$ is simply the image of $\mathcal{F}$: recall that $\mathcal{F}$ takes the convolution product to the pointwise product. So $VN(\mathbb{R})$ is simply $L^\infty(\mathbb{R})$.
- This idea works for any *abelian* locally compact group G. The group of all *characters* $G \to \mathbb{T}$ is called the *dual group*, denoted by $\hat{G}$. The Pontrjagin duality theorem tells us that $\hat{\hat{G}} = G$ canonically.
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Homological properties

- A group G is *amenable* when there is a *mean* on $L^\infty(G)$. That is, a state m on $L^\infty(G)$ which is left-invariant.
- All compact and abelian groups are amenable.
- There is a notion of *amenable* for Banach algebras as introduced by Johnson.
- The group algebra $L^1(G)$ is amenable if and only if G is amenable.
- So $A(G) = L^1(\hat{G})$ is amenable for all abelian G.
- Problem: $A(SO(3))$ is not amenable, but $SO(3)$ is certainly compact!
- Runde: $A(G)$ is amenable if and only if G contains an abelian subgroup of finite index.

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- Runde: $A(G)$ is amenable if and only if G contains an abelian subgroup of finite index.

Operator spaces and complete boundedness

- Let H be a Hilbert space and identify

$$\mathbb{M}_n(\mathcal{B}(H)) = \mathcal{B}(H \oplus \cdots \oplus H).$$

- We hence have a norm on $\mathbb{M}_n(\mathcal{B}(H))$.
- Given a map $T : \mathcal{B}(H) \to \mathcal{B}(H)$, let

$$(T)_n : \mathbb{M}_n(\mathcal{B}(H)) \to \mathbb{M}_n(\mathcal{B}(H)); \big(a_{ij}\big) \mapsto \big(T(a_{ij})\big).$$

- We say that T is *completely-bounded* when

$$\|T\|_{cb} := \sup_n \|(T)_n\| < \infty.$$

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Ruan's Theorem

- When $E \subseteq \mathcal{B}(H)$ is an operator space, is E'? How can we embed E' into $\mathcal{B}(H)$?
- Ruan proved an abstract characterisation of an operator space.
- Let E be a Banach space, and for each n, let $\|\cdot\|_n$ be a norm on the vector space $\mathbb{M}_n(E)$, such that:

$$\left\| \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \right\|_{n+m} = \max\left(\|A\|_n, \|B\|_m\right), \quad \|\alpha A \beta\|_n \leq \|\alpha\| \|A\|_n \|\beta\|,$$

  where $\alpha, \beta \in \mathbb{M}_n(\mathbb{C}) = \mathcal{B}(H_n)$, where H_n is an n-dimensional Hilbert space.
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Mapping and dual spaces

- We write $\mathcal{CB}(E, F)$ for the space of completely bounded maps between two operator spaces E and F.
- We can turn $\mathcal{CB}(E, F)$ into an operator space (by Ruan's Theorem) by setting

$$\mathbb{M}_n(\mathcal{CB}(E, F)) = \mathcal{CB}(E, \mathbb{M}_n(F)).$$

- As the dual space E' is simply $\mathcal{CB}(E, \mathbb{C})$ (this is a lemma) then we get an operator space structure on E'. Without Ruan's Theorem, this is very hard to see!
- Everything we expect to work does: the canonical map $E \to E''$ is a complete isometry, and a map $T : E \to F$ is a complete isometry if and only if $T' : F' \to E'$ is a complete quotient map.

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Ruan's Fourier Algebra Theorem

- The dual of the Fourier algebra is $VN(G)$, which as a C^*-algebra carries a natural operator space structure.
- So $A(G)$ gets an operator space structure, by treating it as a subspace of the dual of $VN(G)$.
- In fact, $A(G)$ is a completely contractive Banach algebra:
- that is, $A(G)$ acts on itself in a completely contractive way, by the Left Regular representation.
- We can define a "completely bounded" notion of amenable for a completely contractive Banach algebra.
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Recall: Figa-Talamanca–Herz algebras

Recall the algebra $A_p(G)$, which is a (non closed) subalgebra of $C_0(G)$, and is the predual of a space of operators $PM_p(G) \subseteq \mathcal{B}(L^p(G))$.

- Idea: if an operator space is a subspace of $\mathcal{B}(H)$ for a Hilbert space H, then
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- Then $A_p(G)$ will become a p-operator space by duality: does Ruan's Theorem hold in this case?

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Input from Pisier and Le Merdy

- Pisier has already studied a notion of p-completely bounded maps.
- Using this, Le Merdy essentially found a definition of p-operator space, and proved a version of Ruan's Representation Theorem.
- However, we need to move to a larger class of Banach spaces. Let SQ_p be the collection of quotients of subspaces of L^p spaces. Notice that SQ_2 is simply the class of Hilbert spaces.
- Then a p-operator space is a subspace of $\mathcal{B}(X)$ for some $X \in SQ_p$.
- For $n \geq 1$, we norm $\mathbb{M}_n$ by identifying this with $\mathcal{B}(\ell_n^p)$. Then $\mathbb{M}_n(X)$ is normed by identifying with $\mathcal{B}(\ell_n^p(X))$.

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Problems ahead

- The main problem comes from the following result: if E is a p-operator space, then E' can be embedded into $\mathcal{B}(\ell^p(I))$ for some set I.
- So as soon as we move to dual spaces, we can dispense with SQ_p and just work with L^p spaces.
- However, if $\kappa : E \to E''$ is still a complete isometry, then every E arises as a subspace of $\mathcal{B}(L^p)$.
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$A_p(G)$ algebras are well behaved!

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- However, $A_p(G)$ is defined to be a quotient of $L^p(G)\widehat{\otimes}L^{p'}(G)$, which is the predual of $\mathcal{B}(L^p(G))$.
- So we could give $A_p(G)$ a p-operator space structure making it a complete quotient of $L^p(G)\widehat{\otimes}L^{p'}(G)$.
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It all works, just!

- There are various other facts, like commutation relations, which work for $p = 2$, and luckily hold for $A_p(G)$, at least when G is amenable.
- When $A_p(G)$ is amenable, it has a bounded approximate identity, so by Leptin's Theorem, G is amenable.
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Other properties, other approaches

- The impression we get is that when G is amenable, then $A_p(G)$ behaves well, but that otherwise we are in trouble.
- Lambert, Neufang and Runde did something similar by finding a fairly natural *operator space* structure on $\mathcal{B}(L^p(G))$.
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Multipliers

- A *multiplier* of a commutative Banach algebra $\mathcal{A}$ is a linear map $m : \mathcal{A} \to \mathcal{A}$ with $m(ab) = am(b)$. We write $\mathcal{M}(\mathcal{A})$ for the collection of multipliers.
- By properties of $A_p(G)$, one can show that every multiplier is bounded, and is given by pointwise multiplication by some continuous function.
- de Canniere and Haagerup introduced the notion of a *completely bounded multiplier* without explicitly using operator spaces, although the definition is as expected, leading to $\mathcal{M}_{cb}(A(G))$.
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Future ideas

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