Fourier algebras to quantum groups

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Compact spaces

- Given a space X, how might we study X?
- We might study maps on X, or maps from X to something else.
- As an analyst, I'm interested in "distance", "continuity", "topology".
- So I might look at the space $C(X,\mathbb{C}) = C(X)$ of continuous functions from X to the complex numbers.
- Complex because I'm also interested in algebra.

Theorem (Urysohn)

If X is a normal (in particular, compact Hausdorff) topological space then give disjoint closed subsets A, B there is a continuous function $f : X \to [0,1]$ with $f \equiv 0$ on A and $f \equiv 1$ on B.

So $C(X)$ "sees the topology".

C^*-Algebras

From now on, X is a compact (Hausdorff) space.

- We turn $C(X)$ into a *vector space* via pointwise operations.
- We turn $C(X)$ into an *algebra* via pointwise operations.
- We give $C(X)$ a *norm* via $\|f\| = \sup_{x\in X} |f(x)|$.
  1. $\|f\| \geq 0$ and $= 0$ iff $f = 0$;
  2. $\|tf\| = |t|\|f\|$ for scalars t;
  3. $\|f + g\| \leq \|f\| + \|g\|$.
- $d(f, g) = \|f - g\|$ defines a *metric*.
- Then $C(X)$ is *complete*.
- We give $C(X)$ an involution $f \mapsto f^*$ via pointwise complex conjugation.
- C^*-identity: $\|f^*f\| = \|f\|^2$.

Abstract C^*-Algebras

- A complex algebra A,
- which has a *norm* with $\|ab\| \leq \|a\|\|b\|$,
- which is *complete*,
- which satisfies the C^*-condition: $\|a^*a\| = \|a\|^2$.

Theorem (Gelfand)

Let A be a unital commutative C^*-algebra. Then there is a compact Hausdorff space X such that A is isomorphic to $C(X)$.

- "isomorphic" means all the structure is preserved.

Gelfand theory

- A *character* on A is a non-zero *homomorphism* $\phi : A \to \mathbb{C}$.
- (Characters are always continuous, indeed, $\|\phi\| \leq 1$ always.)
- The collection of all characters forms our space X.
- Give X the "topology of pointwise convergence".
- Little exercise: If X is compact, then every character on $C(X)$ is of the form: "evaluate at some point of X".

Example

Let X be a non-locally compact metric space. This is a "nice" space, and we can form $C_b(X)$ the algebra of *bounded* continuous functions. The "character space" of $C_b(X)$ is then the *Stone-Cech compactification* of X, the largest compact space containing a dense copy of X.

A little category theory

Suppose X and Y are compact, and $\alpha : X \to Y$ is a continuous map. Then we get an algebra homomorphism $\alpha^* : C(Y) \to C(X)$ given by

$$\alpha^*(f)(x) = f(\alpha(x)) \qquad (f \in C(Y), x \in X).$$

Theorem

Let $\phi : C(Y) \to C(X)$ be a unital $*$-homomorphism. Then there is a continuous map $\alpha : X \to Y$ with $\phi = \alpha^*$.
In this way, the category of compact Hausdorff spaces and the opposite to the category of unital commutative C^*-algebras are isomorphic.

To construct α, just observe that ϕ, composed with evaluation at $x \in X$, gives a character on $C(Y)$, that is, a point $\alpha(x) \in Y$.

A dictionary...

Will I have time??

Towards groups

Suppose our spaces have extra structure: for example, let G be a compact group.

- If we form $C(G)$ then we lose a huge amount of information.
- For example, if G is finite $C_0(G)$ only "remembers" the cardinality of G.

Instead, let's take a representation theory approach.

- Fix a finite group G.

Inner-product spaces

- Consider (column) vectors in $\mathbb{C}^n$ with the *inner-product*

$$(f|g) = \sum_{i=1}^{n} f_i \overline{g_i} \qquad (f, g \in \mathbb{C}^n).$$

- This induces a norm by $\|f\| = \sqrt{(f|f)}$.
- Linear maps on $\mathbb{C}^n$ "are" matrices in $\mathbb{M}_n(\mathbb{C})$.
- Have the notion of the "hermitian transpose" or "adjoint" of a matrix. Satisfies: $(Af|g) = (f|A^*g)$.
- Instead of using $1, 2, \cdots, n$ as a basis, use G as a basis.
- Write $L^2(G)$ for $\mathbb{C}^{|G|}$ with this inner-product and norm.
- Write $\mathcal{L}(L^2(G))$ for the matrix algebra $\mathbb{M}_{|G|}$ with the *operator norm*

$$\|A\| = \max\Big\{\|Af\| : f \in L^2(G), \|f\| \leq 1\Big\}.$$

Left regular representation

- Form the space $L^2(G)$. For $f \in L^2(G), s \in G$ let

$$\lambda(s)(f) \in L^2(G) \text{ be the function } t \mapsto f(s^{-1}t).$$

- Then $\lambda(s)$ is linear.
- Also $\lambda(s)$ is an *isometry*:

$$\|\lambda(s)(f)\| = \Big(\sum_{t\in G} |f(s^{-1}t)|^2 \Big)^{1/2} = \Big(\sum_{t\in G} |f(t)|^2 \Big)^{1/2} = \|f\|.$$

- $\lambda(s)\lambda(t) = \lambda(st)$; $s \mapsto \lambda(s)$ is a homomorphism.
- $\lambda(s^{-1}) = \lambda(s)^*$.

Group algebras

- Consider the group algebra $\mathbb{C}[G]$.
- This has each element of G as a basis element, and we then take the $\mathbb{C}$ linear span. Just a different take on $\mathbb{C}^{|G|}$.
- Typical element: $a = \sum_{s\in G} a_s s$ with $(a_s) \subseteq \mathbb{C}$.
- Turn into an algebra by "multiplying in G":

$$ab = \Big(\sum_{s\in G} a_s s\Big)\Big(\sum_{t\in G} b_t t\Big) = \sum_{s,t} a_s b_t st = \sum_s \Big(\sum_t a_t b_{t^{-1}s}\Big) s,$$

  "convolution product".
- $\mathbb{C}[G]$ is a $*$-algebra:

$$\Big(\sum_s a_s s\Big)^* = \sum_s \overline{a_s} s^{-1}.$$

The group C^*-algebra

- Left-regular representation: represent $\mathbb{C}[G]$ on $L^2(G)$,

$$\sum_s a_s s \mapsto \sum_s a_s \lambda(s).$$

- This is a $*$-homomorphism $\mathbb{C}[G] \to \mathcal{L}(L^2(G))$.
- Let $C_r^*(G)$ be the image: "reduced group C^*-algebra".
- So this is just a copy of $\mathbb{C}[G]$, but now equipped with the norm coming from $\mathcal{L}(L^2(G))$.

Dual spaces and norms.

- Let E be a normed vector space.
- The *dual space* of E, denoted E^*, is the vector space of all continuous linear maps $\mu : E \to \mathbb{C}$.
- A linear map is continuous if and only if it is *bounded*: for some $K > 0$ we have $|\mu(x)| \leq K\|x\|$ for all $x \in E$.
- $\|\mu\| = \sup\{|\mu(x)| : \|x\| \leq 1\}$ defines a *complete norm* on E^*.

The "Fourier Algebra"

- Let's consider $C_r^*(G)^*$ the dual space.
- Call this $A(G)$, the Fourier Algebra.
- As a vector space, is just $\mathbb{C}^{|G|}$ again; but has a new norm!
- (Slightly big stick $\implies$) For any $\omega \in A(G)$, we can find vectors $f, g \in L^2(G)$ so the functional is

$$\omega : C_r^*(G) \to \mathbb{C}; \quad x \mapsto (x(f)|g).$$

- Furthermore, $\|\omega\| = \min \|f\|\|g\|$.
- As $\{\lambda(s) : s \in G\}$ spans $C_r^*(G)$, any such ω is determined by the values

$$\omega(s) := \omega(\lambda(s))$$

- So can view $A(G)$ as (being equal to) $C(G)$, but with a new norm.

Why Fourier, I

Fourier transform: for f a function on $\mathbb{T} = [0, 1]$,

$$\mathcal{F}(f)(n) = \int_0^1 f(t) e^{2\pi i n t} \, dt \qquad (n \in \mathbb{Z}).$$

Get a "sequence" aka a function $\mathbb{Z} \to \mathbb{C}$.

- Pontryagin duality theory: a "character" on G is a (continuous) group homomorphism $G \to \mathbb{T}$.
- E.g. every character on $\mathbb{Z}$ is of the form $n \mapsto e^{2\pi i n t}$ for some $0 \leq t < 1$.
- E.g. every character on $\mathbb{T}$ is of the form $[0, 1] \ni t \mapsto e^{2\pi i n t}$ for some $n \in \mathbb{Z}$.
- Write $\widehat{G}$ for the collection of characters on G, made into a (semi)group for the pointwise operations.
- If G a finite abelian group, then $\widehat{\widehat{G}} = G$ in a canonical way.

Why Fourier, II

Let G be a finite abelian group. Define $\mathcal{F} : L^2(G) \to L^2(\widehat{G})$

$$\mathcal{F}(f)(\gamma) = \sum_{s\in G} f(s)\gamma(s) \qquad (f \in L^2(G), \gamma \in \widehat{G}).$$

- (Plancherel:) This sets up a unitary between the Hilbert spaces $L^2(G)$ and $L^2(\widehat{G})$.
- So we can conjugate to move an operator on $L^2(\widehat{G})$ to an operator on $L^2(G)$; say $x \mapsto \mathcal{F}^{-1}x\mathcal{F}$.
- This gives a $*$-isomorphism between $C^*_r(\widehat{G})$ and $C(G)$; here $C(G)$ acts on $L^2(G)$ by mutliplication.
- So also get an isometry between the dual spaces: $A(\widehat{G}) \cong L^1(G)$.
- $L^1(G)$ is the functions $G \to \mathbb{C}$ with the norm

$$\|f\|_1 = \sum_{s\in G} |f(s)|.$$

- $L^1(G)$ is an algebra for the convolution product, $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

General case: why an algebra?

- The ($\mathbb{C}$-linear) tensor product of algebras is easy to interpret:

$$\mathbb{C}[G] \otimes \mathbb{C}[G] \cong \mathbb{C}[G \times G].$$

- There is an injective $*$-homomorphism

$$\Delta : \mathbb{C}[G] \to \mathbb{C}[G \times G]; \quad \Delta(s) = s \otimes s \qquad (s \in G).$$

- As a vector space, $A(G) = \mathbb{C}[G]^*$, and so the "transpose" of Δ gives a map

$$\Delta_* : A(G) \otimes A(G) \to A(G).$$

- Notice that as $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$ the binary product given by Δ_* is associative.
- The product is "pointwise": $\Delta_*(\omega_1 \otimes \omega_2)(s) = (\omega_1 \otimes \omega_2)(\Delta(s)) = (\omega_1 \otimes \omega_2)(s \otimes s) = \omega_1(s)\omega_2(s)$.

Remembering the group

- So $A(G) = C(G)$ as an algebra.
- But with a different norm!

Theorem (Walter)

Let $A(G)$ be isometrically isomorphic to $A(H)$. Then G is isomorphic to either H or the opposite group to H.

$C(G)$ fails to remember G; but $A(G)$ does rememeber G.

Discrete groups

Now let G be a discrete, maybe infinite, group.

- We now let $L^2(G)$ be the space of functions $f : G \to \mathbb{C}$ with
$$\|f\|_2^2 = \sum_{s \in G} |f(s)|^2 < \infty.$$
- This is still an inner-product space, $(f|g) = \sum_s f(s)\overline{g(s)}$.
- We now get a *Hilbert Space*: a *complete* inner-product space.
- Now let $\mathcal{L}(L^2(G))$ be the space of *continuous* linear maps on $L^2(G)$.
- A linear map is continuous if and only if it is bounded: there is $K > 0$ with $\|T(f)\| \leq K\|f\|$ for all f. The minimal K is the norm $\|T\|$. $\mathcal{L}(L^2(G))$ becomes a complete normed space.
- (Riesz) Any $T \in \mathcal{L}(L^2(G))$ has an adjoint: $T^* \in \mathcal{L}(L^2(G))$ such that $(T(f)|g) = (f|T^*(g))$.

Group algebras

- Can still form $\mathbb{C}[G]$, where we just consider families (a_s) which are non-zero for only finitely many choices of s.
- Still have the representation $\mathbb{C}[G] \to \mathcal{L}(L^2(G))$, which is injective.
- The image is a subalgebra of $\mathcal{L}(L^2(G))$, but it's not (in general) *closed*.
- So we can take the closure (add in all limit points) to get $C_r^*(G)$ the *reduced group C^*-algebra*.
- We could instead consider all possible C^*-algebra norms on $\mathbb{C}[G]$ and take the biggest one.
- Same as considering all unitary representations of G on any Hilbert space.
- This gives $C^*(G)$.
- G is *amenable* if and only if $C_r^*(G) = C^*(G)$.
- $\mathbb{Z}$ is amenable; $\mathbb{F}_2$ is not.

Different topologies

- The norm topology on $\mathcal{B}(L^2(G))$ is very strong.
- A suitable "pointwise" topology is the *strong operator topology* (SOT): $T_n \to T$ if and only if

$$\|T_n(f) - T(f)\| \to 0 \qquad (f \in \ell^2(G)).$$

- Suppose we close up $\mathbb{C}[G]$ in the SOT? Then we get the *group von Neumann algebra* $VN(G)$.
- Consider the functionals $VN(G) \to \mathbb{C}$, which are SOT continuous.
- (Slightly big stick $\implies$) For any such ω, we can find vectors $f, g \in \ell^2(G)$ so the functional is

$$\omega : VN(G) \to \mathbb{C}; \quad x \mapsto (x(f)|g).$$

- (Can you write down a functional not of this form?)

The "Fourier Algebra"

- Let $A(G)$ be the collection of all these functionals.
- Each $\omega \in A(G)$ defines a function

$$G \to \mathbb{C}; \quad s \mapsto \omega(\lambda(s)) =: \omega(s).$$

- Given the special form of such ω, a calculation shows that $\omega(s) \to 0$ as $s \to \infty$.
- So $A(G)$ is a subspace of $C_0(G)$, but is not all of $C_0(G)$.
- But it has a norm coming from action on $VN(G)$.

Why an algebra?

- We need to *complete* our tensor products.
- For Hilbert spaces, we define the obvious inner-product on $L^2(G) \otimes L^2(G)$, and then we complete the inner-product space to get a Hilbert space.
- In this concrete setting, we get $L^2(G \times G)$.
- For von Neumann algebras, we have that $VN(G)$ acts on $L^2(G)$, so $VN(G) \otimes VN(G)$ acts on $L^2(G \times G)$, and the SOT closure gives $VN(G) \overline{\otimes} VN(G)$, which is just $VN(G \times G)$.
- The induced norm on $A(G) \otimes A(G)$ is more mysterious and leads to the area called "Operator Spaces".
- Can still form $\Delta : VN(G) \rightarrow VN(G) \overline{\otimes} VN(G); \lambda(s) \mapsto \lambda(s) \otimes \lambda(s)$ which induces an associative product $A(G) \otimes A(G) \rightarrow A(G)$.

Even for locally compact groups

Let G be a locally compact group.

- Need to replace sums with *integrals*.
- A topological group is locally compact if and only if it admits a "nice" left-invariant measure.
- Replace $\mathbb{C}[G]$ with e.g. continuous functions with compact support, still with the convolution product.
- Run the same programme.

Theorem (Eymard, Walter)

$A(G)$ is a Banach algebra of functions, dense in $C_0(G)$. Every character on $A(G)$ is given by point evaluation at some point of G. The character space of $A(G)$ is G.
$A(G)$ is isometrically isomorphic to $A(H)$ if and only if G is isomorphic to H (or its opposite).

“Topological” quantum groups

- Some of you will now be thinking “Bialgebras”.
- A “Hopf von Neumann algebra” is a pair (M, Δ) where:
  - M is a von Neumann algebra, and
  - $\Delta : M \to M \overline{\otimes} M$ is a (normal) injective $*$-homomorphism, with
  - $(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta$.
- After a long history, Kustermans and Vaes gave an axiomatisation, subsuming Woronowicz’s earlier work on compact quantum groups.
- Turns out the correct extra piece of data to assume is analogues of the left (and right) Haar measures– these are now *axioms* not the result of a *theorem*
- Can then construct maps which behave like an antipode, and counit– but these are in general unbounded, and not everywhere defined.