Banach algebras of operators

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Abstract

In this thesis we investigate algebraic questions about the structure of $\mathcal{B}(E)$ and ideals thereof, where $\mathcal{B}(E)$ is the Banach algebra of all operators on a Banach space E.

Chapter 1 details the necessary background material from the theories of Banach spaces, Banach algebras and C^*-algebras. We also define the Arens products. Given a Banach space E we have the canonical isometry $\kappa_E : E \to E''$ of E into its bidual, this map being an isomorphism only when E is reflexive. The Arens products are the two natural ways to extend the algebra product from a Banach algebra $\mathcal{A}$ to its bidual $\mathcal{A}''$ making $\kappa_{\mathcal{A}}$ into a homomorphism. The topological centres of $\mathcal{A}''$ are the subsets of $\mathcal{A}''$ where the two Arens products agree "on one side". An algebra $\mathcal{A}$ is Arens regular when the Arens products agree on the whole of $\mathcal{A}''$.

In Chapter 2 we review the notion of a tensor norm, sketching a modern approach to the classical work of Grothendieck. Tensor norms give us a way of defining an algebra norm on the algebra of finite-rank operators, $\mathcal{F}(E)$, on a Banach space E. The completion of such an algebra is the algebra of α-nuclear operators, for a tensor norm α. Such an algebra is always an ideal in $\mathcal{B}(E)$, but it need not be closed. We then go on to study the Arens products and, in particular, the topological centres of the biduals of ideals of α-nuclear operators. This generalises and extends the work of, for example, Dales, Lau ([Dales, Lau, 2004]), Ülger ([Laustsen, Loy, 2003]), Palmer ([Palmer, 1985]) and Grosser ([Grosser, 1987]). We also examine ideals of compact operators by using a factorisation scheme which allows us to partially reduce the problem to ideals of ε-nuclear, or approximable, operators.

Chapter 3 reviews the concept of an ultrapower of a Banach space, $(E)_{\mathcal{U}}$. We study how tensor products and ultrapowers interact, and derive a new representation of $\mathcal{B}(l^p)'$ as a quotient of the projective tensor product $(l^p)_{\mathcal{U}}\widehat{\otimes}(l^q)_{\mathcal{U}}$. We define the class of super-reflexive Banach spaces and show that they are stable under taking ultrapowers. This allows us to show that $\mathcal{B}(E)$ is Arens regular for a super-reflexive Banach space E, and

that $\mathcal{B}(E)''$ can be identified with a subalgebra of $\mathcal{B}(F)$ for some super-reflexive Banach space F. We finish the chapter by studying some abstract questions about ultrapowers of Banach modules, culminating in showing that every Arens regular Banach algebra arises, in a weak*-topology manner, as an ultrapower of the original algebra.

Chapter 4 is devoted to constructing counter-examples. We show that there exist many examples of reflexive Banach spaces E such that $\mathcal{B}(E)$ is not Arens regular, improving upon work by Young in [Young, 1976]. We then present some joint work with C. J. Read which shows, in particular, that $\mathcal{B}(l^p)''$ is a semi-simple algebra if and only if $p = 2$. This builds upon the above representation of $\mathcal{B}(l^p)'$.

In Chapter 5 we consider the algebras $\mathcal{B}(l^p)$, for $1 \leq p < \infty$, and $\mathcal{B}(c_0)$. It is well-known that the only proper, closed ideal in these algebras is the ideal of compact operators. Furthermore, this result has been generalised (see [Gohberg et al., 1967]) in the case of $\mathcal{B}(H)$ for an arbitrary Hilbert space H, via the consideration of κ-compact operators, for a cardinal κ. We extend this result to the non-separable analogies of l^p and c_0.

Contents

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Chapter 1

Preliminaries

1.1 Introduction

The study of Banach algebras entails combining the use of algebraic and analytical (or topological) methods. It is common to find that, say, algebraic conditions imply topological ones, and the paradigm of trying to combining algebra and analysis underlies the whole subject.

Banach spaces also exhibit some of this behaviour; for example, the open mapping theorem gives topological conditions (continuity of an inverse) out of purely algebraic hypotheses (the existence of a linear inverse). It should not surprise us that if we look at Banach algebras which are closely related to Banach spaces (for example, the Banach algebra of all operators on a Banach space) then we find that geometrical properties of the Banach spaces have close links to algebraic properties of the Banach algebras. This thesis will explore some questions in this direction.

This first chapter sketches the background material that we shall need from the theory of Banach spaces and Banach algebras. Proofs and references are omitted except for the more unusual results, or when a proof will shed light on later work. Useful references for the material on Banach spaces are [Diestel, 1984], [Megginson, 1998], [Guerre-Delabriére, 1992] and [Habala et al., 1996]. References for Banach algebras are [Dales, 2000] or [Palmer, 1994].

1.2 Banach spaces

We shall denote the real numbers by $\mathbb{R}$, the complex numbers by $\mathbb{C}$, the integers by $\mathbb{Z}$ and the natural numbers by $\mathbb{N}$. For us, $\mathbb{N} = \{1, 2, 3, \ldots\}$ and $\mathbb{N}_0 = \mathbb{Z}^{\geq 0} = \{0, 1, 2, 3, \ldots\}$.

For us, a *vector space* V will be an additive group, together with a scalar multiplication over the complex numbers. A *basis* in V is a linearly independent spanning set. We say that V is *finite-dimensional* if it has a finite basis; the cardinality of this basis is well-defined, and is denoted by $\dim V$. All finite-dimensional vector spaces are isomorphic to $\mathbb{C}^n$ for some $n \in \mathbb{N}$. When X is a subset of V, we write $\operatorname{lin} X$ for the *linear span* of X in V.

A *norm* on a vector space is a map $\rho : V \to \mathbb{R}$ such that:

1. $\rho(v) \geq 0$ for each $v \in V$, and $\rho(v) = 0$ only if $v = 0$;
2. $\rho(\alpha v) = |\alpha|\rho(v)$ for each $\alpha \in \mathbb{C}$ and $v \in V$;
3. $\rho(u + v) \leq \rho(u) + \rho(v)$ for each $u, v \in V$.

We will generally write $\|v\| = \rho(v)$ for a norm. Then $(V, \|\cdot\|)$ is a *normed space*. A norm on V induces a metric d on V by $d(u, v) = \|u - v\|$. We say that $(V, \|\cdot\|)$ is *complete* if the metric space (V, d) is (Cauchy) complete. Complete normed vector spaces are called *Banach spaces*. The collection of finite-dimensional normed spaces is denoted by FIN, and we shall see below that each member of FIN is a Banach space.

For a normed space V and $t \in \mathbb{R}$, we write

$$V_{[t]} = \{v \in V : \|v\| \leq t\}.$$

Thus $V_{[1]}$ is the *closed unit ball* of V.

The correct category of maps between Banach spaces to study is the collection of maps which preserve both the linear structure and the topology. Such maps are bounded linear maps, or *operators*.

Lemma 1.2.1. *Let E and F be Banach spaces and $T : E \to F$ be a linear map. Then the following are equivalent, and define what is means for T to be bounded.*

1. *T is continuous with respect to the norms on E and F;*
2. *T is continuous at 0;*
3. *for some $M \in \mathbb{R}$, we have $\|T(x)\| \leq M\|x\|$ for each $x \in E$.*

The minimal valid value for M above is the *operator norm* of T, denoted by $\|T\|$. Thus

$$\|T\| = \sup\{\|T(x)\| : x \in E_{[1]}\} = \sup\{\|T(x)\|\|x\|^{-1} : x \in E, x \neq 0\}.$$

We write $\mathcal{B}(E, F)$ for the normed vector space of operators between E and F, with the operator norm. Then we can verify that $\mathcal{B}(E, F)$ is complete whenever F is a Banach space and E is a normed space (in particular, when E is a Banach space). When $\|T(x)\| = \|x\|$ for each $x \in E$, we say that T is an *isometry* (noting that we do not require T to be a surjection).

We write the composition of two linear maps T and S by $T \circ S$, or just TS. For a Banach space E, we write Id_E for the identity map on E. Then, when $T \in \mathcal{B}(E, F)$ is such that for some $S \in \mathcal{B}(F, E)$, we have $TS = \mathrm{Id}_F$ and $ST = \mathrm{Id}_E$, we say that T is an *isomorphism*. This is equivalent to T being a bijection, and being *bounded below*; that is, for some $m \in \mathbb{R}$ with $m > 0$, we have

$$\|T(x)\| \geq m\|x\| \qquad (x \in E).$$

The open mapping theorem, to be shown later, tells us that if T is bounded and bijective, then it is automatically bounded below.

For $T \in \mathcal{B}(E, F)$, we write $T(E)$ for the *image* of T in F; more generally, if $X \subseteq E$ is a subset, then $T(X) = \{T(x) : x \in X\} \subseteq F$. We write $\ker T$ for the *kernel* of T, $\ker T = \{x \in E : T(x) = 0\}$, which is a closed subspace of E.

As $\mathbb{C}$ is certainly a complete normed space in the norm induced by setting $\|1\| = 1$, we see that $\mathcal{B}(E, \mathbb{C})$ is a Banach space. This is the most basic space of operators on E, and we denote it by $E' = \mathcal{B}(E, \mathbb{C})$, the *dual space* of E. A member of E' is called a *functional* on E (or *bounded functional* if necessary). For $x \in E$ and $\mu \in E'$, we write

$$\langle \mu, x \rangle = \mu(x)$$

as this notation will be easier to handle later on. We adopt to the convention that the left-hand member of $\langle \cdot, \cdot \rangle$ is a member of the dual space to the space which contains the right-hand member of $\langle \cdot, \cdot \rangle$. Thus $\langle \mu, x \rangle$ and not $\langle x, \mu \rangle$. We write $E^{[1]} = E'$ and define $E^{[n+1]} = (E^{[n]})'$ for $n \in \mathbb{N}$.

For $x \in E$, we can form a functional $\kappa_E(x) \in E''$ by

$$\langle \kappa_E(x), \mu \rangle = \langle \mu, x \rangle \qquad (\mu \in E').$$

We can check that then $\|\kappa_E(x)\| = \|x\|$ and that κ_E is a linear map $\kappa_E : E \to E''$. Thus κ_E is an isometry from E to E''. When κ_E is surjective, we say that E is *reflexive*. All members of FIN are reflexive.

We now collect together some basic properties of Banach spaces and operators between them.

Theorem 1.2.2. (Open mapping theorem) *Let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$ by a surjection. Then T is an open mapping (that is, T maps open sets to open sets). If T is also an injection, then T is an isomorphism.* □

The *graph* of $T \in \mathcal{B}(E, F)$ is the subset

$$G_T = \{(x, T(x)) : x \in E\} \subseteq E \times F,$$

where we give $E \times F$ the product topology.

Theorem 1.2.3. (Closed graph theorem) *Let E and F be Banach spaces, and $T \in \mathcal{B}(E, F)$. Then the following are equivalent:*

1. *T is bounded;*
2. *the graph of T is closed in $E \times F$;*
3. *if $(x_n)_{n=1}^{\infty}$ is a sequence in E such that $x_n \to 0$ in E, and $T(x_n) \to y$ in F, then $y = 0$.*

□

Theorem 1.2.4. (Uniform boundedness theorem) *Let E and F be Banach spaces and let $(T_i)_{i \in I}$ be a family in $\mathcal{B}(E, F)$. Suppose that for each $x \in E$, the family $(T_i(x))_{i \in I}$ is bounded in F. Then for some $M \in \mathbb{R}$, we have $\|T_i\| \leq M$ for each $i \in I$.*

Let (T_n) be a sequence in $\mathcal{B}(E, F)$, and suppose that $\lim_n T_n(x) = T(x)$ for each $x \in E$. Then (T_n) is bounded in $\mathcal{B}(E, F)$, $T \in \mathcal{B}(E, F)$ and $\|T\| \leq \liminf_n \|T_n\|$. □

When E is a Banach space and F is a subspace of E, we can form the linear space E/F, the *quotient* of E by F. We can then define

$$\|x + F\| = \inf\{\|x + y\| : y \in F\} \qquad (x + F \in E/F).$$

This is a semi-norm on E/F (that is, $\|a\|$ can be zero without a being zero), and when F is a closed subspace of E, this is a norm, called the *quotient norm*. We say that F has *finite co-dimension* in E when E/F is finite-dimensional.

Definition 1.2.5. Let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$. Then T is a *quotient operator* if T is surjective, and for $y \in F$, we have $\|y\| = \inf\{\|x\| : x \in E, T(x) = y\}$. Thus, by the open mapping theorem, T factors to give an isometric isomorphism $\tilde{T} : E/\ker(T) \to F$.

We now define some classical examples of Banach spaces. Let (X, τ) be a locally compact Hausdorff topological space, so that every point of X has a compact neighbourhood. Then let $C(X)$ be the vector space of all continuous maps from X to $\mathbb{C}$, under pointwise operations. Let $C_0(X)$ be the subset of $C(X)$ defined by $f \in C_0(X)$ if and only if, for each $\varepsilon > 0$, the set $\{x \in X : |f(x)| \geq \varepsilon\}$ is compact in X. Then $C_0(X)$ is a Banach space under the *supremum norm*

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

When I is a set, let d_I be the discrete topology on I, and define $c_0(I) = C_0(I, d_I)$.

Let S be a set, let 2^S be the *power set* of S and let Σ be a subset of 2^S. Then Σ is a *σ-algebra* if: (i) $S \in \Sigma$; (ii) if $X \in \Sigma$, then $S \setminus X \in \Sigma$; and (iii) if (X_n) is a sequence in Σ then $\bigcup_n X_n \in \Sigma$. The members of Σ are called *measurable sets*. A *complex measure* on (S, Σ) is a map $\nu : \Sigma \to \mathbb{C}$ such that if (X_n) is a sequence of pairwise-disjoint members of Σ, then $\nu\left(\bigcup_n X_n\right) = \sum_n \nu(X_n)$. A *measure* is a map $\nu : \Sigma \to [0, \infty) \cup \{\infty\}$ such that if (X_n) is a sequence of pairwise-disjoint members of Σ, then $\nu\left(\bigcup_n X_n\right) = \sum_n \nu(X_n)$, where divergence of the sum in interpreted to be ∞. A *measure space* is a triple (S, Σ, ν) where Σ is a σ-algebra on S and ν is a measure. A function $f : S \to \mathbb{C}$ is *measurable* if $f^{-1}(U) \in \Sigma$ for each open set $U \subseteq \mathbb{C}$. The set of all measurable functions on a measure space (S, Σ, ν) forms a vector space under pointwise operations. In fact, they form a complex lattice under pointwise suprema and infima, and the taking of real and imaginary parts, and absolute values.

As a technical point, we note that we need a more general definition of measure space than is often used, to allow us to make sense of certain constructions involving ultrafilters. For example, see [Haydon et al., 1991, Chapter 2], which sketches measure theory without the need for σ-finiteness.

For a measurable set $X \subseteq S$, let χ_X be the *characteristic function* of X, defined by

$$\chi_X(x) = \begin{cases} 1 & : x \in X, \\ 0 & : x \notin X, \end{cases} \qquad (x \in S)$$

so that χ_X is a measurable function. Then the linear span of $\{\chi_X : X \in \Sigma\}$ are the *simple functions*. Given a simple function

$$f = \sum_{n=1}^{N} a_n \chi_{X_n}$$

we define the *integral* of f with respect to a measure ν to be

$$\int f \, \mathrm{d}\nu = \int f(x) \, \mathrm{d}\nu(x) = \sum_{n=1}^{N} a_n \nu(X_n).$$

For a general positive measurable function $f : S \to \mathbb{R}$, we define

$$\int f \, \mathrm{d}\nu = \sup \left\{ \int g \, \mathrm{d}\nu : g \text{ simple, and } g \leq f \right\}.$$

By $g \leq f$, we mean that $g(x) \leq f(x)$ for each $x \in S$. We then define the integral of a general measurable function by taking real and imaginary parts, and positive and negative parts.

For $1 \leq p < \infty$ and a measure space (S, Σ, ν), we define the *p-norm* of a measurable function $f : S \to \mathbb{C}$ by

$$\|f\|_p = \left(\int |f(x)|^p \, \mathrm{d}\nu(x) \right)^{1/p}.$$

This is actually a semi-norm, so we identify equivalence classes of functions (and henceforth, when we talk about a function, we will generally mean the equivalence class of the said function). Then the set of functions with finite p-norm form a Banach space, denoted by $L^p(S, \Sigma, \nu) = L^p(S, \nu) = L^p(\nu)$. When I is a set, we define $l^p(I) = L^p(I, \Sigma, \nu)$, where $\Sigma = 2^I$ and ν is the counting measure.

For $Y \in \Sigma$, we say that Y is *ν-locally null* when $\nu(Y \cap X) = 0$ for each $X \in \Sigma$ with $\nu(X) < \infty$. We define $L^\infty(S, \Sigma, \nu)$ to be the Banach space of measurable functions with finite supremum norm, where

$$\|f\|_\infty = \inf\{M \in \mathbb{R} : \{s \in S : |f(s)| > M\} \text{ is } \nu\text{-locally null}\}.$$

Then $l^\infty(I)$ is defined similarly.

When I is a finite set, $l^p(I)$ is isometrically isomorphic to l^p_n, for $1 \leq p \leq \infty$, where $n = |I|$, the cardinality of I, and $l^p_n = (\mathbb{C}^n, \|\cdot\|_p)$, where

$$\|(x_i)\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \qquad ((x_i)_{i=1}^n \in \mathbb{C}^n),$$

with the obvious definition for $p = \infty$.

An *inner product* on a vector space H is a map $[\cdot,\cdot] : H \times H \to \mathbb{C}$ such that:

1. the map $x \mapsto [x, y]$ is a linear map for each $y \in H$;
2. for $x, y \in H$, we have $[x, y] = \overline{[y, x]}$;
3. for $x \in H$, we have $[x, x] \geq 0$ and $[x, x] = 0$ only if $x = 0$.

We call such an H an *inner product space*. Define $\|x\| = [x,x]^{1/2}$, so that $\|\cdot\|$ is a norm on H. When $(H, \|\cdot\|)$ is a Banach space, we say that H is a *Hilbert space*. Thus the spaces l_n^2, $l^2(I)$ and $L^2(\nu)$ are all Hilbert spaces.

Let $p \in [1,\infty]$. We generalise the idea of l^p spaces from taking values in the complex numbers to taking values in Banach spaces (we can also do this for L^p spaces, but this will not interest us). Specifically, let (E_n) be a sequence of Banach spaces and define, for $1 \leq p < \infty$,

$$l^p(E_n) = \left\{ (x_n)_{n=1}^\infty : \|(x_n)\| := \left(\sum_{n=1}^\infty \|x_n\|^p \right)^{1/p} < \infty, x_n \in E_n \ (n \in \mathbb{N}) \right\}.$$

We also write $l^p(E_n) = l^p\Big(\bigoplus_{n=1}^\infty E_n \Big)$. The spaces $l^\infty(E_n)$ and $c_0(E_n)$ have similar definitions. Then we can check that $l^p(E_n)' = l^q(E_n')$ for $p^{-1} + q^{-1} = 1$, with duality defined by

$$\langle (\mu_n), (x_n) \rangle = \sum_{n=1}^\infty \langle \mu_n, x_n \rangle \qquad ((\mu_n) \in l^q(E_n'), (x_n) \in l^p(E_n)).$$

We can now show that if $E \in \mathrm{FIN}$ then E is complete, by showing that E is isomorphic to the Hilbert space with dimension $\dim E$.

Proposition 1.2.6. *Let $(E, \|\cdot\|)$ be a normed space of dimension n. Then $(E, \|\cdot\|)$ is isomorphic to l_n^2 and thus is a Banach space.*

Proof. Let E have a basis $(x_1, \ldots, x_n)$, where we may suppose that $\|x_i\| = 1$ for each i. Define $T : l_n^2 \to E$ by

$$T((a_i)) = \sum_{i=1}^n a_i x_i \qquad ((a_i)_{i=1}^n \in l_n^2),$$

so that T is a linear isomorphism. We shall show that T is a topological isomorphism as well. Indeed, for $(a_i) \in l_n^2$, we have

$$\begin{aligned} \|T((a_i))\| &= \left\| \sum_{i=1}^n a_i x_i \right\| \leq \sum_{i=1}^n |a_i| \|x_i\| = \sum_{i=1}^n |a_i| \\ &\leq \sqrt{n} \left(\sum_{i=1}^n |a_i|^2 \right)^{1/2} = \sqrt{n} \|(a_i)\|_2 \end{aligned}$$

The non-obvious inequality here can be proved by induction, for example. Thus we see that T is bounded.

Suppose that T is not bounded below, so that for some sequence (a_k) in l_n^2 with $\|a_k\| = 1$ for each k, we have $T(a_k) \to 0$ in E. Now, the unit ball of n-dimensional Euclidean space (that is, l_n^2) is compact, so by moving to a subsequence, we may suppose that $a_k \to a$. Thus, as T is continuous, $T(a_k) \to T(a) = 0$. As $(x_1, \ldots, x_n)$ is a basis for E, $T(a) = 0$ means that $a = 0$, a contradiction, as a lies in the unit sphere of l_n^2. $\square$

The key point of this proof is that the unit ball of l_n^2 is compact. Indeed, the above proof then shows that the unit ball of any finite-dimensional normed space is compact. This is not true for any infinite-dimensional normed space.

1.3 Nets, filters and ultrafilters

We will find it useful to generalise the idea of a sequence, where we increase the cardinality of the index set, and also change the ordering structure. The correct tools for this are nets, filters and ultrafilters.

Definition 1.3.1. Let Λ be a set and $\leq$ be a binary relation on Λ.

1. $(\Lambda, \leq)$ is a *partially ordered set* and $\leq$ is a *partial order* when: (i) if $\alpha, \beta, \gamma \in \Lambda$ with $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$; (ii) for $\alpha \in \Lambda$, we have $\alpha \leq \alpha$; and (iii) if $\alpha, \beta \in \Lambda$ with $\alpha \leq \beta$ and $\beta \leq \alpha$, then $\alpha = \beta$.

2. A partially ordered set $(\Lambda, \leq)$ is a *directed set* if, for each $\alpha, \beta \in \Lambda$, there exists $\gamma \in \Lambda$ with $\alpha \leq \gamma$ and $\beta \leq \gamma$.

3. A *net* in a set X is a family $(x_\alpha)_{\alpha \in \Lambda}$ where the index set Λ is a directed set.

4. A partially ordered set $(\Lambda, \leq)$ is *totally ordered* and $\leq$ is a *total order* if, for each $\alpha, \beta \in \Lambda$, either $\alpha \leq \beta$ or $\beta \leq \alpha$.

5. A subset S of a partially ordered set $(\Lambda, \leq)$ is a *chain* if $(S, \leq|_S)$ is a totally ordered set, where $\leq|_S$ is the restriction of the partial order $\leq$ to S.

A *maximal* element in $(\Lambda, \leq)$ is an element α such that, if $\beta \in \Lambda$ with $\alpha \leq \beta$, then $\alpha = \beta$. A similar definition holds for a *minimal* element.

Theorem 1.3.2. (Zorn's Lemma) *Let $(\Lambda, \leq)$ be a partially ordered set so that for each chain S in Λ, we can find an upper bound $\alpha \in \Lambda$ for S (that is, $\beta \leq \alpha$ for each $\beta \in S$). Then Λ contains a maximal element.* $\square$

A *filter* on a set S is a subset $\mathcal{F}$ of 2^S such that: (i) $\emptyset \notin \mathcal{F}$; (ii) if $A \in \mathcal{F}$ and $B \subseteq S$ with $A \subseteq B$, then $B \in \mathcal{F}$; and (iii) if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. We can partially order the collection of filters on a set by inclusion, and then a simple application of Zorn's Lemma shows that maximal filters exist. Such filters are called *ultrafilters*. If $s \in S$ then the collection

$$\mathcal{U}_s = \{A \subseteq S : s \in A\}$$

is an ultrafilter (by the next lemma). Such an ultrafilter is the *principal ultrafilter* at s. Ultrafilters which are not principal are called *non-principal*, and will be the ones which interest us.

Lemma 1.3.3. *Let $\mathcal{U}$ be a filter on a set S. Then $\mathcal{U}$ is an ultrafilter if and only if, for each $X \subseteq S$, either $X \in \mathcal{U}$ or $S \setminus X \in \mathcal{U}$.* □

Given a directed set $(\Lambda, \leq)$ and $\alpha \in \Lambda$, let $A_\alpha = \{\beta \in \Lambda : \alpha \leq \beta\}$, and then let

$$\mathcal{F}_\leq = \{B \subseteq \Lambda : \exists\, \alpha \in \Lambda, A_\alpha \subseteq B\}.$$

As $\alpha \in A_\alpha$ for each α, $\mathcal{F}_\leq$ does not contain the empty set. As Λ is directed, $A_\alpha \cap A_\beta$ contains A_γ for some $\gamma \in \Lambda$. Thus $\mathcal{F}_\leq$ is a filter, called the *order filter* on $(\Lambda, \leq)$.

Let X be a topological space, and $(x_\alpha)_{\alpha\in\Lambda}$ be a net in X. Then we say that (x_α) converges to x, written $x_\alpha \to x$ or $\lim_{\alpha\in\Lambda} x_\alpha = x$, if, for each open neighbourhood U of x, there exists $\beta \in \Lambda$ so that if $\beta \leq \alpha$, then $x_\alpha \in U$. This is equivalent to asking that, for each open neighbourhood U of x, we have $\{\alpha \in \Lambda : x_\alpha \in U\} \in \mathcal{F}_\leq$.

We use this to define the notion of a limit along a filter. Thus if $\mathcal{F}$ is a filter on a set I and $(x_i)_{i\in I}$ is a family in a topological space, we write $x = \lim_{i\in\mathcal{F}} x_i$ if, for each open neighbourhood U of x, we have $\{i \in I : x_i \in U\} \in \mathcal{F}$. The use of ultrafilters can now be stated.

Proposition 1.3.4. *Let X be a compact topological space, let $\mathcal{U}$ be an ultrafilter on a set I, and let $(x_i)_{i\in I}$ be a family in X. Then there exists $x \in X$ with $x = \lim_{i\in\mathcal{U}} x_i$. Furthermore, if X is Hausdorff, then x is unique.* □

In fact, the converse to the above proposition is also true, in that if every family converges along every ultrafilter, then X must be compact.

Ultrafilters also respect continuous functions. To state this fully, recall that if $(X_j)_{j\in J}$ is a family of topological spaces, then the axiom of choice tells us that the set $X = \prod_{j\in J} X_j$ is non-empty. For each $j \in J$ let $\pi_j : X \to X_j$ be the canonical projection map. Then the *product topology* on X is the weakest topology making each π_j continuous. That is, open sets are arbitrary unions of sets of the form

$$\bigcap_{k=1}^{n} \pi_{j_k}^{-1}(U_k),$$

where $n \in \mathbb{N}$, $(j_k)_{k=1}^{n} \subseteq J$ and for each k, U_k is an open subset of X_{j_k}.

Proposition 1.3.5. *Let $(X_j)_{j\in J}$ be a family of topological spaces, let Y be a topological space, let $X = \prod_{j\in J} X_j$ with the product topology, and let $f : X \to Y$ be continuous. Let I be an index set, let $\mathcal{F}$ be a filter on I, and for each $j \in J$, let $(x_i^{(j)})_{i\in I}$ be a family in X_j. Suppose that, for each $j \in J$, $z_j = \lim_{i\in\mathcal{F}} x_i^{(j)} \in X_j$ exists. Then $\lim_{i\in\mathcal{F}} f((x_i^{(j)})_{j\in J})$ exists and equals $f((z_j)_{j\in J})$.* □

Thus, for example, if $(x_i)_{i\in I}$ and $(y_i)_{i\in I}$ are families in a topological vector space such that

$$x = \lim_{i\in\mathcal{F}} x_i \quad , \quad y = \lim_{i\in\mathcal{F}} y_i,$$

then $x + y = \lim_{i\in\mathcal{F}}(x_i + y_i)$.

Filter limits also respect other “natural structures”, for example, the order structure of $\mathbb{R}$, so that if $x_i \leq y_i$ for each i, we have $\lim_{i\in\mathcal{F}} x_i \leq \lim_{i\in\mathcal{F}} y_i$, assuming that these limits exist.

Another way of seeing the use of ultrafilters is to consider the following. Suppose we have bounded sequences of reals (a_n) and (b_n). Then the Bolzano-Weierstrass theorem states that there are convergent subsequences of (a_n) and (b_n). However, $(a_n + b_n)$ is also a bounded sequence of reals, so some subsequence of $(a_n + b_n)$ also converges. We would obviously like these subsequences to all be the same (or, rather, to be nested in some fashion) but there is no particular reason why they should be. What an ultrafilter on $\mathbb{N}$ does is to provide a *consistent way of picking a subsequence* so that we can then add, multiply etc. It should come as no surprise that we thus need Zorn’s Lemma to show that non-principal ultrafilters exist.

1.4 Dual spaces and weak topologies

We only state the definitions in a restricted case.

Definition 1.4.1. Let E be a Banach space. Then the *weak-topology* on E is defined by the family of open sets

$$\mathcal{O}(x, \mu_1, \ldots, \mu_n, \varepsilon) = \{y \in E : |\langle \mu_i, x - y\rangle| < \varepsilon \ (1 \leq i \leq n)\},$$

where $x \in E$, $\varepsilon > 0$ and $(\mu_i)_{i=1}^n$ is a finite family in E'.

The *weak***-topology* on E' is defined by the family of open sets

$$\mathcal{O}(\mu, x_1, \ldots, x_n, \varepsilon) = \{\lambda \in E' : |\langle \mu - \lambda, x_i \rangle| < \varepsilon \; (1 \leq i \leq n)\},$$

where $\mu \in E'$, $\varepsilon > 0$ and $(x_i)_{i=1}^n$ is a finite family in E.

Equivalently, we can state what it means for a net (and thus a filter or ultrafilter) to converge. Thus a net $(x_\alpha)_{\alpha \in \Lambda}$ in E converges weakly to $x \in E$ if and only if

$$\lim_{\alpha \in \Lambda} \langle \mu, x_\alpha \rangle = \langle \mu, x \rangle \qquad (\mu \in E').$$

A net $(\mu_\alpha)_{\alpha \in \Lambda}$ converges weak* to $\mu \in E'$ if and only if

$$\lim_{\alpha \in \Lambda} \langle \mu_\alpha, x \rangle = \langle \mu, x \rangle \qquad (x \in E).$$

The weak- and weak*-topologies are only given by a norm in the finite-dimensional case. They are sometimes given by a metric, but in general are just a topology. However, scalar multiplication and vector space addition are clearly continuous with respect to these topologies. In fact, these topologies are locally convex, in that each point has a neighbourhood base consisting of convex sets.

The importance of these topologies can be summed up in a few results.

Theorem 1.4.2. (Banach-Alaoglu theorem) *Let E be a Banach space. Then $E'_{[1]}$, with the weak*-topology, is a compact topological space.* □

Theorem 1.4.3. (Mazur's theorem) *Let E be a Banach space, and let $X \subseteq E$ be a bounded, convex subset. Then the weak and norm closures of X coincide.* □

Of course, we do not yet know that we have a rich supply of linear functionals. This is solved by the Hahn-Banach theorem.

Theorem 1.4.4. (Hahn-Banach theorem) *Let E be a complex Banach space, let F a subspace of E, and let $\mu : F \to \mathbb{C}$ be a bounded linear functional. Then there exists $\lambda \in E'$ such that*

$$\|\lambda\| = \|\mu\| \quad , \quad \langle \lambda, x \rangle = \langle \mu, x \rangle \quad (x \in F).$$

Thus λ is an isometric extension of μ to the whole space. We can also replace complex scalars by real scalars. □

For example, we can now show that for each $x \in E$, there exists $\mu \in E'$ with $\langle \mu, x \rangle = \|x\|$ and $\|\mu\| = 1$. There exist a whole family of Hahn-Banach theorems, dealing with extensions of other type of maps (convex not linear, for example) and with various "separation" properties. These variants are required to prove Mazur's Theorem, for example.

For a Banach space E, a subset $X \subseteq E$ is *weakly-compact* if it is compact in E with the weak-topology. Similarly, X is *relatively weakly-compact* if its closure, in the weak-topology, is weakly-compact. We say that X is *weakly sequentially compact* if every sequence in X has a sub-sequence which is convergent in the weak-topology.

Theorem 1.4.5. (Eberlein-Smulian) *Let E be a Banach space, and let X be a subset of E. Then X is weakly-compact if and only if it is weakly sequentially compact.* □

Definition 1.4.6. Let E and F be Banach spaces and let $T \in \mathcal{B}(E, F)$. We write $T \in \mathcal{W}(E, F)$, and say that T is *weakly-compact*, if T maps bounded subsets of E to relatively weakly-compact subsets of F.

Recall that a Banach space E is reflexive if and only if $\kappa_E : E \to E''$ is an isomorphism.

Theorem 1.4.7. *Let E be a Banach space. Then the following are equivalent:*

1. *E is reflexive;*
2. *$E_{[1]}$ is weakly-compact;*
3. *$E_{[1]}$ is weakly sequentially compact;*
4. *the identity map Id_E is weakly-compact;*
5. *whenever (x_n) and (μ_n) are bounded sequences in E and E' respectively,*

$$\lim_m \lim_n \langle \mu_n, x_m \rangle = \lim_n \lim_m \langle \mu_n, x_m \rangle,$$

whenever these limits exist.

Proof. This is standard, apart from perhaps (5), which shall be shown in Theorem 4.1.1. □

Thus we immediately see that E is reflexive if and only if E' is reflexive, and that if E is isomorphic to F, then E is reflexive if and only if F is.

The next two results tell us that, from a certain perspective, a Banach space E and its bidual E'' are not too different.

Theorem 1.4.8. (Goldstine) *Let E be a Banach space, and identify $E_{[1]}$ with its image, under the map $\kappa_E : E \to E''$, in $E''_{[1]}$. Then $E_{[1]}$ is dense, in the weak*-topology, in $E''_{[1]}$.* $\square$

Theorem 1.4.9. (Principle of local reflexivity) *Let E be a Banach space, $X \subseteq E''$ be finite-dimensional and $Y \subseteq E'$ be finite-dimensional. For each $\varepsilon > 0$ there exists a map $T : X \to E$ such that:*

1. *for $\Phi \in X$, $(1-\varepsilon)\|\Phi\| \leq \|T(\Phi)\| \leq (1+\varepsilon)\|\Phi\|$;*
2. *for $\kappa_E(x) \in X \cap \kappa_E(E)$, $T(\kappa_E(x)) = x$;*
3. *for $\Phi \in X$ and $\mu \in Y$, $\langle \Phi, \mu \rangle = \langle \mu, T(\Phi) \rangle$.* $\square$

We can interpret this result as saying that the finite-dimensional subspaces of E'' are, with respect to duality, the same as the finite-dimensional subspaces of E. Note also that this result trivially proves Goldstine's theorem, if we let X be one-dimensional. We shall see later that actually a variant of Goldstine's theorem can be used to prove the principle of local reflexivity.

We can use the Principle of local reflexivity (or Goldstein's Theorem) to show that for a Banach space E, and $\Phi \in E''$, there is a net $(x_\alpha) \subseteq E$ such that $\|x_\alpha\| \leq \|\Phi\|$ for each α, and such that $\kappa_E(x_\alpha) \to \Phi$ in the weak*-topology on E''. Indeed, let Λ be the collection of finite-dimensional subspaces of E', partially ordered by inclusion, so that Λ is a directed set. Then, for $M \in \Lambda$, let $y_M \in E$ be given by the Principle of local reflexivity, with $\varepsilon_M = (\dim M)^{-1}$, so that

$$(1+\varepsilon_M)\|\Phi\| \leq \|y_M\| \leq (1+\varepsilon_M)\|\Phi\| \quad , \quad \langle \mu, y_M \rangle = \langle \Phi, \mu \rangle \qquad (\mu \in M).$$

Then let $x_M = y_M\|\Phi\|\|y_M\|^{-1}$ so that $\|x_M\| = \|\Phi\|$, and as $\lim_{M\in\Lambda} \varepsilon_M = 0$, for each $\mu \in E'$, we have

$$\lim_{M\in\Lambda} \langle \mu, x_M \rangle = \lim_{M\in\Lambda} \langle \mu, y_M \rangle \|y_M\|^{-1}\|\|\Phi\| = \langle \Phi, \mu \rangle \lim_{M\in\Lambda} \|y_M\|^{-1}\|\|\Phi\| = \langle \Phi, \mu \rangle,$$

as required.

We can express the Hahn-Banach theorem in a more algebraic manner. For a Banach space E, a subspace F of E and a subspace G of E', we define

$$F^\circ = \{\mu \in E' : \langle \mu, x \rangle = 0 \ (x \in F)\},$$
$${}^\circ G = \{x \in E : \langle \mu, x \rangle = 0 \ (\mu \in G)\}.$$

Thus F° is a closed subspace of E' and ${}^\circ G$ is a closed subspace of E. Furthermore, $({}^\circ G)^\circ$ is the weak*-closure of G in E', and ${}^\circ(F^\circ)$ is the norm-closure of F in E. Thus, if G is finite-dimensional, then $({}^\circ G)^\circ = G$.

Theorem 1.4.10. *Let E be a Banach space, and let F be a closed subspace of E.*

1. *For each $\lambda \in F'$ let $\Lambda \in E'$ be such that $\|\lambda\| = \|\Lambda\|$ and $\langle \lambda, x\rangle = \langle \Lambda, x\rangle$ for each $x \in F$. Then the map $F' \to E'/F^\circ$, $\lambda \mapsto \Lambda + F^\circ$, is an isometric isomorphism.*

2. *Let $\pi : E \to E/F$ be the quotient map. Then $\pi' : (E/F)' \to F^\circ$ is an isometric isomorphism.*

3. *We identify F'' with $F^{\circ\circ}$ which is the same as the weak*-closure of $\kappa_E(F)$ in E''.*

4. *Thus also, $(E/F)''$ is isometrically isomorphic to $E''/F^{\circ\circ}$.* □

Finally we will consider how operators and subspaces interact with dual spaces. For Banach spaces E and F, and $T \in \mathcal{B}(E, F)$, define the *adjoint* or *dual* of T to be $T' : F' \to E'$, where

$$\langle T'(\mu), x\rangle = \langle \mu, T(x)\rangle \qquad (\mu \in F', x \in E).$$

We can check that then $T' \in \mathcal{B}(F', E')$, and that $\|T'\| = \|T\|$.

Theorem 1.4.11. *Let E and F be Banach spaces, and $T \in \mathcal{B}(E, F)$. Then we have:*

1. $\ker T = {}^\circ(T'(F'))$ *and so T is an injection if and only if $T'(F')$ is weak*-dense in E';*

2. $\ker T' = T(E)^\circ$ *and so T' is an injection if and only if $T(E)$ is dense in F;*

3. *the following are equivalent: (i) $T(E)$ is closed in F; (ii) $T'(F')$ is weak*-closed in E'; and (iii) $T'(F')$ is closed in E';*

4. *when T is an isometry onto its range, T' is a quotient map $F' \to E'$, and factors to give an isometric isomorphism $F'/T(E)^\circ \to E'$;*

5. *when T is a quotient map, $T' : F' \to (\ker T)^\circ$ is an isometric isomorphism.* □

1.5 Algebras

An *algebra* is a vector space $\mathcal{A}$ (we only consider the case of complex scalars) together with a multiplication, called an *algebra product*, $\mathcal{A} \times \mathcal{A} \to \mathcal{A}$; $(a, b) \mapsto ab$, which is associative and respects the vector space operations:

$$(ab)c = a(bc), a(b+c) = ab + ac, (b+c)a = ba + ca \qquad (a, b, c \in \mathcal{A})$$
$$(\alpha a)b = a(\alpha b) = \alpha(ab) \qquad (\alpha \in \mathbb{C}, a, b \in \mathcal{A}).$$

We say that $\mathcal{A}$ is *commutative* if the multiplication is abelian. We say that $\mathcal{A}$ is *unital* if there is a multiplicative identity, a *unit*, $e = e_{\mathcal{A}}$ in $\mathcal{A}$. We define the *unitisation* of $\mathcal{A}$ to be $\mathcal{A}^\flat = \mathcal{A} \oplus \mathbb{C}$ with multiplication

$$(a, \alpha)(b, \beta) = (ab + \beta a + \alpha b, \alpha\beta) \qquad (a, b \in \mathcal{A}, \alpha, \beta \in \mathbb{C}).$$

Thus $\mathcal{A}^\flat$ is a unital algebra with unit $(0, 1)$. We define $\mathcal{A}^\sharp$ to be the conditional unitisation; that is, $\mathcal{A}^\sharp$ is $\mathcal{A}$ if $\mathcal{A}$ is unital, and is $\mathcal{A}^\flat$ otherwise.

An element $a \in \mathcal{A}$ in a unital algebra $\mathcal{A}$ is *invertible* if for some (necessarily unique) element $b \in \mathcal{A}$, we have $ab = ba = e_{\mathcal{A}}$. In this case we write $b = a^{-1}$. Then $\operatorname{Inv} \mathcal{A}$ is the set of invertible elements in $\mathcal{A}$, which forms a group with the multiplication product.

A *subalgebra* of an algebra $\mathcal{A}$ is a linear subspace $\mathcal{B} \subseteq \mathcal{A}$ such that $ab \in \mathcal{B}$ for each $a, b \in \mathcal{B}$. An *ideal* in an algebra $\mathcal{A}$ is a subalgebra $\mathcal{I} \subseteq \mathcal{A}$ such that, if $a \in \mathcal{A}$ and $b \in \mathcal{I}$, then $ab, ba \in \mathcal{I}$. Similarly we define the notion of a *left-ideal* and a *right-ideal*. An ideal $\mathcal{I}$ is *proper* if $0 \subsetneq \mathcal{I} \subsetneq \mathcal{A}$.

For algebras $\mathcal{A}$ and $\mathcal{B}$, let $T : \mathcal{A} \to \mathcal{B}$ be a linear map. Then T is a *homomorphism* if $T(ab) = T(a)T(b)$ for each $a, b \in \mathcal{A}$. If T is a bijection, then T is an *isomorphism*. When T is injective, we can identify $\mathcal{A}$ with its image in $\mathcal{B}$, so that $\mathcal{A}$ can be viewed as a subalgebra of $\mathcal{B}$. The kernel of a homomorphism is an ideal.

Let V be a vector space and $\mathcal{L}(V)$ be the vector space of all linear maps from V to V. Then $\mathcal{L}(V)$ is an algebra with the composition product. Let $\mathcal{A}$ be an algebra, and $\theta : \mathcal{A} \to \mathcal{L}(V)$ be a homomorphism. Then θ is a *representation* of $\mathcal{A}$ on V. As $\mathcal{A}$ is a vector space, for $a \in \mathcal{A}$, define $T_a \in \mathcal{L}(\mathcal{A}^\sharp)$ by

$$T_a(b) = ab \qquad (b \in \mathcal{A}^\sharp).$$

Then the map $\mathcal{A} \to \mathcal{L}(\mathcal{A}^\sharp); a \mapsto T_a$ is an injective homomorphism (as $T_a(e_{\mathcal{A}^\sharp}) = a$), called the *left regular representation*. Thus we see that every algebra arises as a subalgebra of $\mathcal{L}(V)$ for some vector space V.

Subalgebras are thus too general an object to study, which is (roughly speaking) why ideals are useful. Unfortunately, some algebras have a lack of "good" ideals to study, which leads to the concept of *semi-simplicity*.

Definition 1.5.1. Let V be a vector space, let $\mathcal{A}$ be an algebra, and suppose we have a bilinear map $\mathcal{A} \times V \to V; (a, v) \mapsto a \cdot v$, such that

$$(ab) \cdot v = a \cdot (b \cdot v) \qquad (a, b \in \mathcal{A}, v \in V).$$

Then V is a *left $\mathcal{A}$-module*. Similarly we have the notion of a *right-module*, or a *bi-module*, where we also insist that

$$(a \cdot v) \cdot b = a \cdot (v \cdot b) \qquad (a, b \in \mathcal{A}, v \in V).$$

Given a left $\mathcal{A}$-module V, define $T : \mathcal{A} \to \mathcal{L}(V)$ by

$$T(a)(v) = a \cdot v \qquad (a \in \mathcal{A}, v \in V).$$

Then T is a representation of V on $\mathcal{A}$. Conversely, given a representation $T : \mathcal{A} \to \mathcal{L}(V)$, we can make V into a left $\mathcal{A}$-module by setting $a \cdot v = T(a)(v)$. Thus left-modules and representations are the same thing, and the following definitions hence also apply to representations.

Definition 1.5.2. Let $\mathcal{A}$ be an algebra and V be a left $\mathcal{A}$-module. Then a *submodule* of V is a subspace U of V such that $a \cdot u \in U$ for each $a \in \mathcal{A}$ and $u \in U$.

We say that V is *non-trivial* if $\{0\} \neq \mathcal{A} \cdot V = \{a \cdot v : a \in \mathcal{A}, v \in V\}$; we say that V is *simple* if it is non-trivial and if $\{0\}$ and V are the only submodules of V.

We can view the simple representations as being the most basic types of representation, as there is no "redundancy", in the sense that

$$\operatorname{lin} \mathcal{A} \cdot v = \mathcal{A} \cdot v = \{a \cdot v : a \in \mathcal{A}\} = V$$

for every non-zero $v \in V$, as $\operatorname{lin} \mathcal{A} \cdot v = \mathcal{A} \cdot v$ is a submodule of V. We can now give the definition of a useful class of ideals.

Definition 1.5.3. We say that an ideal $\mathcal{I}$ in an algebra $\mathcal{A}$ is a *primitive* ideal when $\mathcal{I}$ is the kernel of a simple representation.

An algebra $\mathcal{A}$ is *primitive* if $\{0\}$ is a primitive ideal.

The trivial representation of $\mathcal{L}(V)$ on V shows that $\mathcal{L}(V)$ is a primitive algebra.

Definition 1.5.4. The *radical* of an algebra $\mathcal{A}$, denoted $\operatorname{rad} \mathcal{A}$, is the intersection of the primitive ideals of $\mathcal{A}$. An algebra $\mathcal{A}$ is *semi-simple* if $\operatorname{rad} \mathcal{A} = \{0\}$ and is *radical* if $\operatorname{rad} \mathcal{A} = \mathcal{A}$.

Thus an algebra $\mathcal{A}$ is semi-simple if and only if, for each non-zero $a \in \mathcal{A}$, we can find a simple representation $T : \mathcal{A} \to \mathcal{L}(V)$ with $T(a) \neq 0$. Equivalently, this is the case if and only if we can find a left $\mathcal{A}$-module V and $v \in V$ so that $a \cdot v \neq 0$ and $\mathcal{A} \cdot v$ has no proper submodules. Thus, in some sense, we can study $\mathcal{A}$ by just looking at the simple representations. We immediately see that $\mathcal{L}(V)$ is semi-simple for every vector space V.

Let $\mathcal{A}$ be a unital algebra, and $a \in \mathcal{A}$. Then the *spectrum* of a is

$$\sigma_{\mathcal{A}}(a) = \{z \in \mathbb{C} : ze_{\mathcal{A}} - a \notin \operatorname{Inv}\mathcal{A}\},$$

and the *spectral radius* of a is

$$\nu_{\mathcal{A}}(a) = \sup\{|z| : z \in \sigma_{\mathcal{A}}(a)\}.$$

For an arbitrary algebra $\mathcal{A}$ and $a \in \mathcal{A}$, we set $\sigma_{\mathcal{A}}(a) = \sigma_{\mathcal{A}^\sharp}(a)$. When V is a finite-dimensional vector space, we see that the spectrum of $T \in \mathcal{L}(V)$ is simply the set of eigenvalues of T.

Theorem 1.5.5. *Let $\mathcal{A}$ be an algebra. Then* $\operatorname{rad}\mathcal{A}$ *is an ideal in $\mathcal{A}$,* $\operatorname{rad}\mathcal{A} = \operatorname{rad}\mathcal{A}^\sharp$*, and*

$$\begin{aligned}
\operatorname{rad}\mathcal{A}^\sharp &= \{a \in \mathcal{A} : e_{\mathcal{A}^\sharp} - ba \in \operatorname{Inv}\mathcal{A}^\sharp \ (b \in \mathcal{A}^\sharp)\} \\
&= \{a \in \mathcal{A} : \sigma_{\mathcal{A}}(ba) = \{0\} \textit{ or } \emptyset \ (b \in \mathcal{A}^\sharp)\} \\
&= \{a \in \mathcal{A} : e_{\mathcal{A}^\sharp} - ab \in \operatorname{Inv}\mathcal{A}^\sharp \ (b \in \mathcal{A}^\sharp)\} \\
&= \{a \in \mathcal{A} : \sigma_{\mathcal{A}}(ab) = \{0\} \textit{ or } \emptyset \ (b \in \mathcal{A}^\sharp)\}.
\end{aligned}$$

□

Proposition 1.5.6. *Let $\mathcal{A}$ be an algebra and $\mathcal{I}$ be an ideal in $\mathcal{A}$. Then* $\operatorname{rad}\mathcal{I} = \mathcal{I} \cap \operatorname{rad}\mathcal{A}$.

□

1.6 Banach algebras

A *Banach algebra* is an algebra $\mathcal{A}$ with a norm $\|\cdot\|$ such that $(\mathcal{A}, \|\cdot\|)$ is a Banach space and

$$\|ab\| \leq \|a\|\|b\| \qquad (a, b \in \mathcal{A}).$$

The definitions of homomorphism, representation, module etc. all follow over, where we insist on bounded maps and Banach spaces, in the appropriate places. In particular, when $\mathcal{A}$ is a Banach algebra, we give $\mathcal{A}^\flat$ the norm

$$\|(a, \alpha)\| = \|a\| + |\alpha| \qquad (a \in \mathcal{A}, \alpha \in \mathbb{C}).$$

A Banach space E is a *Banach left $\mathcal{A}$-module* if E is a left $\mathcal{A}$-module, and

$$\|a \cdot x\| \leq \|a\|\|x\| \qquad (a \in \mathcal{A}, x \in E).$$

Similarly, we get the notion of a *Banach right $\mathcal{A}$-module* and a *Banach $\mathcal{A}$-bimodule*. If the module action is not norm-decreasing, then a simple re-norming gives an equivalent norm for which the module action is norm-decreasing.

For a Banach space E, $\mathcal{B}(E)$ is a Banach algebra with respect to the operator norm. The left regular representation $\mathcal{A} \to \mathcal{B}(\mathcal{A}^\sharp); a \mapsto T_a$ is a norm-decreasing homomorphism, and so $\mathcal{A}$ is a left bimodule over itself. In fact, $\|T_a\| = \|a\|$, so that every Banach algebra can be isometrically identified with a subalgebra of $\mathcal{B}(E)$ for some Banach space E.

The topological structure of a Banach algebra implies some algebraic structure.

Proposition 1.6.1. *Let $\mathcal{A}$ be a unital Banach algebra, and suppose that $a \in \mathcal{A}$ is such that $\|e_{\mathcal{A}} - a\| < 1$. Then $a \in \operatorname{Inv} \mathcal{A}$. Consequently,* $\operatorname{Inv} \mathcal{A}$ *is an open subset of $\mathcal{A}$.*

Proof. As $\|e_{\mathcal{A}} - a\| < 1$, the sum $b = e_{\mathcal{A}} + \sum_{n=1}^{\infty}(e_{\mathcal{A}} - a)^n$ converges in $\mathcal{A}$. Then we have

$$b(e_{\mathcal{A}} - a) = (e_{\mathcal{A}} - a)b = \lim_{N\to\infty} \sum_{n=1}^{N}(e_{\mathcal{A}} - a)^n = b - e_{\mathcal{A}},$$

so that $ab = ba = e_{\mathcal{A}}$, and hence $a \in \operatorname{Inv} \mathcal{A}$.

Suppose that $a \in \operatorname{Inv} \mathcal{A}$ and $b \in \mathcal{A}$ with $\|b\| < \|a^{-1}\|^{-1}$. Then $\|a^{-1}b\| \leq \|a^{-1}\|\|b\| < 1$, so that $e_{\mathcal{A}} + a^{-1}b \in \operatorname{Inv} \mathcal{A}$. Thus, as $a \in \operatorname{Inv} \mathcal{A}$, we have that $a(e_{\mathcal{A}} + a^{-1}b) = a + b \in \operatorname{Inv} \mathcal{A}$. Hence $\operatorname{Inv} \mathcal{A}$ is an open subset of $\mathcal{A}$. □

Theorem 1.6.2. *Let $\mathcal{A}$ be a unital Banach algebra, and let $a \in \mathcal{A}$. Then $\sigma_{\mathcal{A}}(a)$ is a non-empty, compact subset of $\mathbb{C}$. Furthermore,*

$$\nu_{\mathcal{A}}(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \inf_{n\in\mathbb{N}} \|a^n\|^{1/n}.$$ □

Theorem 1.6.3. (Spectral mapping theorem) *Let $\mathcal{A}$ be a Banach algebra, let $a \in \mathcal{A}$, and let $p \in \mathbb{C}[X]$ be a polynomial. Then $\sigma_{\mathcal{A}}(p(a)) = \{p(z) : z \in \sigma_{\mathcal{A}}(a)\}$.* □

Consequently, we see that

$$\operatorname{rad} \mathcal{A} = \{a \in \mathcal{A} : \lim_n \|(ab)^n\|^{1/n} = 0 \ (b \in \mathcal{A})\},$$

giving a purely topological characterisation of the radical.

For a Banach algebra $\mathcal{A}$, we can turn $\mathcal{A}'$ into a Banach $\mathcal{A}$-bimodule by setting

$$\langle a \cdot \mu, b \rangle = \langle \mu, ba \rangle = \langle \mu \cdot b, a \rangle \qquad (a, b \in \mathcal{A}, \mu \in \mathcal{A}').$$

More generally, if E is a Banach left $\mathcal{A}$-module, E' becomes a Banach right $\mathcal{A}$-module by setting

$$\langle \mu \cdot a, x \rangle = \langle \mu, a \cdot x \rangle \qquad (a \in \mathcal{A}, x \in E, \mu \in E').$$

In a similar fashion, if E is a Banach right $\mathcal{A}$-module, then E' becomes a Banach left $\mathcal{A}$-module. Thus we can also make $\mathcal{A}''$ into a Banach $\mathcal{A}$-bimodule, and so forth.

When E and F are Banach left $\mathcal{A}$-modules, and $T \in \mathcal{B}(E, F)$, we say that T is a *left $\mathcal{A}$-module homomorphism* if we have

$$a \cdot T(x) = T(a \cdot x) \qquad (a \in \mathcal{A}, x \in E).$$

We define *right $\mathcal{A}$-module homomorphisms* and *$\mathcal{A}$-bimodule homomorphisms* in a similar fashion. We can show that if T is a left $\mathcal{A}$-module homomorphism, then $T' : F' \to E'$ is a right $\mathcal{A}$-module homomorphism.

While conditional unitisations are a useful tool for dealing with non-unital Banach algebras, often we need a more nuanced approach.

Definition 1.6.4. A net (a_α) is a Banach algebra $\mathcal{A}$ is a *left approximate identity* if $a_\alpha a \to a$ in norm, for each $a \in \mathcal{A}$. Similarly we have the notions of *right approximate identity* and *approximate identity*. If (a_α) is norm bounded, then we have a *bounded (left/right) approximate identity*.

Often we can construct a bounded net (a_α) such that, say, $aa_\alpha \to a$ in the weak-topology on $\mathcal{A}$, for each $a \in \mathcal{A}$. As the weak and norm closures of a bounded convex set coincide, a standard argument allows us to find a bounded net (b_β), formed out of convex combinations of the a_α, such that $ab_\beta \to a$ in norm, for each $a \in \mathcal{A}$.

Definition 1.6.5. Let $\mathcal{A}$ be a Banach algebra and E be a closed submodule of $\mathcal{A}'$, so that $E' = \mathcal{A}''/E^\circ$. Let $Q : \mathcal{A}'' \to \mathcal{A}''/E^\circ = E'$ be the quotient map. When $Q \circ \kappa_{\mathcal{A}}$ is an isomorphism, we say that $\mathcal{A}$ is a *dual Banach algebra* (see [Runde, 2002, Section 4.4]).

Proposition 1.6.6. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is a dual Banach algebra if and only if $\mathcal{A}$ is isomorphic, as a $\mathcal{A}$-bimodule, to F' for some Banach $\mathcal{A}$-bimodule F.*

Proof. Let $\mathcal{A}$ be a dual Banach algebra with respect to $E \subseteq \mathcal{A}'$. It is clear that $\kappa_{\mathcal{A}}$ and the map Q in the definition are both $\mathcal{A}$-bimodule homomorphisms. Thus $\mathcal{A}$ is isomorphic, as a $\mathcal{A}$-bimodule, to E'.

Conversely, let F be as in the hypothesis, and let $T : \mathcal{A} \to F'$ be an $\mathcal{A}$-bimodule isomorphism. Then let $S = T' \circ \kappa_F : F \to \mathcal{A}'$, so that S is an $\mathcal{A}$-bimodule isomorphism onto its range. Let $E = S(F)$, so that E is a submodule of $\mathcal{A}'$, and we can treat S as an isomorphism from F to E. For $x \in E$, let $x = S(y)$ for some $y \in F$, so that for $a \in \mathcal{A}$, we have

$$\begin{aligned}\langle Q(\kappa_{\mathcal{A}}(a)), x\rangle &= \langle \kappa_{\mathcal{A}}(a) + E^\circ, S(y)\rangle = \langle S(y), a\rangle = \langle T'(\kappa_F(y)), a\rangle = \langle T(a), y\rangle \\ &= \langle T(a), S^{-1}(x)\rangle = \langle (S^{-1})'(T(a)), x\rangle.\end{aligned}$$

Hence $Q \circ \kappa_{\mathcal{A}} = (S^{-1})' \circ T$, and is thus an isomorphism, as required. □

1.7 Arens products

We have seen that the bidual E'' of a Banach space E is a useful object to study: it has compact bounded subsets in a useful topology, yet has the same finite-dimensional structure as E, for example. As a Banach algebra $\mathcal{A}$ is also a Banach space, we can form $\mathcal{A}''$. However, it would obviously be natural to extend, not just the linear structure, but also the algebraic structure to $\mathcal{A}''$. This is what the Arens products do.

We proceed with a little generality, noting that the map $\mathcal{A} \times \mathcal{A} \to \mathcal{A}; (a, b) \mapsto ab$ is certainly a continuous bilinear map. The norm of a bilinear map $B : E \times F \to G$ is simply

$$\|B\| = \sup\{\|B(x, y)\| : \|x\|\|y\| = 1\}.$$

The following ideas were first studied in [Arens, 1951].

Given a bilinear map $B : E\times F \to G$, define $C_1 : G'\times E \to F'$ and $D_1 : F''\times G' \to E'$ by

$$\begin{aligned}\langle C_1(\phi, x), y\rangle &= \langle \phi, B(x, y)\rangle \qquad (x \in E, y \in F, \phi \in G') \\ \langle D_1(\Psi, \phi), x\rangle &= \langle \Psi, C_1(\phi, x)\rangle \qquad (x \in E, \phi \in G', \Psi \in F'').\end{aligned}$$

Then we define $B_1 : E'' \times F'' \to G''$ by

$$\langle B_1(\Phi, \Psi), \phi\rangle = \langle \Phi, D_1(\Psi, \phi)\rangle \qquad (\Phi \in E'', \Psi \in F'', \phi \in G').$$

Similarly, we swap the roles of E and F to define $C_2 : G' \times F \to E'$ and $D_2 : E'' \times G' \to F'$ by

$$\begin{aligned}\langle C_2(\phi, y), x\rangle &= \langle \phi, B(x, y)\rangle \quad (x \in E, y \in F, \phi \in G') \\ \langle D_2(\Phi, \phi), y\rangle &= \langle \Phi, C_2(\phi, y)\rangle \quad (y \in F, \phi \in G', \Phi \in E'').\end{aligned}$$

Then we define $B_2 : E'' \times F'' \to G''$ by

$$\langle B_2(\Phi, \Psi), \phi\rangle = \langle \Psi, D_2(\Phi, \phi)\rangle \quad (\Phi \in E'', \Psi \in F'', \phi \in G').$$

Proposition 1.7.1. *Let E, F and G be Banach spaces, let $B : E\times F \to G$ be a continuous bilinear map, and let $B_1, B_2 : E'' \times F'' \to G''$ be as above. Then, for $i = 1, 2$, B_i is a bilinear map, $\|B_i\| = \|B\|$ and $B_i(\kappa_E(x), \kappa_F(y)) = \kappa_G(B(x, y))$ for $x \in E$ and $y \in F$.*

Proof. It is simple to check that, for $i = 1, 2$, the maps C_i, D_i and B_i are bilinear. Then we have

$$\begin{aligned}\|C_1\| &= \sup\{\|C_1(\phi, x)\| : \|\phi\| = \|x\| = 1\}\\ &= \sup\{|\langle C_1(\phi, x), y\rangle| : \|\phi\| = \|x\| = \|y\| = 1\}\\ &= \sup\{\|B(x, y)\| : \|x\| = \|y\| = 1\} = \|B\|.\end{aligned}$$

Similarly, $\|D_1\| = \|C_1\|$ and $\|B_1\| = \|D_1\|$, so that $\|B_1\| = \|B\|$. Analogously, $\|C_2\| = \|B\|$, $\|D_2\| = \|C_2\|$, $\|B_2\| = \|D_2\|$, and so $\|B_2\| = \|B\|$.

For $x \in E$, $y \in F$ and $\phi \in G'$, we have

$$\langle B_1(\kappa_E(x), \kappa_F(y)), \phi\rangle = \langle D_1(\kappa_F(y), \phi), x\rangle = \langle C_1(\phi, x), y\rangle = \langle \kappa_G B(x, y), \phi\rangle,$$
$$\langle B_2(\kappa_E(x), \kappa_F(y)), \phi\rangle = \langle D_2(\kappa_E(x), \phi), y\rangle = \langle \kappa_E(x), C_2(\phi, y)\rangle = \langle \kappa_G B(x, y), \phi\rangle,$$

as required. □

As in the remark after Theorem 1.4.9, for $\Phi \in E''$, we can find a bounded net (x_α) in E such that $\kappa_E(x_\alpha) \to \Phi$ in the weak*-topology on E''. Similarly, let (y_β) be a bounded net such that $\kappa_F(y_\beta) \to \Psi \in F''$. Then we have

$$B_1(\Phi, \Psi) = \lim_\alpha \lim_\beta \kappa_G B(x_\alpha, y_\beta) \quad , \quad B_2(\Phi, \Psi) = \lim_\beta \lim_\alpha \kappa_G B(x_\alpha, y_\beta)$$

where the limits converge in the weak*-topology on G'' (if you are unhappy about the existence of a limit, take the limits along ultrafilters). We now see clearly why we have two extensions B_1 and B_2.

As the algebra product on a Banach algebra $\mathcal{A}$ is a bilinear map, we can apply this result to find two bilinear maps from $\mathcal{A}'' \times \mathcal{A}''$ to $\mathcal{A}''$ which extend the product. Following the construction through, and recalling that $\mathcal{A}'$ is a Banach $\mathcal{A}$-bimodule, we first define bilinear maps $\mathcal{A}'' \times \mathcal{A}' \to \mathcal{A}'$ and $\mathcal{A}' \times \mathcal{A}'' \to \mathcal{A}'$ by

$$\langle \Phi \cdot \mu, a\rangle = \langle \Phi, \mu \cdot a\rangle \quad , \quad \langle \mu \cdot \Phi, a\rangle = \langle \Phi, a \cdot \mu\rangle \quad (a \in \mathcal{A}, \mu \in \mathcal{A}', \Phi \in \mathcal{A}'').$$

Then we define bilinear maps $\Box, \Diamond : \mathcal{A}'' \times \mathcal{A}'' \to \mathcal{A}''$, corresponding to B_1 and B_2, by

$$\langle \Phi \Box \Psi, \mu \rangle = \langle \Phi, \Psi \cdot \mu \rangle \quad , \quad \langle \Phi \Diamond \Psi, \mu \rangle = \langle \Psi, \mu \cdot \Phi \rangle \quad (\mu \in \mathcal{A}', \Phi, \Psi \in \mathcal{A}'').$$

We can check that $\Box$ and $\Diamond$ are actually algebra products (checking associativity is non-trivial) so that

$$\kappa_{\mathcal{A}} : \mathcal{A} \to (\mathcal{A}'', \Box) \quad , \quad \kappa_{\mathcal{A}} : \mathcal{A} \to (\mathcal{A}'', \Diamond)$$

are isometric isomorphisms onto their ranges. Treating $\mathcal{A}''$ as a $\mathcal{A}$-bimodule in the canonical way, we have

$$a \cdot \Phi = \kappa_{\mathcal{A}}(a) \Box \Phi = \kappa_{\mathcal{A}}(a) \Diamond \Phi \qquad (a \in \mathcal{A}, \Phi \in \mathcal{A}''),$$

and a similar result "on the right". When $\Box = \Diamond$, we say that $\mathcal{A}$ is *Arens regular*.

There is a standard characterisation of when $B : E \times F \to G$ gives rise to $B_1 = B_2$.

Theorem 1.7.2. *Let E, F and G be Banach spaces and let $B : E \times F \to G$ be a bounded bilinear map. Let $B_i : E'' \times F'' \to G''$ be the extensions defined as above. Then the following are equivalent:*

1. *$B_1 = B_2$;*
2. *for each $\phi \in G'$, the map $E \to F'; x \mapsto C_1(\phi, x)$ is weakly-compact;*
3. *for each $\phi \in G'$, the map $F \to E'; y \mapsto C_2(\phi, y)$ is weakly-compact;*
4. *for each pair of bounded sequences (x_n) in E and (y_n) in F, and each $\phi \in G'$, we have*
$$\lim_n \lim_m \langle \phi, B(x_n, y_m) \rangle = \lim_m \lim_n \langle \phi, B(x_n, y_m) \rangle,$$
whenever both the iterated limits exist. $\Box$

An element $\Xi \in \mathcal{A}''$ is a *mixed identity* if we have

$$\Phi \Box \Xi = \Xi \Diamond \Phi = \Phi \qquad (\Phi \in \mathcal{A}'').$$

Equivalently, this is if and only if

$$\Xi \cdot \mu = \mu \cdot \Xi = \mu \quad (\mu \in \mathcal{A}'), \ \text{ or } \ a \cdot \Xi = \Xi \cdot a = \kappa_{\mathcal{A}}(a) \quad (a \in \mathcal{A}).$$

Proposition 1.7.3. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}''$ has a mixed identity if and only if $\mathcal{A}$ has a bounded approximate identity.*

Proof. We just give a sketch. If we have a mixed identity $\Xi \in \mathcal{A}''$, then let (a_α) be a bounded net in $\mathcal{A}$ converging to Ξ in the weak*-topology on $\mathcal{A}''$. For $a \in \mathcal{A}$ and $\mu \in \mathcal{A}'$, we then have

$$\langle \mu, a\rangle = \langle \kappa_{\mathcal{A}}(a), \mu\rangle = \langle \kappa_{\mathcal{A}}(a)\Diamond\Xi, \mu\rangle = \langle a \cdot \Xi, \mu\rangle = \lim_\alpha \langle \mu, aa_\alpha\rangle$$

and similarly, $\lim_\alpha \langle \mu, a_\alpha a\rangle = \langle \mu, a\rangle$, so that (a_α) is, weakly, a bounded approximate identity. Using Mazur's theorem, we see that a suitable convex combination of (a_α) forms a bounded approximate identity.

Conversely, let (a_α) be a bounded approximate identity, and let Ξ be the (ultrafilter) limit of (a_α) in $\mathcal{A}''$. Then, for $a \in \mathcal{A}''$ and $\mu \in \mathcal{A}'$, we have

$$\langle \Xi, a \cdot \mu\rangle = \lim_\alpha \langle a \cdot \mu, a_\alpha\rangle = \lim_\alpha \langle \mu, a_\alpha a\rangle = \langle \mu, a\rangle = \lim_\alpha \langle \mu, aa_\alpha\rangle = \langle \Xi, \mu \cdot a\rangle.$$

We see immediately that $\Xi \cdot a = a \cdot \Xi = \kappa_{\mathcal{A}}(a)$, so that Ξ is a mixed identity. $\square$

At one extreme, we have Arens regular Banach algebras; at the other, we always have that $\Box$ and $\Diamond$ agree on $\kappa_{\mathcal{A}}(\mathcal{A})$, though there are examples of Banach algebras when $\Box$ and $\Diamond$ only agree here (see, for example, [Lau, Ülger, 1996, Corollary 5.5]).

Definition 1.7.4. The *topological centres*, $\mathfrak{Z}_t^{(1)}(\mathcal{A}'')$ and $\mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, are defined as

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Phi\Box\Psi = \Phi\Diamond\Psi \ (\Psi \in \mathcal{A}'')\},$$
$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \{\Phi \in \mathcal{A}'' : \Psi\Box\Phi = \Psi\Diamond\Phi \ (\Psi \in \mathcal{A}'')\}.$$

We have that, for $i = 1, 2$, $\kappa_{\mathcal{A}}(\mathcal{A}) \subseteq \mathfrak{Z}_t^{(i)}(\mathcal{A}'')$. We can easily show that $\mathcal{A}$ is Arens regular if and only if $\mathfrak{Z}_t^{(i)}(\mathcal{A}'') = \mathcal{A}''$ for one (or equivalently both) of $i = 1$ or $i = 2$. We say that $\mathcal{A}$ is *left strongly Arens irregular* if $\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \mathcal{A}$; similarly we have *right strongly Arens irregular* and *strongly Arens irregular*. These definitions were introduced in [Lau, Ülger, 1996], and studied in detail in [Dales, Lau, 2004], for example.

A good survey of results about the Arens products is [Duncan, Hosseiniun, 1979]; [Civin, Yood, 1961] is the first systematic study of Arens products. For example, let E be a Banach space and let $\mathcal{A}(E)$ be the algebra of *approximable operators* (see Section 2.1). Then $\mathcal{A}(E)$ is Arens regular if and only if E is reflexive (see Theorem 2.5.2 and Theorem 2.7.36). Similarly, in the positive directly, every C*-algebra (see the next section) is Arens regular

Definition 1.7.5. Let G be a group. Form $l^1(G)$ as a Banach space, so that we can write each $x \in l^1(G)$ in the form

$$x = \sum_{g \in G} a_g e_g$$

where $(a_g)_{g\in G}$ is an absolutely summable sequence in $\mathbb{C}$. We define an algebra product on $l^1(G)$ by *convolution*, that is, for $x = \sum_{g\in G} a_g e_g$ and $y = \sum_{g\in G} b_g e_g$, we have

$$x * y = \sum_{g,h\in G} a_g b_h e_{gh} = \sum_{g\in G} \Big(\sum_{h\in G} a_h b_{g^{-1}h} \Big) e_g.$$

Then we have

$$\|x * y\| = \sum_{g\in G} \Big| \sum_{h\in G} a_h b_{g^{-1}h} \Big| \leq \sum_{g\in G} \sum_{h\in G} |a_h||b_{g^{-1}h}| = \sum_{h\in G} |a_h| \sum_{g\in G} |b_g| = \|x\|\|y\|,$$

so that $(l^1(G), *)$ is a Banach algebra, called the *group algebra* of G.

For more information about group algebras, see [Dales, 2000, Section 3.3]. When G is a locally compact group, we can form $L^1(G)$ is a similar manner to the above, using the *Haar measure*. Again, see the reference for details. Then $l^1(G)$ is Arens regular if and only if G is a finite group (the same holds for $L^1(G)$), as first shown by Young (see [Duncan, Hosseiniun, 1979, Section 2]). Indeed, when G is infinite, $l^1(G)$ is strongly Arens irregular.

Conversely, consider $l^1(\mathbb{Z})$ as a Banach space. Then we can define a multiplication pointwise, and this clearly gives rise to a Banach algebra. A simple argument shows that this Banach algebra is Arens regular, whereas the convolution algebra on $l^1(\mathbb{Z})$ is not. Hence the nature of the algebra product on a Banach algebra, and not just the geometry of the underlying Banach space, is important as far as Arens regularity is concerned.

1.8 C*-algebras

We shall briefly sketch some results about C*-algebras, which can be thought of as special Banach algebras. We shall not directly study C*-algebras, but shall instead use them as motivating examples. For more details, see, for example, [Arveson, 1976].

Definition 1.8.1. Let $\mathcal{A}$ be an algebra. A map $* : \mathcal{A} \to \mathcal{A}$, written $a \mapsto a^*$, is an *involution* when we have:

1. $(\alpha a + b)^* = \overline{\alpha} a^* + b^*$ for $\alpha \in \mathbb{C}$ and $a, b \in \mathcal{A}$;
2. $(ab)^* = b^* a^*$ for $a, b \in \mathcal{A}$;
3. $(a^*)^* = a$ for $a \in \mathcal{A}$.

When $\mathcal{A}$ is a Banach algebra and $\|a^* a\| = \|a\|^2$ for each $a \in \mathcal{A}$, we say that $(\mathcal{A}, *)$ is a C*-algebra.

For example, let H be a Hilbert space and $T \in \mathcal{B}(H)$. Define $T^* \in \mathcal{B}(H)$ by

$$[T^*(x), y] = [x, T(y)] \qquad (x, y \in H).$$

We can check that $T \mapsto T^*$ is an involution, and that $\|T^*T\| = \|T\|^2$, so that $(\mathcal{B}(H), *)$ is a C*-algebra. In fact, let $\mathcal{A}$ be a subalgebra of $\mathcal{B}(H)$ such that $\mathcal{A} = \mathcal{A}^* := \{T^* : T \in \mathcal{A}\}$. Then $(\mathcal{A}, *)$ is a C*-algebra. We shall see that every C*-algebra arises in this way.

Let $\mathcal{A}$ be a C*-algebra with a unit $e_{\mathcal{A}}$. Then, for $a \in \mathcal{A}$, we have $(e_{\mathcal{A}}^* a)^* = a^* e_{\mathcal{A}} = a^*$, so that $e_{\mathcal{A}}^* a = (a^*)^* = a$, and similarly $a e_{\mathcal{A}}^* = a$, so that $e_{\mathcal{A}}^* = e_{\mathcal{A}}$.

Theorem 1.8.2. *Let $\mathcal{A}$ be a C*-algebra, and $a \in \mathcal{A}$. Then we have:*

1. $\sigma_{\mathcal{A}}(a^*) = \overline{\sigma_{\mathcal{A}}(a)} := \{\overline{z} : z \in \sigma_{\mathcal{A}}(a)\}$ *and* $\nu_{\mathcal{A}}(a^*a) = \|a^*a\| = \|a\|^2$*;*
2. *suppose that $a^*a = aa^*$, that is, a is* normal. *Then $\nu_{\mathcal{A}}(a) = \|a\|$;*
3. *suppose that $a^* = a$, that is, a is* self-adjoint. *Then $\sigma_{\mathcal{A}}(a) \subseteq \mathbb{R}$.* □

Let $\mathcal{A}$ be a C*-algebra which is not unital. Then we see that, in general, our unitisation, $\mathcal{A}^\flat$, is not a C*-algebra (as the norm does not satisfy the correct condition). However, for $\alpha \in \mathbb{C}$ and $a \in \mathcal{A}$, we have

$$\sigma_{\mathcal{A}}(\alpha e_{\mathcal{A}} + a) = \{z \in \mathbb{C} : (z - \alpha)e_{\mathcal{A}} - a \notin \operatorname{Inv} \mathcal{A}\} = \{z + \alpha : z \in \sigma_{\mathcal{A}}(a)\}.$$

We then simply define

$$\begin{aligned} \|\alpha e_{\mathcal{A}^\flat} + a\|_{\mathcal{A}^\flat} &= \nu_{\mathcal{A}^\flat}(|\alpha|^2 e_{\mathcal{A}^\flat} + \alpha a^* + \overline{\alpha} a + aa^*)^{1/2} \\ &= \sup\{|z + |\alpha|^2|^{1/2} : z \in \sigma_{\mathcal{A}}(\alpha a^* + \overline{\alpha} a + aa^*)\}. \end{aligned}$$

We can check (using the spectral mapping theorem) that $(\mathcal{A}^\flat, \|\cdot\|_{\mathcal{A}^\flat})$ is a C*-algebra.

In particular, by Theorem 1.5.5, we see that every C*-algebra is semi-simple.

Proposition 1.8.3. *Let $\mathcal{A}$ be a C*-algebra. Then $\mathcal{A}$ has a bounded approximate identity (e_α) which satisfies $e_\alpha = e_\alpha^*$ and $\|e_\alpha\| \leq 1$ for each α.* □

Let H be a Hilbert space. A *representation* of a C*-algebra $\mathcal{A}$ is a homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$ which preserves $*$. By moving to a subspace of H if necessary, we may suppose that $\operatorname{lin}\{\pi(a)(x) : a \in \mathcal{A}, x \in H\}$ is dense in H. Then it is simple to show that when $\mathcal{A}$ is unital, we must have $\pi(e_{\mathcal{A}}) = \mathrm{Id}_H$. For the next theorem to make sense, we need to note that every ideal in a C*-algebra is automatically self-adjoint, so that the quotient is a C*-algebra in a natural way.

Theorem 1.8.4. *Let $\mathcal{A}$ and $\mathcal{B}$ be C*-algebras and $\pi : \mathcal{A} \to \mathcal{B}$ be a $*$-homomorphism. Then π is norm-decreasing, $\pi(A)$ is a C*-subalgebra of $\mathcal{B}$, and π factors to give an isometric $*$-isomorphism $\mathcal{A}/\ker\pi \to \pi(\mathcal{A})$. Consequently, every injective C*-algebra representation is an isometry.* $\square$

Let $\mu \in \mathcal{A}'$ for a unital C*-algebra $\mathcal{A}$. We say that μ is *positive* when $\langle \mu, a^*a\rangle \geq 0$ for each $a \in \mathcal{A}$, and that μ is a *state* when μ is positive and $\langle \mu, e_{\mathcal{A}}\rangle = 1$. Indeed, when μ is positive, we have $\|\mu\| = \langle \mu, e_{\mathcal{A}}\rangle$, and we can show that

$$|\langle \mu, b^*a\rangle|^2 \leq \langle \mu, a^*a\rangle\langle \mu, b^*b\rangle \qquad (a, b \in \mathcal{A}),$$

the Cauchy-Schwarz inequality. Furthermore, for $\mu \in A'$, μ is a state if and only if $\|\mu\| = \langle \mu, e_{\mathcal{A}}\rangle = 1$.

Let $\pi : \mathcal{A} \to \mathcal{B}(H)$ be a representation, and let $x \in H$ be such that $\|x\| = 1$. Define $\mu \in \mathcal{A}'$ by

$$\langle \mu, a\rangle = [\pi(a)(x), x] \qquad (a \in \mathcal{A}).$$

Then we have $\langle \mu, a^*a\rangle = [\pi(a^*a)(x), x] = [\pi(a)^*\pi(a)(x), x] = \|\pi(a)(x)\|^2$, so that μ is positive.

Theorem 1.8.5. (Gelfand-Naimark-Segal construction) *Let μ be a positive linear functional on a C*-algebra $\mathcal{A}$. Then there is a representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ and $x \in H$ such that $\langle \mu, a\rangle = [\pi(a)(x), x]$ for each $a \in \mathcal{A}$. We may suppose that x is* cyclic*, that is, $\mathrm{lin}\{\pi(a)(x) : a \in \mathcal{A}\}$ is dense in H.* $\square$

Notice that the states in $\mathcal{A}'$ form a convex set. We say that a state μ is *pure* when μ is an extreme point of this convex set, that is, whenever $\mu = t\mu_1 + (1-t)\mu_2$ for some $t \in (0, 1)$ and states μ_1 and μ_2, we must have that $\mu_1 = \mu_2 = \mu$.

Theorem 1.8.6. *Let $\pi : \mathcal{A} \to \mathcal{B}(H)$ be a representation with a cyclic vector x, and let μ be the state $\langle \mu, a\rangle = [\pi(a)(x), x]$. Then μ is a pure state if and only if π is simple (that is, there are no non-trivial invariant subspaces of H for π).* $\square$

Theorem 1.8.7. *Let $\mathcal{A}$ be a C*-algebra and $a \in \mathcal{A}$ be self-adjoint. Then there is a pure state μ such that $|\langle \mu, a\rangle| = \|a\|$.*

For each $a \in \mathcal{A}$, there exists a simple representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ and $x \in H$ with $\|x\| = 1$ and $\|\pi(a)(x)\| = \|a\|$. $\square$

We can now prove the Gelfand-Naimark theorem, which states that every C*-algebra $\mathcal{A}$ is isometrically a $*$-subalgebra of $\mathcal{B}(H)$ for some Hilbert space H. Indeed, for each non-zero $a \in \mathcal{A}$, let $\pi_a : \mathcal{A} \to \mathcal{B}(H_a)$ and $x_a \in H_a$ be given as above, so that $\|\pi_a(a)(x_a)\| = \|a\| > 0$. Then define

$$H = l^2\Big(\bigoplus_{a\in\mathcal{A}} H_a\Big) \quad , \quad \pi : \mathcal{A} \to H;\ a \mapsto \big(\pi_a(a)\big).$$

Thus, for $y = (y_b)_{b\in\mathcal{A}} \in H$ and $a \in \mathcal{A}$, we have

$$\|\pi(a)(y)\|^2 = \sum_{b\in\mathcal{A}} \|\pi_b(a)(y_b)\|^2 \leq \|a\|^2 \sum_{b\in\mathcal{A}} \|y_b\|^2 = \|a\|^2\|y\|^2,$$

so that π is norm-decreasing and is a representation. Then, treating x_a as a member of H, we have $\|\pi(a)(x_a)\| = \|\pi_a(a)(x_a)\| = \|a\|$, so that π is actually an isometry onto its range, as required.

Notice that, in general, H will be very large. For example, starting with $\mathcal{A} = \mathcal{B}(l^2)$, we do not have that $H = l^2$, a fact certainly seen by noting that there are operators $T \in \mathcal{B}(l^2)$ which do not attain their norm, and hence that there are pure states on $\mathcal{B}(l^2)$ which do not have the form $T \mapsto [T(x), x]$ for some $x \in l^2$.

Chapter 2

Operator ideals and Arens products

In this chapter we introduce tensor products of Banach spaces, and the associated idea of a tensor norm. When E and F are finite-dimensional Banach spaces, the tensor product of E with F, $E \otimes F$, can be thought of as the space of operators $\mathcal{B}(E', F)$, which in turn is simply a space of matrices (with respect to a choice of bases for E and F). Then the dual space is also a space of matrices: we shall see that the duality can be realised by the trace functional. If we have a norm on $\mathcal{B}(E', F)$ then we get the dual norm on $\mathcal{B}(F, E')$, the nuclear or integral norm. Tensor norms provide a framework in which to extend these ideas to the infinite-dimensional setting.

Tensor products reflect the local structure of Banach spaces, that is, the structure of finite-dimensional subspaces, and have proved to be useful in the study of such structure. However, tensor products also give rise to an interesting class of Banach algebras: essentially by taking the finite-rank operators on a Banach space E and then completing under a suitable norm (which may or may not be the operator norm). The behaviour of the Arens products on the bidual of such algebras has been shown to be closely linked to certain properties of the Banach space. For example, let $\mathcal{A}(E)$ be the approximable operators, the closure of $\mathcal{F}(E)$ with respect to the operator norm. Then $\mathcal{A}(E)$ is Arens regular if and only if E is reflexive.

In this chapter we shall study the topological centres of the bidual of the algebra of nuclear operators (with respect to an arbitrary tensor norm, so making this more general than the usual meaning of nuclear operator).

The first six sections in this chapter detail the results we shall need. We refer the reader to [Ryan, 2002] or [Defant, Floret, 1993] for more details on the topics in these sections. There is a short, self-contained account in [Diestel, Uhl, 1977] of many of the more important ideas in these initial sections. Proofs are omitted where they are standard (and can be found in the above references). We sketch the proofs of some results which are not standard; such results are surely known to experts in the area, but may not be presented in a suitable form in the literature.

2.1 Tensor products and tensor norms

The *tensor product* of vector spaces V and U is a vector space $V \otimes U$ together with a bilinear map $\iota : V \times U \to V \otimes U; (v, u) \mapsto v \otimes u$, such that whenever W is a vector space, and $T : V \times U \to W$ is a bilinear map, there is a unique linear map $\hat{T} : V \otimes U \to W$ such that the following diagram commutes:

$$\begin{tikzcd}
V \times U \arrow[r, "T"] \arrow[d, "\iota"'] & W \\
V \otimes U \arrow[ur, "\hat{T}"'] &
\end{tikzcd}$$

Of course, it can be shown that $V \otimes U$ always does exist, and it is easy to show that it is unique up to isomorphism. For $u \in V \otimes U$, we can write

$$u = \sum_{i=1}^{n} v_i \otimes u_i,$$

where $n \in \mathbb{N}$, $(v_i)_{i=1}^n \subseteq V$ and $(u_i)_{i=1}^n \subseteq U$. Such a representation is not unique.

When E and F are normed vector spaces, it seems natural to put a norm on $E \otimes F$. However, there are many ways to do this, not all of them agreeing with the algebraic structure of $E \otimes F$. We follow an approach which can be traced back to work of Schatten, [Schatten, 1950].

Recall that FIN is the class of finite-dimensional normed vector spaces. For a normed vector space E, let $\mathrm{FIN}(E)$ be all subspaces F of E such that $F \in \mathrm{FIN}$.

Definition 2.1.1. Let E and F be normed vectors spaces and α be a norm on $E \otimes F$. Then α is a *reasonable crossnorm* if we have:

1. $\alpha(x \otimes y) \leq \|x\|\|y\|$ for each $x \in E$ and $y \in F$;

2. for $\mu \in E'$ and $\lambda \in F'$, let $\mu \otimes \lambda$ be the linear functional on $E \otimes F$ defined by

$$\langle \mu \otimes \lambda, x \otimes y \rangle = \langle \mu, x \rangle \langle \lambda, y \rangle \quad (x \in E, y \in F)$$

   and linearity. Then we require that $\|\mu \otimes \lambda\|_\alpha \leq \|\mu\|\|\lambda\|$.

In fact, if α is a reasonable crossnorm, then $\alpha(x \otimes y) = \|x\|\|y\|$ and $\|\mu \otimes \lambda\|_\alpha = \|\mu\|\|\lambda\|$.

Definition 2.1.2. A *uniform crossnorm* is an assignment to each pair, (E, F), of Banach spaces, of a reasonable crossnorm α on $E \otimes F$ such that we have the following. Let D, E, F and G be Banach spaces, and let $T \in \mathcal{B}(D, E)$, $S \in \mathcal{B}(G, F)$. Then we form the bilinear map

$$T \otimes S : D \times G \to E \otimes F; (x, y) \mapsto T(x) \otimes S(y) \quad (x \in D, y \in G),$$

which extends to $D \otimes G$ by the tensorial property. Then we insist that $\|T \otimes S\| \leq \|T\|\|S\|$ with respect to the norm α on $D \otimes G$ and on $E \otimes F$.

For $u \in E \otimes F$, we write $\alpha(u, E \otimes F)$ to avoid confusion. Let D be a closed subspace of E, let G be a closed subspace of F, and let $u \in D \otimes G$. By considering the inclusion maps $D \to E$ and $G \to F$, we identify u with its image in $E \otimes F$, and for a uniform crossnorm α, we see that

$$\alpha(u, E \otimes F) \leq \alpha(u, D \otimes G).$$

Definition 2.1.3. Let α be a uniform crossnorm. Then α is *finitely generated* if, for each pair of Banach spaces E and F, and each $u \in E \otimes F$, we have

$$\alpha(u, E \otimes F) = \inf\{\alpha(u, M \otimes N) : M \in \mathrm{FIN}(E), N \in \mathrm{FIN}(F), u \in M \otimes N\}.$$

We call a finitely generated uniform crossnorm a *tensor norm*. We denote the completion of the normed space $(E \otimes F, \alpha)$ by $E \widehat{\otimes}_\alpha F$.

Definition 2.1.4. For Banach spaces E and F, and $u \in E \otimes F$, let $u^t \in F \otimes E$ be defined by $u^t = \sum_{i=1}^n y_i \otimes x_i$ when $u = \sum_{i=1}^n x_i \otimes y_i$. We call u^t the *transpose* of u and often refer to the map $u \mapsto u^t$ as the *swap map*. For a tensor norm α, define α^t by $\alpha^t(u, E \otimes F) = \alpha(u^t, F \otimes E)$, so that α^t is a tensor norm.

The two most common tensor norms, the injective and projective tensor norms (defined shortly) are *symmetric*, in that the swap map leaves them invariant, but this is not true for general tensor norms.

Let E and F be Banach spaces, and consider the bilinear map $E' \times F \to \mathcal{B}(E, F)$ given by

$$(\mu, y) : x \mapsto \langle \mu, x \rangle y \qquad (x \in E, y \in F, \mu \in E').$$

This extends to a map $E' \otimes F \to \mathcal{B}(E, F)$, whose image is $\mathcal{F}(E, F)$, the set of all bounded linear maps from E to F whose image is finite-dimensional, the *finite-rank operators*. We use this bijection to define a norm on $E' \otimes F$; more generally we use the map $\kappa_E \otimes \mathrm{Id}_F : E \otimes F \to E'' \otimes F = \mathcal{F}(E', F)$ to define a norm on $E \otimes F$. This gives the *injective tensor norm* ε.

Theorem 2.1.5. *Let E and F be Banach spaces, and $u \in E \otimes F$. Let $u = \sum_{i=1}^n x_i \otimes y_i$ be a representative of u. We then have*

$$\varepsilon(u, E \otimes F) = \sup \left\{ \left| \sum_{i=1}^n \langle \mu, x_i \rangle \langle \lambda, y_i \rangle \right| : \mu \in E'_{[1]}, \lambda \in F'_{[1]} \right\}.$$

Notice that this is independent of the representative chosen for u. The injective tensor norm is finitely generated and is a uniform crossnorm. If D is a closed subspace of E and G a closed subspace of F, then for $u \in D \otimes G$, we have $\varepsilon(u, D \otimes G) = \varepsilon(u, E \otimes F)$.

We can thus identify $E' \widehat{\otimes}_\varepsilon F$ with the closure of $\mathcal{F}(E, F)$ in $\mathcal{B}(E, F)$, the *approximable operators* $\mathcal{A}(E, F)$. It is common to write $\check{\otimes}$ for $\widehat{\otimes}_\varepsilon$.

Let E, F and G be Banach spaces and $T : E \times F \to G$ be a bounded bilinear map. Then there is a linear extension $\hat{T} : E \otimes F \to G$. It would be natural to give $E \otimes F$ a norm so that $\|\hat{T}\| = \|T\|$. This leads to the *projective tensor norm* π (indeed, it is enough to look at all maps $T : E \times F \to \mathbb{C}$).

Theorem 2.1.6. *Let E and F be Banach spaces and $u \in E \otimes F$. Then we have*

$$\pi(u, E \otimes F) = \inf \left\{ \sum_{i=1}^n \|x_i\| \|y_i\| : u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

The projective tensor norm is finitely generated and is a uniform crossnorm. For each $u \in E \widehat{\otimes}_\pi F$, and $\varepsilon > 0$, we can write, with convergence with respect to π,

$$u = \sum_{i=1}^\infty x_i \otimes y_i \quad , \quad \sum_{i=1}^\infty \|x_i\| \|y_i\| < \pi(u, E \widehat{\otimes}_\pi F) + \varepsilon.$$

Let $T : E \to D$ and $S : F \to G$ be quotient operators. Then $T \otimes S : E \widehat{\otimes}_\pi F \to D \widehat{\otimes}_\pi G$ is a quotient operator.

It is common to write $\widehat{\otimes}$ for $\widehat{\otimes}_\pi$. The projective tensor norm does not respect subspaces in the way that the injective tensor norm does. However, we have the following useful exception.

Proposition 2.1.7. *Let E and F be Banach spaces. Then the map $\kappa_E \otimes \kappa_F : E \widehat{\otimes} F \to E'' \widehat{\otimes} F''$ is an isometry, so that $E \widehat{\otimes} F$ is a subspace of $E'' \widehat{\otimes} F''$.*

Proof. This is [Ryan, 2002, Corollary 2.14]. □

The injective and projective tensor norms are the "extreme" reasonable crossnorms.

Proposition 2.1.8. *Let E and F be Banach spaces and α be a norm on $E \otimes F$. Then α is a reasonable crossnorm if and only if $\varepsilon(u) \leq \alpha(u) \leq \pi(u)$ for each $u \in E \otimes F$.*

Proof. This is a simple calculation given the definition of a reasonable crossnorm. □

2.2 Duals of tensor products and operator ideals

Let $E, F \in$ FIN so that $E \otimes F$ is finite-dimensional, and thus all norms on $E \otimes F$ are equivalent. Let $\mu \in (E \otimes F)'$ (where $E \otimes F$ has some tensor norm on it) so that, for $u \in E \otimes F$ with $u = \sum_{i=1}^n x_i \otimes y_i$, we have

$$\langle \mu, u \rangle = \sum_{i=1}^n \langle \mu, x_i \otimes y_i \rangle.$$

Define $T_\mu : E \to F'$ by $\langle T_\mu(x), y \rangle = \langle \mu, x \otimes y \rangle$ for $x \in E$ and $y \in F$. Thus we have

$$\langle \mu, u \rangle = \sum_{i=1}^n \langle T_\mu(x_i), y_i \rangle.$$

Hence we have $(E \otimes F)' = \mathcal{B}(E, F')$, as vector spaces. As $F \in$ FIN, $\mathcal{B}(E, F') = \mathcal{F}(E, F') = E' \otimes F'$, so that $(E \otimes F)' = E' \otimes F'$. Explicitly, the duality is

$$\langle u, v \rangle = \sum_{i=1}^n \sum_{j=1}^m \langle \mu_i, x_j \rangle \langle \lambda_i, y_j \rangle,$$

for $u = \sum_{i=1}^n \mu_i \otimes \lambda_i \in E' \otimes F'$ and $v = \sum_{j=1}^m x_j \otimes y_j \in E \otimes F$. As $E' \otimes F' = \mathcal{F}(E, F')$ and $E \otimes F = \mathcal{F}(E', F)$, let $T_u \in \mathcal{F}(E, F')$ and $T_v \in \mathcal{F}(E', F)$ be the operators represented by u and v, respectively. Then we have

$$T_u \circ T_v' = \sum_{j=1}^m y_j \otimes T_u(x_j) \in \mathcal{F}(F') \quad , \quad \langle u, v \rangle = \sum_{j=1}^m \langle T_u(x_j), y_j \rangle = \mathrm{Tr}(T_u \circ T_v'),$$

the *trace* of $T_u \circ T_v'$. The duality using the trace is often referred to as *trace duality*. Note that $\mathrm{Tr}(T_u \circ T_v') = \mathrm{Tr}(T_v' \circ T_u)$, a property which is useful in calculations.

Definition 2.2.1. Let α be a tensor norm. Then the *dual tensor norm* to α is α', and is given by setting

$$(E \widehat{\otimes}_\alpha F)' = E' \widehat{\otimes}_{\alpha'} F'$$

for $E, F \in$ FIN, and extending α' to all Banach spaces by finite-generation. Define $\breve{\alpha}$ to be the tensor norm $(\alpha')^t$, called the *adjoint* of α.

Of course, we can show that α' is a tensor norm. We then have that $\alpha'' = \alpha$, $\varepsilon' = \pi$ and $\pi' = \varepsilon$. So for $E, F \in \mathrm{FIN}$, we have $(E\widehat{\otimes}F)' = \mathcal{B}(E, F') = E'\check{\otimes}F'$ and that $\mathcal{B}(E,F)' = (E'\check{\otimes}F)' = E\widehat{\otimes}F'$.

The picture is more complicated for infinite-dimensional Banach spaces, due to our insisting that tensor norms are finitely generated (which is necessary to ensure that, for example, $\alpha'' = \alpha$). For a tensor norm α define α^s by the embedding $E\widehat{\otimes}_{\alpha^s}F \to (E'\widehat{\otimes}_\alpha F')'$ for *any* Banach spaces E and F. Thus $\alpha^s = \alpha'$ on FIN, but not, in general, on infinite-dimensional spaces, and we can check that α^s need not be a tensor norm, as it need not be finitely-generated.

Definition 2.2.2. Let α be a tensor norm such that $(\alpha')^s = \alpha'' = \alpha$ on $E \otimes F$ whenever at least one of E and F are in FIN. Then α is said to be *accessible*.

Suppose further that we always have $(\alpha')^s = \alpha$. Then α is *totally accessible*.

We can show that ε is totally accessible, that π is accessible, and that α is accessible if and only if α' is accessible. Indeed, most common tensor norms are accessible; certainly any defined in [Ryan, 2002] are. However, as shown in [Defant, Floret, 1993, Section 31.6], there do exist tensor norms which are not accessible.

Proposition 2.2.3. *Let E and F be Banach spaces. Then the map $\Gamma : \mathcal{B}(E, F') \to (E\widehat{\otimes}F)'$ given by*

$$\langle \Gamma(T), x \otimes y\rangle = \langle T(x), y\rangle \qquad (T \in \mathcal{B}(E, F'), x \in E, y \in F)$$

extends by continuity and linearity to an isometric isomorphism.

Proof. For $u \in E\widehat{\otimes}F$, for each $\varepsilon > 0$ we can write

$$u = \sum_{i=n}^{\infty} x_n \otimes y_n \quad , \quad \pi(u) \leq \sum_{n=1}^{\infty} \|x_n\|\|y_n\| \leq \pi(u) + \varepsilon.$$

Thus we have

$$|\langle \Gamma(T), u\rangle| \leq \sum_{n=1}^{\infty} |\langle T(x_n), y_n\rangle| \leq \|T\|(\pi(u) + \varepsilon).$$

As $\varepsilon > 0$ was arbitrary, we see that Γ is norm-decreasing.

Conversely, for $\mu \in (E\widehat{\otimes}F)'$, define $T : E \to F'$ by $\langle T(x), y\rangle = \langle \mu, x \otimes y\rangle$ for $x \otimes y \in E\widehat{\otimes}F$. Then $T \in \mathcal{B}(E, F')$, $\|T\| \leq \|\mu\|$ and by linearity and continuity, we see that $\mu = \Gamma(T)$. Thus Γ is an isometric isomorphism. $\square$

As the swap map $E\widehat{\otimes}F \to F\widehat{\otimes}E$ is an isometry, we can naturally identify $(E\widehat{\otimes}F)'$ with $\mathcal{B}(F, E')$ as well as with $\mathcal{B}(E, F')$.

When E is reflexive, we see that $(E\widehat{\otimes}E')' = \mathcal{B}(E)$, and hence that $\mathcal{B}(E)$ is a dual Banach algebra (recall Definition 1.6.5).

Let α be some tensor norm. As $\alpha \leq \pi$ for each pair of Banach spaces E and F, the formal identity map $I_\alpha : E\widehat{\otimes}F \to E\widehat{\otimes}_\alpha F$ is norm decreasing. For $\mu \in (E\widehat{\otimes}_\alpha F)'$, we then have

$$T := I'_\alpha(\mu) \in (E\widehat{\otimes}F)' = \mathcal{B}(E, F').$$

A check shows that

$$\langle \mu, \sum_{i=1}^n x_i \otimes y_i \rangle = \sum_{i=1}^n \langle T(x_i), y_i \rangle \quad \Big(\sum_{i=1}^n x_i \otimes y_i \in E \otimes F \Big),$$

so that we can identify $(E\widehat{\otimes}_\alpha F)'$ with a subspace of $\mathcal{B}(E, F')$, denoted by $\mathcal{B}_{\alpha'}(E, F')$, the *$\alpha'$-integral operators*, and give it the norm $\|\cdot\|_{\alpha'}$ induced by the identification of $\mathcal{B}_{\alpha'}(E, F')$ with $(E\widehat{\otimes}_\alpha F)'$. This notation is chosen because we have

$$\mathcal{B}_{\alpha'}(E, F') = (E\widehat{\otimes}_\alpha F)' = E'\widehat{\otimes}_{\alpha'} F' \qquad (E, F \in \mathrm{FIN}).$$

Again, the duality can be explicitly defined by using the trace, at least when dual spaces are being used. For let $u = \sum_{i=1}^n \mu_i \otimes y_i \in E'\widehat{\otimes}_\alpha F$ and $T \in \mathcal{B}_{\alpha'}(E', F')$, and let $S \in \mathcal{F}(E, F)$ be the operator induced by u. Then we have

$$\begin{aligned} \langle T, u \rangle &= \sum_{i=1}^n \langle T(\mu_i), y_i \rangle = \mathrm{Tr}\Big(\sum_{i=1}^n \kappa_F(y_i) \otimes T(\mu_i) \Big) = \mathrm{Tr}(T \circ S') \\ &= \sum_{i=1}^n \langle T' \kappa_F(y_i), \mu_i \rangle = \mathrm{Tr}\Big(\sum_{i=1}^n T'\kappa_F(y_i) \otimes \mu_i \Big) = \mathrm{Tr}(S' \circ T). \end{aligned}$$

The ε-integral operators are just the bounded operators. We call the π-integral operators just the *integral operators* and denote them by $\mathcal{I}(E, F') = \mathcal{B}_\pi(E, F')$.

For a tensor norm α, the finite-generation property of α passes over to α-integral operators. Suppose that E and F are Banach spaces, and that $T \in \mathcal{B}_\alpha(E, F')$. For $M \in \mathrm{FIN}(E)$ and $N \in \mathrm{FIN}(F)$, let $I_M : M \to E$ be the inclusion map and, noting that $N' = F'/N^\circ$, let $Q_N : F' \to N'$ by the quotient map. Then $Q_N \circ T \circ I_M \in \mathcal{F}(M, N') = M' \otimes N'$,

$$\begin{tikzcd} E \arrow[r, "T"] & F' \arrow[d, two heads, "Q_N"] \\ M \arrow[u, hook, "I_M"] \arrow[r, "Q_N T I_M"] & N' \end{tikzcd}$$

Suppose that $u \in M \otimes N$, and let $S \in \mathcal{F}(M', N)$ be the operator induced by u. Then we have

$$\begin{aligned}\langle Q_N T I_M, u\rangle &= \mathrm{Tr}(Q_N T I_M S') = \mathrm{Tr}(T I_M S' Q_N) = \langle T, (I_M S' Q_N)'\rangle \\ &= \langle T, Q_N' S I_M'\rangle = \langle T, (I_M \otimes Q_N')(u)\rangle,\end{aligned}$$

so that

$$|\langle Q_N T I_M, u\rangle| \leq \|T\|_\alpha \alpha'((I_M \otimes Q_N')(u), E \otimes F'') \leq \|T\|_\alpha \alpha'(u, M \otimes N),$$

as $\|I_M\| = \|Q_N\| = 1$. Thus $Q_N T I_M \in M' \widehat{\otimes}_\alpha N'$ with $\alpha(Q_N T I_M) \leq \|T\|_\alpha$.

Proposition 2.2.4. *Let E and F be Banach spaces, let α be a tensor norm, and let $T \in \mathcal{B}(E, F')$. Then the following are equivalent:*

1. *$T \in \mathcal{B}_\alpha(E, F')$;*
2. *For some $C > 0$, for $M \in \mathrm{FIN}(E)$ and $N \in \mathrm{FIN}(F)$, we have*

$$\|Q_N T I_M\|_\alpha = \alpha(Q_N T I_M, M' \otimes N') \leq C.$$

Furthermore, the minimal value for C is $\|T\|_\alpha$. □

We make the definition that $T \in \mathcal{B}(E, F)$ is α-integral if and only if $\kappa_F \circ T \in \mathcal{B}(E, F'')$ is α-integral. Then an argument using the principle of local reflexivity, and noting that $N \in \mathrm{FIN}(F)$ if and only if N° has finite co-dimension in N', leads us to the following. Write $\mathrm{COFIN}(F)$ for the collection of closed subspaces of F with finite co-dimension.

Proposition 2.2.5. *Let E and F be Banach spaces, let α be a tensor norm, and let $T \in \mathcal{B}(E, F)$. Then $T \in \mathcal{B}_\alpha(E, F)$ if and only if, for some $C > 0$, we have the following. For each $M \in \mathrm{FIN}(E)$ and $N \in \mathrm{COFIN}(F)$, letting I_M be as before, and $Q_N : F \to F/N$ be the quotient map, we have $\|Q_N T I_M\|_\alpha \leq C$. As before, $Q_N T I_M \in \mathcal{F}(M, F/N) = M \otimes F/N$, and the minimal value for C is $\|T\|_\alpha$.* □

Proposition 2.2.6. *Let E and F be Banach spaces, let $T \in \mathcal{B}(E, F)$, and let α be a tensor norm. The following are equivalent:*

1. *T is an α-integral operator;*
2. *$\kappa_F T : E \to F''$ is an α-integral operator;*
3. *$T'' : E'' \to F''$ is an α-integral operator;*

4. *$T' : F' \to E'$ is an α^t-integral operator.*

Furthermore $\|T\|_\alpha = \|\kappa_F T\|_\alpha = \|T''\|_\alpha = \|T'\|_{\alpha^t}$.

Let D and G be Banach spaces, let $S \in \mathcal{B}(D,E)$ and $R \in \mathcal{B}(F,G)$. Then $RTS \in \mathcal{B}_\alpha(D,G)$ and $\|RTS\|_\alpha \leq \|R\|\|T\|_\alpha\|S\|$. □

Hence we have the following isometric inclusions

$$\mathcal{B}_\alpha(E,F) \subseteq (E\widehat{\otimes}_{\alpha'}F')' = \mathcal{B}_\alpha(E,F'') \subseteq \mathcal{B}_\alpha(E'',F''),$$

noting that $T''\kappa_E = \kappa_F T$ for any $T \in \mathcal{B}(E,F)$. The final part of the above proposition shows that the α-integral operators are an *operator ideal* in the sense of Pietsch (see [Pietsch, 1980]). In particular, $\mathcal{B}_\alpha(E)$ is, algebraically, an ideal in $\mathcal{B}(E)$.

Definition 2.2.7. An *operator ideal* $\mathfrak{U}$ is an assignment, to each pair of Banach spaces E and F, a subspace $\mathfrak{U}(E,F) \subseteq \mathcal{B}(E,F)$ such that:

1. there is a norm u on $\mathfrak{U}(E,F)$ such that $(\mathfrak{U}(E,F),u)$ is a Banach space;
2. $\mathcal{F}(E,F) \subseteq \mathfrak{U}(E,F)$, and for $\mu \in E'$ and $x \in F$, for the one-dimensional operator $\mu \otimes x \in \mathcal{F}(E,F)$, we have $u(\mu \otimes x) = \|\mu\|\|x\|$;
3. for Banach spaces D and G, $T \in \mathfrak{U}(E,F)$, $S \in \mathcal{B}(D,E)$ and $R \in \mathcal{B}(F,G)$, $RTS \in \mathfrak{U}(D,G)$, and $u(RTS) \leq \|R\|u(T)\|S\|$.

If $\mathfrak{U}(E,F)$ is always a closed subspace of $\mathcal{B}(E,F)$, then we say that $\mathfrak{U}$ is a *closed operator ideal*.

Note that some sources give a more general definition for the term "operator ideal". For each tensor norm α, we see that $\mathcal{B}_\alpha$ is an operator ideal for the norm $\|\cdot\|_\alpha$; it is rarely closed. The assignment $\mathcal{A}(E,F)$ is a closed operator ideal, and by condition (2) we see that it is the smallest closed operator ideal. We say that an operator $T : E \to F$ is *compact* if $T(E_{[1]})$ is a relatively norm-compact subset of F, denoted by $T \in \mathcal{K}(E,F)$. Then $\mathcal{K}$ is a closed operator ideal, the *compact operators*. Similarly, the collection of weakly-compact operators, $\mathcal{W}(E,F)$, is also a closed operator ideal.

Theorem 2.2.8. *Let E and F be Banach spaces, and $T \in \mathcal{B}(E,F)$. Then $T \in \mathcal{K}(E,F)$ if and only if $T' \in \mathcal{K}(F',E')$. Moreover, the following are equivalent: (1) $T \in \mathcal{W}(E,F)$; (2) $T' \in \mathcal{W}(F',E')$; and (3) $T''(E'') \subseteq \kappa_F(F)$.*

Proof. The statement about compact operators is Schauder's Theorem, [Megginson, 1998, Theorem 3.4.15]. The statement about weakly-compact operators is Gantmacher's Theorem, [Megginson, 1998, Theorem 3.5.8, Theorem 3.5.13]. □

Theorem 2.2.9. *Let E and F be Banach spaces, and $T \in \mathcal{B}(E, F)$. Then $T \in \mathcal{W}(E, F)$ if and only if there exists a reflexive Banach space G, $R \in \mathcal{B}(E, G)$ and $S \in \mathcal{B}(G, F)$ with $T = S \circ R$. Furthermore, we can choose G, S and R so that R has dense range and the same norm and kernel as T, and so that S is norm-decreasing and injective.*

Proof. This is [Davis et al., 1974]. We will give a brief sketch of the construction, following the presentation given in [Palmer, 1994, Section 1.7.8]. For $n \in \mathbb{N}$, define a new norm on F by

$$\|y\|_n = \inf\{2^{-n/2}\|T\|\|x\| + \|y - T(x)\| : x \in E\} \qquad (y \in F).$$

Setting $x = 0$ gives $\|y\|_n \leq \|y\|$ for each $y \in F$. Similarly, as $\|y\| \leq \|y - T(x)\| + \|T(x)\| \leq 2^{n/2}\|y - T(x)\| + \|T\|\|x\|$, we have $\|y\| \leq 2^{n/2}\|y\|_n$ for each $y \in F$. Thus, for each $n \in \mathbb{N}$, $\|\cdot\|_n$ and $\|\cdot\|$ are equivalent norms on F. We also have, for $x \in E$, $\|T(x)\|_n \leq 2^{-n/2}\|T\|\|x\|$. Let

$$G = \Big\{y \in F : \sum_{n=1}^{\infty} \|y\|_n^2 < \infty\Big\},$$

with the norm $\|y\|_G = \big(\sum_{n=1}^{\infty} \|y\|_n^2\big)^{1/2}$. Then $\|T(x)\|_G \leq \|T\|\|x\|$, so that $T(E) \subseteq G$. We can thus define R and S by $R(x) = T(x)$, for $x \in E$, and $S(y) = y$, for $y \in G$. The remaining claims follow by calculation (we can ensure that R has dense range by simply replacing G with the image of R). □

Corollary 2.2.10. *Let E and F be Banach spaces and $T \in \mathcal{K}(E, F)$. Then there exists a reflexive Banach space G, $R \in \mathcal{B}(E, G)$ and $S \in \mathcal{K}(G, F)$ with $T = S \circ R$.*

Proof. We follow the (sketch) proof above, and claim that the map S constructed in actually compact. This will follow if we can show that $S(G_{[1]})$ is contained in the closure of $T(E_{[\|T\|^{-1}]})$, which is a compact subset of F, as T is a compact operator.

Let $y \in G$ be such that $\|y\|_G < 1$, so that $y \in F$ and $\sum_{n=1}^{\infty} \|y\|_n^2 < 1$. Hence we can find a sequence (x_n) in E with

$$\sum_{n=1}^{\infty} \Big(2^{-n/2}\|T\|\|x_n\| + \|y - T(x_n)\|\Big)^2 < 1.$$

We have $\|y - T(x_n)\| \geq \|y\| - \|T(x_n)\| \geq \|y\| - \|T\|\|x_n\|$, so that

$$1 > \sum_{n=1}^{\infty} \Big(\|y\| + (2^{-n/2} - 1)\|T\|\|x_n\|\Big)^2.$$

In particular, $\|y\| = \lim_n \|T\|\|x_n\|$. As $\lim_n \|y - T(x_n)\| = 0$, it is clear that y is in the closure of $T(E_{[\|y\|\|T\|^{-1}]})$. As $\|y\| \leq 2^{n/2}\|y\|_n$ for each $n \in \mathbb{N}$, we have

$$\|y\|_G^2 = \sum_{n=1}^{\infty} \|y\|_n^2 \geq \sum_{n=1}^{\infty} 2^{-n}\|y\|^2 = \|y\|^2,$$

so that $\|y\| \leq \|y\|_G < 1$, and thus $T(E_{[\|y\|\|T\|^{-1}]}) \subseteq T(E_{[\|T\|^{-1}]})$. As y was arbitrary, we are done. □

2.3 Nuclear and integral operators; the approximation property

Definition 2.3.1. Let E and F be Banach spaces, and let α be a tensor norm. Then there is a natural, norm-decreasing map $J_\alpha : E'\widehat{\otimes}_\alpha F \to \mathcal{B}(E, F)$ given by

$$J_\alpha(\mu \otimes y)(x) = \langle \mu, x\rangle y \qquad (x \in E, y \in F, \mu \in E'),$$

and linearity and continuity. The image of J_α, equipped with the quotient norm, is the set of *α-nuclear operators*, denoted $\mathcal{N}_\alpha(E, F)$, with norm $\|\cdot\|_{\mathcal{N}_\alpha}$. The *nuclear operators*, $\mathcal{N}(E, F)$, are the π-nuclear operators.

We can check that the α-nuclear operators form an operator ideal. In particular, for $y \in F$ and $\mu \in E'$, suppose that $u \in E'\widehat{\otimes}_\alpha F$ is such that $J_\alpha(u) = \mu \otimes y$. Then

$$\|\mu\|\|y\| = \|J_\alpha(u)\| \leq \alpha(u, E'\widehat{\otimes}_\alpha F),$$

so we see that $\|\mu \otimes y\|_{\mathcal{N}_\alpha} = \|\mu\|\|y\|$. Furthermore, for $E, F \in \mathrm{FIN}$, we have $E'\widehat{\otimes}_\alpha F = (E\widehat{\otimes}_{\alpha'} F')' = \mathcal{B}_\alpha(E, F)$, so that the α-integral and α-nuclear operators coincide for finite-dimensional spaces.

Proposition 2.3.2. *Let E and F be Banach spaces. Then the map $J_\alpha : E'\widehat{\otimes}_\alpha F \to \mathcal{B}(E, F)$ maps into $\mathcal{B}_\alpha(E, F)$, and the arising inclusion $\mathcal{N}_\alpha(E, F) \to \mathcal{B}_\alpha(E, F)$ is norm-decreasing; that is, $\|T\|_{\mathcal{N}_\alpha} \geq \|T\|_\alpha$ for each $T \in \mathcal{N}_\alpha(E, F)$.*

Proof. It suffices to show that $\|J_\alpha(u)\|_\alpha \leq \alpha(u)$ for each $u \in E' \otimes F$, the general result following by continuity. Fix $u \in E' \otimes F$. We have $\|J_\alpha(u)\|_\alpha = \|\kappa_F J_\alpha(u)\|_\alpha$, where

$$\|\kappa_F J_\alpha(u)\|_\alpha = \sup\{|\langle \kappa_F J_\alpha(u), v\rangle| : v \in E \otimes F', \alpha'(v) \leq 1\}.$$

Fix $v \in E \otimes F'$ with $\alpha'(v) \leq 1$ and let $v = \sum_{i=1}^n x_i \otimes \mu_i$.

For $\varepsilon > 0$, we can find $M \in \mathrm{FIN}(E')$ and $N \in \mathrm{FIN}(F)$ with $u \in M \otimes N$ and $\alpha(u, E' \otimes F) \leq \alpha(u, M \otimes N) \leq \alpha(u, E' \otimes F) + \varepsilon$. Let $Q_M : E \to E/{}^\circ M = M'$ be the quotient map, and $Q_N : F' \to F'/N^\circ = N'$ be the quotient map. Let $u = \sum_{j=1}^m \lambda_j \otimes y_j \in M \otimes N$, so that

$$\begin{aligned}\langle \kappa_F J_\alpha(u), v\rangle &= \sum_{i=1}^n \sum_{j=1}^m \langle \lambda_j, x_i\rangle\langle \mu_i, y_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \lambda_j, x_i + {}^\circ M\rangle\langle \mu_i + N^\circ, y_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^m \langle \lambda_j, Q_M(x_i)\rangle\langle Q_N(\mu_i), y_j\rangle = \langle (Q_M \otimes Q_N)(v), u\rangle,\end{aligned}$$

where $(Q_M \otimes Q_N)(v) \in M' \otimes N'$. We conclude that

$$\begin{aligned}|\langle \kappa_F J_\alpha(u), v\rangle| &\leq \alpha(u, M \otimes N)\alpha'((Q_M \otimes Q_N)(v), M' \otimes N') \\ &\leq \Big(\alpha(u, E' \otimes F) + \varepsilon\Big)\|Q_M\|\|Q_N\|\alpha'(v, E \otimes F') \leq \alpha(u, E' \otimes F) + \varepsilon.\end{aligned}$$

As v and $\varepsilon > 0$ were arbitrary, we must have $\|\kappa_F J_\alpha(u)\|_\alpha \leq \alpha(u)$, as required. □

To say more on the relationship between $\mathcal{N}_\alpha$ and $\mathcal{B}_\alpha$ we need to study ideas which go back to Grothendieck, [Grothendieck, 1953].

Definition 2.3.3. Let E be a Banach space and α be a tensor norm. Then E has the *approximation property* if the map $J_\pi : E'\widehat{\otimes}E \to \mathcal{N}(E)$ is injective.

There is a related, but more complicated, notion for a general tensor norm α (see [Defant, Floret, 1993, Section 21.7]). We shall not have cause to consider this though (and, in the above reference, it is shown that the case where $\alpha = \pi$ is the important one).

Proposition 2.3.4. *Let E be a Banach space. Then the following are equivalent:*

1. *E has the approximation property;*
2. *for each Banach space F, the map $J_\pi \circ (\kappa_E \otimes \mathrm{Id}_F) : E\widehat{\otimes}F \to E''\widehat{\otimes}F \to \mathcal{B}(E', F)$ is injective;*
3. *for each Banach space F, the map $J_\pi \circ (\kappa_F \otimes \mathrm{Id}_E) : F\widehat{\otimes}E \to F''\widehat{\otimes}E \to \mathcal{B}(F', E)$ is injective;*
4. *for each Banach space F, we have $\mathcal{K}(F, E) = \mathcal{A}(F, E)$;*
5. *for each compact set $K \subseteq E$ and each $\varepsilon > 0$, there is $T \in \mathcal{F}(E)$ with $\|T(x) - x\| < \varepsilon$ for each $x \in K$.*

Furthermore, if $E = D'$ for some Banach space D, then the following are equivalent:

1. *E has the approximation property;*

2. *for each Banach space F, the map $J_\pi : E\widehat{\otimes}F = D'\widehat{\otimes}F \to \mathcal{B}(D,F)$ is injective;*

3. *for each Banach space F, the map $J_\pi : F'\widehat{\otimes}E \to \mathcal{B}(F,E)$ is injective.*

If E' has the approximation property, then so does E.

Proof. See, for example, [Ryan, 2002, Proposition 4.6]. $\square$

Using condition (5), it is reasonably simple to show that, for $1 \leq p \leq \infty$ and any measure μ, $L^p(\mu)$ and $L^p(\mu)'$ have the approximation property (indeed, they have the metric approximation property – see Definition 2.3.13). Furthermore, $C(X)$ and $C(X)'$ have the (metric) approximation property for each compact X. There are spaces without the approximation property (the first was constructed in [Enflo, 1973]). In fact, for $p \neq 2$, l^p contains subspaces without the approximation property (see [Szankowski, 1978]), and $\mathcal{B}(l^2)$ does not have the approximation property (see [Szankowski, 1981]).

Proposition 2.3.5. *Let E and F be Banach spaces, and let $T \in \mathcal{B}(E,F)$. Suppose that E' has the approximation property and $T' \in \mathcal{N}(F',E')$. Then $T \in \mathcal{N}(E,F)$.*

Proof. We adapt [Ryan, 2002, Proposition 4.10]. As E' has the approximation property, by Proposition 2.3.4(3) (and a short calculation) we have $\mathcal{N}(F',E') = F''\widehat{\otimes}E'$. Thus $T' = \sum_{n=1}^\infty \Phi_n \otimes \mu_n \in F''\widehat{\otimes}E'$. Let $K : E'\widehat{\otimes}F \to F''\widehat{\otimes}E'$ be the map

$$E'\widehat{\otimes}F \xrightarrow{\ S\ } F\widehat{\otimes}E' \xrightarrow{\ \kappa_F \otimes \mathrm{Id}_{E'}\ } F''\widehat{\otimes}E',$$

where S is the swap map. We wish to show that T' lies in the image of K. By the Hahn-Banach theorem, it suffices to show that each $S \in (K(E'\widehat{\otimes}F))^\circ \subseteq \mathcal{B}(F'',E'')$ satisfies $\langle S, T'\rangle = 0$.

Now, for $S \in \mathcal{B}(F'',E'')$ and $u = \mu \otimes x \in E'\widehat{\otimes}F$, we have

$$\langle S, K(u)\rangle = \langle S(\kappa_F(x)), \mu\rangle,$$

so we see that $S \in (K(E'\widehat{\otimes}F))^\circ$ if and only if $S \circ \kappa_F = 0$. Then, for $S \in (K(E'\widehat{\otimes}F))^\circ$, we have

$$\begin{aligned}
\langle S, T'\rangle &= \sum_{n=1}^\infty \langle S(\Phi_n), \mu_n\rangle = \sum_{n=1}^\infty \langle S''(\kappa_{F''}(\Phi_n)), \kappa_{E'}(\mu_n)\rangle \\
&= \mathrm{Tr}\Big(\kappa_{E'} \circ \kappa_E' \circ \Big(\sum_{n=1}^\infty \kappa_{F''}(\Phi_n) \otimes \kappa_{E'}(\mu_n)\Big) \circ S'\Big) \\
&= \mathrm{Tr}(\kappa_{E'} \circ \kappa_E' \circ T''' \circ S') = \mathrm{Tr}(\kappa_{E'} \circ (S \circ T'' \circ \kappa_E)') \\
&= \mathrm{Tr}(\kappa_{E'} \circ (S \circ \kappa_F \circ T)') = 0,
\end{aligned}$$

as required. Note that, if we expand out the adjoints in the above calculation, then we are always dealing with T' or one of its higher adjoints, all of which we know to be nuclear. Consequently, the trace is well-defined at all points. $\square$

Using [Figiel, Johnson, 1973] and Proposition 2.3.18 below, we see that there is a Banach space E, which has the approximation property, and such that there exists $T \in \mathcal{B}(E) \setminus \mathcal{N}(E)$ with $T' \in \mathcal{N}(E')$.

We wish to give a more concrete description of $\mathcal{I}(E, F)$.

Theorem 2.3.6. *Let E and F be Banach spaces, and let $T \in \mathcal{B}(E, F)$. Then the following are equivalent:*

1. *$T \in \mathcal{I}(E, F)$;*
2. *$T' \in \mathcal{I}(F', E')$;*
3. *there exists a finite measure space (Ω, Σ, ν) and operators $S : E \to L^\infty(\nu)$ and $R : L^1(\nu) \to F''$ such that if $I : L^\infty(\nu) \to L^1(\nu)$ is the formal identity map, then $RIS = \kappa_F T$.*

$$\begin{tikzcd} E \arrow[r, "T"] \arrow[d, "S"'] & F \arrow[r, "\kappa_F"] & F'' \\ L^\infty(\nu) \arrow[rr, "I"] & & L^1(\nu) \arrow[u, "R"'] \end{tikzcd}$$

Furthermore, $\|T\|_\pi = \|T'\|_\pi = \inf \nu(\Omega)\|S\|\|R\|$ where the infimum is taken over all factorisations as above.

Proof. See [Ryan, 2002, Theorem 3.10]. $\square$

Corollary 2.3.7. *Let E be a Banach space, and let $T \in \mathcal{I}(E)$. Then T is weakly-compact and* completely continuous *(that is, T takes weakly-convergent sequences to norm-convergent sequences). Thus the composition of two integral operators is compact, and so $\mathcal{I}(E) \neq \mathcal{B}(E)$ when E is infinite-dimensional.*

Proof. This follows directly from the factorisation given in the above theorem. For further details, see [Ryan, 2002, Proposition 3.20]. $\square$

Note that for an infinite-dimensional Banach space, we have

$$(E'\widehat{\otimes}E)' = \mathcal{B}(E') \quad , \quad (E'\check{\otimes}E)' = \mathcal{I}(E'),$$

so we immediately see that π and ε are not equivalent norms on $E' \otimes E$. A construction by Pisier, [Pisier, 1983], gives a separable Banach space P such that $P\check{\otimes}P = P\widehat{\otimes}P$. The space P does not have the approximation property, but it does satisfy $\mathcal{N}(P) = \mathcal{A}(P)$. In particular, the integral norm on $\mathcal{I}(P')$ is equivalent to the operator norm, as $\mathcal{N}(P)'$ is isometrically a subspace of $\mathcal{B}(E')$, namely $(\ker J_\pi)^\circ = \{T \in \mathcal{B}(E') : \langle T, u\rangle = 0 \ (J_\pi(u) = 0)\}$.

We briefly mention the *Pietsch integral operators*, $\mathcal{PI}(E,F)$, which are defined by using the same factorisation as for the integral operators (that is, property (3) above) but with F'' replaced by F. See, for example, [Ryan, 2002, Section 3.5]. We can see that

$$\mathcal{N}(E,F) \subseteq \mathcal{PI}(E,F) \subseteq \mathcal{I}(E,F),$$

where each inclusion is norm decreasing.

Lemma 2.3.8. *Let E and F be Banach spaces. Then $\mathcal{I}(F,E') = \mathcal{PI}(F,E')$, with the same norm.*

Proof. Just note that $\kappa_E' : E''' \to E'$ is a projection, given that we identify E' with its image in E''' under the map $\kappa_{E'}$. The result now follows by considering factorisations. □

Following the theme of factorising maps, we have the following.

Definition 2.3.9. Let E be a Banach space such that for each $T \in \mathcal{B}(L^1([0,1]), E)$, there exists $S \in \mathcal{B}(L^1([0,1]), l^1)$ and $R \in \mathcal{B}(l^1, E)$ with $R \circ S = T$. Then E has the *Radon-Nikodým property*.

There are many equivalent formulations of the Radon-Nikodým property, see, for example, [Diestel, Uhl, 1977, Chapter VII, Section 6]. In particular, we have the following. Recall that a Banach space F is *separable* if F contains a dense, countable subset.

Theorem 2.3.10. *Let E be a Banach space. Then the following are equivalent.*

1. *E' has the Radon-Nikodým property;*
2. *every separable subspace of E has a separable dual.*

In particular, $l^\infty(I)$ does not have the Radon-Nikodým property for any infinite set I. However, all separable dual spaces do have the Radon-Nikodým property.

Let E be a reflexive Banach space. Then E has the Radon-Nikodým property.

Proof. See [Diestel, Uhl, 1977, Chapter VII, Section 6]. □

To us, the Radon-Nikodým property is important because of the following.

Theorem 2.3.11. *Let E be a Banach space with the Radon-Nikodým property. Then, for each Banach space F, $\mathcal{N}(F,E) = \mathcal{PI}(F,E)$ with the same norm.*

Proof. See [Ryan, 2002, Section 5.3]. □

Corollary 2.3.12. *Let E and F be Banach spaces, with E' or F' having the Radon-Nikodým property. Then $(F\check{\otimes}E)' = \mathcal{I}(F,E') = \mathcal{N}(F,E')$. If E' or F' have the approximation property, then $(F\check{\otimes}E)' = F'\widehat{\otimes}E'$.*

In particular, if E is a Banach space with E' or E'' having the Radon-Nikodým property, then $\mathcal{A}(E)' = (E'\check{\otimes}E)' = \mathcal{N}(E')$.

Proof. For all Banach spaces E and F, we have $(F\check{\otimes}E)' = \mathcal{I}(F,E')$. By Lemma 2.3.8, $\mathcal{I}(F,E') = \mathcal{PI}(F,E')$, so if E' has the Radon-Nikodým property, we are done by the above theorem. As $F\check{\otimes}E$ and $E\check{\otimes}F$ are isometrically isomorphic, we also have the result when F' has the Radon-Nikodým property, so that $\mathcal{I}(F,E') = \mathcal{N}(F,E')$. The comments about the approximation property follow from Proposition 2.3.4. □

We hence see that, if E' has the Radon-Nikodým property and the approximation property, then $\mathcal{A}(E)' = E''\widehat{\otimes}E'$. Thus $\mathcal{A}(E)'' = (E''\widehat{\otimes}E')' = \mathcal{B}(E'')$. If $T \in \mathcal{A}(E)$, and $\Phi \otimes \mu \in E''\widehat{\otimes}E'$, we have

$$\langle \Phi \otimes \mu, T\rangle = \langle \Phi, T'(\mu)\rangle = \langle T''(\Phi), \mu\rangle = \langle \kappa_{\mathcal{A}(E)}(T), \Phi \otimes \mu\rangle.$$

We hence see that $\kappa_{\mathcal{A}(E)}(T) = T''$ for each $T \in \mathcal{A}(E)$. In particular, if E is reflexive and has the approximation property (so that E' has the Radon-Nikodým property and the approximation property) then $\mathcal{A}(E)'' = \mathcal{B}(E)$ and $\kappa_{\mathcal{A}(E)}$ is just the inclusion map $\mathcal{A}(E) \to \mathcal{B}(E)$. We shall shortly study these ideas in far greater detail.

Definition 2.3.13. Let E be a Banach space. Then E has the *bounded approximation property* if, for some $M > 0$, for each compact set $K \subseteq E$ and each $\varepsilon > 0$, there is $T \in \mathcal{F}(E)$ with $\|T\| \leq M$ and $\|T(x) - x\| < \varepsilon$ for each $x \in K$. If we can take $M = 1$, then E has the *metric approximation property*.

There are Banach spaces with the approximation property, but without the bounded approximation property (see [Figiel, Johnson, 1973]).

Theorem 2.3.14. *Let E be a Banach space. Then the following are equivalent:*

1. *E has the bounded approximation property with bound M;*

2. *for each Banach space F, the map*

$$E\widehat{\otimes}F \xrightarrow{\kappa_E \otimes \kappa_F} E''\widehat{\otimes}F'' \xrightarrow{J_\pi} \mathcal{N}(E',F'') \hookrightarrow \mathcal{I}(E',F'') = (E'\check{\otimes}F')' = \mathcal{A}(E,F')'$$

is bounded below by M^{-1};

3. *the map*

$$E\widehat{\otimes}E' \xrightarrow{\kappa_E \otimes \mathrm{Id}_{E'}} E''\widehat{\otimes}E' \xrightarrow{J_\pi} \mathcal{N}(E') \hookrightarrow \mathcal{I}(E') = (E'\check{\otimes}E)' = \mathcal{A}(E)'$$

is bounded below by M^{-1}.

Proof. This follows from the proof of [Ryan, 2002, Theorem 4.14]. □

Corollary 2.3.15. *Let E be a Banach space such that E' has the bounded approximation property. Then E has the bounded approximation property with a smaller (or equal) bound.*

Proof. By (3) above, it is enough to show that the map

$$K_1 : E\widehat{\otimes}E' \xrightarrow{\kappa_E \otimes \mathrm{Id}_{E'}} E''\widehat{\otimes}E' \xrightarrow{J_\pi} \mathcal{N}(E') \hookrightarrow \mathcal{I}(E')$$

is bounded below by M^{-1}. Using (2) above, with $F = E$, we know that the map

$$K_2 : E'\widehat{\otimes}E \xrightarrow{\kappa_{E'} \otimes \kappa_E} E'''\widehat{\otimes}E'' \xrightarrow{J_\pi} \mathcal{N}(E'') \hookrightarrow \mathcal{I}(E'')$$

is bounded below by M^{-1}. For $\mu \otimes x \in E'\widehat{\otimes}E$, we have that

$$K_2(\mu \otimes x)(\Phi) = \langle \Phi, \mu \rangle \kappa_E(x) \qquad (\Phi \in E'').$$

Let $S : E'\widehat{\otimes}E \to E\widehat{\otimes}E'$ be the swap map, $S(\mu \otimes x) = x \otimes \mu$. Then S is an isometry, and we have

$$K_1 S(\mu \otimes x)(\lambda) = \langle \lambda, x \rangle \mu \qquad (\lambda \in E').$$

Thus let $K_3 : \mathcal{I}(E') \to \mathcal{I}(E''); T \mapsto T'$, so that $K_3 K_1 S = K_2$. By Theorem 2.3.6, K_3 is an isometry, so that K_1 must be bounded below by M^{-1} as required. □

Proposition 2.3.16. *Let E be a Banach space such that E' has the bounded approximation property with bound M. Then, for every Banach space F, the map*

$$E'\widehat{\otimes}F \xrightarrow{\mathrm{Id}_{E'} \otimes \kappa_F} E'\widehat{\otimes}F'' \xrightarrow{J_\pi} \mathcal{N}(E,F'') \hookrightarrow \mathcal{I}(E,F'') = (E\check{\otimes}F')'$$

is bounded below by M^{-1}.

Proof. As above, but with fewer details, let $K_1 : E'\widehat{\otimes}F \to \mathcal{I}(E'', F'')$ be the map in Theorem 2.3.14(2), so that K_1 is bounded below by M^{-1}. We wish to show that $K_2 : E'\widehat{\otimes}F \to \mathcal{I}(E, F'')$ is bounded below by M^{-1}. Define $K_3 : \mathcal{I}(E, F'') \to \mathcal{I}(E'', F''); T \mapsto \kappa'_{E'} \circ T'' = (T' \circ \kappa_{E'})'$, so that K_3 is an isometry. We can verify that $K_1 = K_3 K_2$, so that we are done. □

Corollary 2.3.17. *Let E and F be Banach spaces such that at least one of E' or F has the bounded approximation property. Then $\mathcal{N}(E, F) = E'\widehat{\otimes}F$ is a closed subspace of $\mathcal{I}(E, F)$.*

Proof. If E' has the bounded approximation property, then $E'\widehat{\otimes}F = \mathcal{N}(E, F)$, and, by the above proposition, the map $E'\widehat{\otimes}F \to \mathcal{I}(E', F'')$ is bounded below. We can then show that this map takes values in $\mathcal{I}(E', F)$ and that $\mathcal{I}(E', F)$ is a closed subspace of $\mathcal{I}(E', F'')$. The argument in the case when F has the bounded approximation property is similar. □

Proposition 2.3.18. *Let E be a Banach space which has the approximation property, does not have the bounded approximation property, and be such that E' is separable. Then there exists $T \in \mathcal{B}(E) \setminus \mathcal{N}(E)$ with $T' \in \mathcal{N}(E')$.*

Proof. This is [Figiel, Johnson, 1973, Proposition 3], though we can give a simple proof using the ideas we have developed. As E' is separable, it has the Radon-Nikodým property, and so $\mathcal{N}(E') = \mathcal{I}(E')$ with the same norm. Using a now familiar argument, as E does not have the bounded approximation property, but does have the approximation property, the map $\mathcal{N}(E) = E'\widehat{\otimes}E \to \mathcal{N}(E') = \mathcal{I}(E'); T \mapsto T'$ is not bounded below, that is, does not have a closed image. However, the map $\mathcal{I}(E) \to \mathcal{I}(E') = \mathcal{N}(E'); T \mapsto T'$ is an isometry, so the set

$$\{T' : T \in \mathcal{I}(E)\} \setminus \{T' : T \in \mathcal{N}(E)\} = \{T' : T \in \mathcal{I}(E) \setminus \mathcal{N}(E)\} \subseteq \mathcal{N}(E')$$

is non-empty. This completes the proof. □

The metric approximation property also allows us to "assume finite-dimensionality" when dealing with tensor norms.

Proposition 2.3.19. *Let α be a tensor norm. Then α is accessible if and only if $(\alpha')^s = \alpha$ on $E \otimes F$ whenever at least one of E and F has the metric approximation property.*

If E or F has only the bounded approximation property, then $(\alpha')^s$ and α are merely equivalent on $E \otimes F$ for an accessible tensor norm α.

Proof. See [Ryan, 2002, Section 7.1]. The statement about the bounded approximation property is an obvious generalisation. □

This allows us to extend Corollary 2.3.17. First note that this corollary actually states that the π-nuclear operators form a closed subspace of the π-integral operators, at least under some conditions. The property of π which allows this is the fact that π is accessible.

Proposition 2.3.20. *Let α be an accessible tensor norm. Then $\mathcal{N}_\alpha(E,F)$ is a subspace of $\mathcal{B}_\alpha(E,F)$ whenever E' or F has the metric approximation property.*

Proof. As α is accessible, $(\alpha')^s = \alpha$ on $E' \otimes F$ whenever E' or F has the metric approximation property. The proof is complete if we show that for $T \in \mathcal{F}(E,F)$ we have $\alpha(T, E'\otimes F) = \|T\|_\alpha$. We have $\|T\|_\alpha = \|T''\|_\alpha$, where $T'' \in \mathcal{B}_\alpha(E'',F'') = (E''\widehat{\otimes}_{\alpha'}F')'$. Thus the embedding $E' \otimes F = \mathcal{F}(E,F) \rightarrow \mathcal{B}_\alpha(E,F)$ induces the same norm on $E' \otimes F$ as does the embedding $E' \otimes F \rightarrow (E''\widehat{\otimes}_{\alpha'}F')'$. This, however, is precisely the definition of the norm $(\alpha')^s$. We are hence done, as we know that $(\alpha')^s = \alpha$. □

Proposition 2.3.21. *Let α be a totally accessible tensor norm. Then $\mathcal{N}_\alpha(E,F)$ is a subspace of $\mathcal{B}_\alpha(E,F)$ for any Banach spaces E and F.*

Proof. This is exactly the same as the above proof. □

Proposition 2.3.22. *Let α be an accessible tensor norm. Then $\mathcal{N}_\alpha(E,F)$ is a closed (but not necessarily isometric) subspace of $\mathcal{B}_\alpha(E,F)$ whenever E' or F has the bounded approximation property.*

Proof. This is the obvious generalisation of Proposition 2.3.20 given Proposition 2.3.19. □

Finally we give another application of these sorts of argument.

Theorem 2.3.23. *Let E be a reflexive Banach space, or let $E = F'$ for some Banach space F such that F' is separable. If E has the approximation property, then E has the metric approximation property.*

Proof. By Theorem 2.3.10, E' has the Radon-Nikodým property, so by Corollary 2.3.12, we have $\mathcal{A}(E)' = \mathcal{I}(E') = \mathcal{N}(E')$. By Theorem 2.3.14(3), it is enough to show that the map $E\widehat{\otimes}E' \rightarrow \mathcal{I}(E')$ is an isometry onto its range. As E has the approximation property, and $\mathcal{I}(E') = \mathcal{N}(E')$, we need to show that the map $\mathcal{N}(E) = E\widehat{\otimes}E' \rightarrow \mathcal{N}(E'); T \mapsto T'$ is an isometry onto its range. Suppose that we have a norm one projection $P : E'' \rightarrow E$

(by projection, we mean that $P \circ \kappa_E = \mathrm{Id}_E$). Then, for $T \in \mathcal{N}(E)$, we have $T \in \mathcal{I}(E)$, and $\|T\|_{\mathcal{N}(E)} \geq \|T\|_\pi$. Then $P \circ T'' \circ \kappa_E = P \circ \kappa_E \circ T = T$, so that

$$\|T\|_\pi \leq \|T\|_{\mathcal{N}(E)} \leq \|P\|\|\kappa_E\|\|T''\|_{\mathcal{N}(E'')} = \|T''\|_{\mathcal{N}(E'')} \leq \|T'\|_{\mathcal{N}(E')} = \|T'\|_\pi = \|T\|_\pi.$$

We thus have $\|T\|_{\mathcal{N}(E)} = \|T'\|_{\mathcal{N}(E')}$, and so we are done.

Finally, we note that if E is reflexive or a dual space, then E is one-complemented in E'', in the latter case by the map κ_F' where $E = F'$. □

Finally, we collect some miscellaneous results.

Theorem 2.3.24. *Let E, F and G be Banach spaces. Then we have:*

1. *if $T \in \mathcal{I}(E,F)$ and $S \in \mathcal{W}(F,G)$, then $ST \in \mathcal{N}(E,G)$;*
2. *if $S \in \mathcal{W}(E,F)$ and $T \in \mathcal{I}(F,G)$, then $\kappa_G TS \in \mathcal{N}(E,G'')$. Furthermore, if E' has the approximation property, then $TS \in \mathcal{N}(E,G)$.*

Proof. For (1), from Theorem 2.2.9, we see that as S is weakly-compact, we can find a reflexive Banach space D and $S_1 \in \mathcal{B}(F,D)$, $S_2 \in \mathcal{B}(D,G)$ so that $S = S_2S_1$. Then $S_1T \in \mathcal{I}(E,D)$, and as D is reflexive, $\mathcal{I}(E,D) = \mathcal{N}(E,D)$. Thus S_1T is nuclear, so $S_2S_1T = ST$ is also nuclear.

For (2), again factor S through a reflexive space D as $S = S_2S_1$. Then $S_2'T' \in \mathcal{I}(G',D') = \mathcal{N}(G',D')$, as D' is reflexive, so that $S'T' \in \mathcal{N}(G',E')$. Then $\kappa_G TS = T''S''\kappa_G$ is also nuclear. When E' has the approximation property, by Proposition 2.3.5, we see that as $S'T'$ is nuclear, so is TS. □

Theorem 2.3.25. (Grothendieck Composition Theorem) *Let α be a tensor norm, E, F and G be Banach spaces, $T \in \mathcal{B}_{\alpha'}(E,F)$ and $S \in \mathcal{B}_{\alpha^t}(F,G)$. If α is accessible or F has the metric approximation property, then $ST \in \mathcal{I}(E,G) = \mathcal{B}_\pi(E,G)$ with $\|ST\|_\pi \leq \|S\|_{\alpha^t}\|T\|_{\alpha'}$. If F has the bounded approximation property with bound M, then $ST \in \mathcal{I}(E,G) = \mathcal{B}_\pi(E,G)$ with $\|ST\|_\pi \leq M\|S\|_{\alpha^t}\|T\|_{\alpha'}$.*

Proof. See [Ryan, 2002, Theorem 8.5] while considering Proposition 2.3.19. The comment about the bounded approximation property is again an obvious generalisation. □

2.4 Integral operators on $C(X)$ spaces

We shall now quickly sketch some results about integral operators on $C(X)$ spaces, as these will be useful examples later on.

Following [Diestel, Uhl, 1977] and [Ryan, 2002, Chapter 5], let X be a compact Hausdorff topological space and Σ be the collection of Borel subsets of X; that is, the σ-algebra generated by open sets in X. Let E be a Banach space. Then a map $\mu : \Sigma \to E$ is a *vector measure* when μ is countably additive. By the Orlicz-Pettis Theorem ([Ryan, 2002, Proposition 3.12]) we can show that μ is a vector measure if and only if $\lambda \circ \mu : \Sigma \to \mathbb{C}$ is a complex Borel measure for each $\lambda \in E'$.

For a complex Borel measure $\nu : \Sigma \to \mathbb{C}$, we define the *variation* of ν to be

$$|\nu|(A) = \sup \Big\{ \sum_{i=1}^{n} |\nu(A_i)| : (A_i)_{i=1}^{n} \text{ is a partition of } A \Big\} \qquad (A \in \Sigma).$$

Similarly, for a vector measure $\mu : \Sigma \to E$, we define the *variation* by

$$|\mu|_1(A) = \sup \Big\{ \sum_{i=1}^{n} \|\mu(A_i)\| : (A_i)_{i=1}^{n} \text{ is a partition of } A \Big\} \qquad (A \in \Sigma).$$

A vector measure has *finite variation* if $|\mu|_1(X) < \infty$; the set of vector measures with finite variation forms a Banach space with the variation norm $|\cdot|_1$.

A complex Borel measure $\nu : \Sigma \to \mathbb{C}$ is *regular* if, for each $A \subseteq \Sigma$ and $\varepsilon > 0$, we can find an open set $U \supseteq A$ and a compact (equals closed in our case) set $C \subseteq A$ with $|\nu|_1(U \setminus C) < \varepsilon$. We then define (see [Ryan, 2002, Lemma 5.24]) a vector measure μ to be *regular* if $\lambda \circ \mu$ is regular for each $\lambda \in E'$.

Given a regular vector measure $\mu : \Sigma \to E$ we can define a map $T_\mu : C(X) \to E$ by

$$T_\mu(f) = \int_K f \, \mathrm{d}\mu \qquad (f \in C(X)).$$

To make sense of this, we first define $\langle \lambda, T_\mu(f) \rangle = \int_K f \, \mathrm{d}(\lambda \circ \mu)$ where $\lambda \circ \mu : \Sigma \to \mathbb{C}$ is a regular Borel measure, and thus lies in the dual space of $C(X)$ by the Riesz-Representation Theorem, so that the integral has a natural meaning. We have thus defined $T_\mu(f) \in E''$. However, we can show that actually $T_\mu(f) \in E$ for each $f \in C(X)$, and thus that T_μ is a weakly-compact operator (and that every weakly-compact operator from $C(X)$ to E arises in this way: see [Ryan, 2002, Theorem 5.25]).

Theorem 2.4.1. *Let $T : C(X) \to E$ be an operator induced by a regular vector measure μ. Then T is an integral operator if and only if μ has finite variation. Furthermore, in this case, we have $\|T\|_\pi = |\mu|_1$.*

Proof. See [Ryan, 2002, Proposition 5.28]. □

A function $g : X \to E$ is *simple* if

$$g = \sum_{k=1}^{n} x_k \chi_{A_k}$$

for some $(x_k)_{k=1}^n \subseteq E$ and $(A_k)_{k=1}^n \subseteq \Sigma$, where χ_A is the indicator function of $A \in \Sigma$. Given a positive Borel measure $\nu : \Sigma \to \mathbb{R}$, we say that a function $g : X \to E$ is *Bochner integrable* if we can find a sequence a simple functions (g_n) so that $\lim_{n\to\infty} \|g(t) - g_n(t)\| = 0$ for ν-almost every $t \in K$ and, furthermore, we have $\int_K \|g\| \,\mathrm{d}\nu < \infty$.

Given a Bochner-integrable function $g : X \to E$ and a positive Borel measure ν, we can define a vector measure G by

$$G(A) = \int_A g(t) \,\mathrm{d}\nu(t) \qquad (A \in \Sigma).$$

This integral is defined in the obvious way by taking a simple function approximation to g. We say that G has *derivative* g with respect to ν.

Theorem 2.4.2. *Let $T : C(X) \to E$ be an operator induced by a regular vector measure μ. Then T is a nuclear operator if and only if μ has finite variation and there exists a Bochner-integrable function g so that g is the derivative of μ with respect to $|\mu|_1$.*

Proof. See [Diestel, Uhl, 1977, Chapter VI, Theorem 4]. □

We say that a vector measure μ is *compact* if the set $\{\mu(A) : A \in \Sigma\}$ is relatively compact in E. When $E = C(X)$ we have a simple description of relatively compact subsets of E.

Theorem 2.4.3. *Let K be a subset of $C(X)$ for a compact Hausdorff space X. The K is relatively compact if and only if K is bounded and, for each $t \in X$ and $\varepsilon > 0$, we can find an open set $U \subseteq X$ with $t \in U$ and such that $|f(t) - f(s)| < \varepsilon$ for each $s \in U$ and $f \in K$.*

Proof. This is the Arzelá-Ascoli Theorem, [Megginson, 1998, Theorem 3.4.14], for example. □

Example 2.4.4. We shall now construct a compact, integral operator on $C([0,1])$ which is not nuclear. Let m be the usual Lebesgue measure on $[0,1]$, restricted to the Borel σ-algebra Σ. Then define $\mu : \Sigma \to C([0,1])$ by

$$\mu(A)(t) = \int_0^t \chi_A(s) \,\mathrm{d}s = m([0,t] \cap A) \qquad (A \in \Sigma).$$

Then, for each $A \in \Sigma$, $\mu(A)$ is a continuous, increasing function on $[0,1]$ with $\mu(A)(0) = 0$, $\mu(A)(1) = m(A)$, and hence $\|\mu(A)\| = m(A)$. In particular, we see that

$$|\mu|_1(A) = m(A) \qquad (A \in \Sigma),$$

so that μ has bounded variation. Let $T \in \mathcal{B}(C([0,1]))$ be the operator induced by μ, so that T is integral.

For $\varepsilon > 0$ and $t \in [0,1]$, suppose that $s \in [0,1]$ with $|s-t| < \varepsilon$, and $A \in \Sigma$. Then we have, for $t < s$,

$$|\mu(A)(t) - \mu(A)(s)| = |m([0,t] \cap A) - m([0,s] \cap A)| = |m((t,s] \cap A)| \leq |t-s| < \varepsilon,$$

and the same result when $s < t$. Thus we see that $\{\mu(A) : A \in \Sigma\}$ is a relatively compact subset of $C([0,1])$, and so T is a compact operator.

Finally, suppose that, for a Bochner-integrable function $g : [0,1] \to C([0,1])$, we have

$$\mu(A) = \int_A g(t)\,\mathrm{d}m(t) \qquad (A \subseteq \Sigma).$$

For each $s \in [0,1]$, let $g_s(t) = g(t)(s)$, so that $g_s : [0,1] \to \mathbb{C}$. For $r \in (0,1]$ we then have

$$\mu([0,r])(s) = \left(\int_0^r g(t)\,\mathrm{d}m(t)\right)(s) = \int_0^r g_s(t)\,\mathrm{d}m(t),$$

which follows as g is the limit of simple functions. Thus we have

$$\int_0^r g_s(t)\,\mathrm{d}m(t) = \mu([0,r])(s) = \begin{cases} s & : s \leq r, \\ r & : s \geq r, \end{cases}$$

so that we must have

$$g(t)(s) = g_s(t) = \begin{cases} 1 & : t \leq s, \\ 0 & : t > s \end{cases} \qquad (s,t \in [0,1]).$$

In particular, for a fixed $t \in [0,1]$, the function $g(t)$ is not continuous, a contradiction, as g maps into $C([0,1])$. Hence T is not nuclear. □

Example 2.4.5. We shall now show that $\mathcal{N}(l^\infty) \neq \mathcal{I}(l^\infty)$. Notice that l^∞ is a commutative C*-algebra with the product defined pointwise, written $x \cdot y$ for $x, y \in l^\infty$. Then $(l^\infty)'$ becomes a Banach l^∞-bimodule, so that

$$\langle \mu \cdot f, g \rangle = \langle \mu, f \cdot g \rangle \qquad (f, g \in l^\infty, \mu \in (l^\infty)').$$

Let $A_0^0 = \mathbb{N}$. Then, for $n \geq 1$, inductively define A_{2m}^n and A_{2m+1}^n to be infinite disjoint subsets of $\mathbb{N}$ such that $A_{2m}^n \cup A_{2m+1}^n = A_m^{n-1}$, for $0 \leq m < 2^{n-1}$. For $A \subseteq \mathbb{N}$, let χ_A be the indicator function of A, so that $\chi_A \in l^\infty$. By the Hahn-Banach theorem, we can find $\mu \in (l^\infty)'$ with $\|\mu\| = 1$ and

$$\langle \mu, \chi_{A_m^n} \rangle = 2^{-n} \qquad (n \in \mathbb{N}_0, 0 \leq m < 2^n).$$

For $n \geq 0$, let
$$f_n = \sum_{m=0}^{2^n-1} (-1)^m \chi_{A_m^n},$$
so that $f_n \in l^\infty$ and $\|f_n\| = 1$ for each n.

Then define $T \in \mathcal{B}(l^\infty)$ by
$$T(f) = \Big(\langle \mu, f_{n-1} \cdot f \rangle \Big)_{n=1}^{\infty} \qquad (f \in l^\infty).$$
Thus we have $\|T(f)\| = \sup_n |\langle \mu, f_{n-1} \cdot f\rangle| \leq \sup_n \|f_{n-1}\| \|f\| = \|f\|$, so that T is bounded. Then note that $f_n \cdot f_n = \chi_{\mathbb{N}}$ for each $n \geq 0$, and that for $0 \leq n < m$, we have $\chi_{A_k^n} \cdot \chi_{A_j^m} \neq 0$ only if $j \in \{2^{m-n}k, \ldots, 2^{m-n}(k+1)-1\}$. Thus we have, for $0 \leq n < m$,
$$\begin{aligned} \langle \mu, f_n \cdot f_m \rangle &= \sum_{k=0}^{2^n-1} \sum_{j=1}^{2^m-1} (-1)^{k+j} \langle \mu, \chi_{A_k^n} \cdot \chi_{A_j^m} \rangle = \sum_{k=0}^{2^n-1} \sum_{j=2^{m-n}k}^{2^{m-n}(k+1)-1} (-1)^{k+j} \langle \mu, \chi_{A_j^m} \rangle \\ &= \sum_{k=0}^{2^n-1} \sum_{j=2^{m-n}k}^{2^{m-n}(k+1)-1} (-1)^{k+j} 2^{-m} = 0. \end{aligned}$$
Thus $T(f_n) = e_{n+1}$ where e_{n+1} is the usual unit vector in $c_0 \subset l^\infty$. In particular, we see that T is not compact, and so T is not nuclear.

We claim that T is integral, however. For $S \in \mathcal{F}(l^1)$ define $f_S : \mathbb{N} \to l^1$ by $f_S(n) = S(e_n)$. Then, for $x = \sum_{n=1}^\infty x_n e_n \in l^1$, we have $\|S(x)\| \leq \sum_{n=1}^\infty |x_n| \|S(e_n)\| \leq \|x\| \sup_n \|f_S(n)\|$. Thus we see that $\|S\| = \sup_n \|f_S(n)\|$. Now let $S = f \otimes x \in l^\infty \otimes l^1$, so that $f_S(m) = x \langle f, e_m \rangle$. Note that then
$$(\langle e_n, f_S(m) \rangle)_{m=1}^\infty = (\langle e_n, x \rangle \langle f, e_m \rangle)_{m=1}^\infty = \langle e_n, x \rangle f \in l^\infty,$$
and so we have
$$\begin{aligned} \mathrm{Tr}(TS') = \langle T(f), x \rangle &= \Big| \sum_{n=1}^\infty \langle e_n, x \rangle \langle \mu, f_{n-1} \cdot f \rangle \Big| \\ &= \Big| \sum_{n=1}^\infty \langle \mu, f_{n-1} \cdot (\langle e_n, f_S(m) \rangle)_{m=1}^\infty \rangle \Big|. \end{aligned}$$
By linearity, this holds for all $S \in \mathcal{F}(l^1)$, so that
$$\begin{aligned} |\mathrm{Tr}(TS')| &\leq \|\mu\| \Big\| \sum_{n=1}^\infty f_{n-1} \cdot (\langle e_n, f_S(m) \rangle)_{m=1}^\infty \Big\| \\ &= \sup_{m \in \mathbb{N}} \Big| \sum_{n=1}^\infty \langle f_{n-1}, e_m \rangle \langle e_n, f_S(m) \rangle \Big| \leq \sup_{m \in \mathbb{N}} \sum_{n=1}^\infty |\langle e_n, f_S(m) \rangle| \\ &= \sup_{m \in \mathbb{N}} \|f_S(m)\| = \|S\|. \end{aligned}$$
Thus we have that T is integral, and $\|T\|_\pi \leq 1$. $\square$

2.5 Arens products on operator ideals

We will study some questions related to Arens products on operators ideals.

The following is essentially laid out in [Pietsch, 1980, Section 6], and the conclusions match those in [Dales, 2000, Section 2.5], although our definitions are slightly different.

Proposition 2.5.1. *Let $\mathfrak{U}$ be an operator ideal and E be a Banach space.*

1. *$\mathfrak{U}(E)$ is an ideal in $\mathcal{B}(E)$, and $(\mathfrak{U}(E), u)$ is a Banach algebra;*
2. *The inclusion map $(\mathfrak{U}(E), u) \to \mathcal{B}(E)$ is continuous;*
3. *The bilinear map $E' \times E \to (\mathfrak{U}(E), u);\ (\mu, x) \mapsto \mu \otimes x \in \mathcal{F}(E) \subseteq \mathfrak{U}(E)$ is continuous.*

Proof. We prove (1) and (2) together. From the definition, it is immediate that $\mathfrak{U}(E)$ is an algebraic ideal in $\mathcal{B}(E)$. For $x \in E$, $\mu \in E'$ and $T \in \mathfrak{U}(E)$, we have

$$u(T)\|\mu \otimes x\| \geq u(T \circ (\mu \otimes x)) = u(\mu \otimes T(x)) = \|\mu\|\|T(x)\|,$$

so that $u(T) \geq \|T(x)\|\|x\|^{-1}$. Thus, as x was arbitrary, we see that $u(T) \geq \|T\|$ for $T \in \mathfrak{U}(E)$, and thus we have (2). Furthermore, for $T, S \in \mathfrak{U}(E)$, we have $u(TS) \leq \|T\|u(S) \leq u(T)u(S)$, which completes the proof of (1).

For (3), we simply have $u(\mu \otimes x) = \|\mu\|\|x\|$, so this is immediate. □

Note that (3) allows us to define a norm-decreasing map $E'\widehat{\otimes}E \to \mathfrak{U}$.

The Arens regularity of $\mathfrak{U}(E)$ is closely related to the topology of E, a fact first shown (in less generality) in [Young, 1976, Theorem 3]. See also [Dales, 2000, Section 2.6].

Theorem 2.5.2. *Let $\mathfrak{U}$ be an operator ideal, and let E be a Banach space such that $\mathfrak{U}(E)$ is Arens regular. Then E is reflexive.*

Proof. If E is not reflexive, then, by Theorem 1.4.7, we can find bounded sequences (x_n) in E and (μ_n) in E' such that the following iterated limits exist, but such that

$$\lim_m \lim_n \langle \mu_n, x_m \rangle \neq \lim_n \lim_m \langle \mu_n, x_m \rangle.$$

Pick $y \in E$ and $\lambda \in E'$ with $\langle \lambda, y \rangle = 1$, and let $T_n = \mu_n \otimes y \in \mathcal{F}(E) \subseteq \mathfrak{U}(E)$ and $S_m = \lambda \otimes x_m \in \mathcal{F}(E) \subseteq \mathfrak{U}(E)$. Then, by (3) above, (T_n) and (S_m) are bounded sequences in $\mathfrak{U}$.

Define $\Lambda \in \mathfrak{U}(E)'$ by $\langle \Lambda, T \rangle = \langle \lambda, T(y) \rangle$ for $T \in \mathfrak{U}(E)$. Then

$$|\langle \Lambda, T \rangle| \leq \|\lambda\|\|y\|\|T\| \leq \|\lambda\|\|y\|u(T), \qquad (T \in \mathfrak{U}(E))$$

by (2) above, so that Λ is a bounded linear functional on $\mathfrak{U}(E)$. We then have

$$\langle \Lambda, T_n S_m \rangle = \langle \lambda, T_n S_m(y) \rangle = \langle \lambda, T_n(x_m) \rangle \langle \lambda, y \rangle = \langle \lambda, y \rangle \langle \mu_n, x_m \rangle = \langle \mu_n, x_m \rangle,$$

so that by Theorem 1.7.2, $\mathfrak{U}(E)$ cannot be Arens regular. $\square$

The converse is not true in full generality, for there exist reflexive Banach spaces E such that $\mathcal{B}(E)$ is not Arens regular (see [Young, 1976, Corollary 1] or Proposition 4.1.2). However, for $\mathcal{A}(E)$ and $\mathcal{K}()$, we do have a converse, again first shown in [Young, 1976]. This will be proved below, in Theorem 2.7.36.

2.6 Arens products on ideals of approximable operators

The ideals of compact and approximable operators are reasonably accessible objects to study, and a fair amount is know about the Arens products on their biduals. We first start by studying the so called *Arens representations* as detailed in, for example, [Palmer, 1994, Section 1.4]. We use the language of modules, but the results (once translated) are the same.

Let $\mathcal{A}$ be a Banach algebra, and let F be a Banach left $\mathcal{A}$-module. Then F' is a Banach right $\mathcal{A}$-module, and F'' is a Banach left $\mathcal{A}$-module. Thus $F'\widehat{\otimes}F$ and $F''\widehat{\otimes}F'$ become Banach $\mathcal{A}$-bimodules for the module actions

$$(\mu \otimes x) \cdot a = \mu \cdot a \otimes x \ , \ a \cdot (\mu \otimes x) = \mu \otimes a \cdot x \ ,$$
$$(\Lambda \otimes \mu) \cdot a = \Lambda \otimes \mu \cdot a \ , \ a \cdot (\Lambda \otimes \mu) = a \cdot \Lambda \otimes \mu \ ,$$

for $a \in \mathcal{A}$, $\mu \otimes x \in F'\widehat{\otimes}F$ and $\Lambda \otimes \mu \in F''\widehat{\otimes}F'$.

Define a bilinear map $\phi_1 : F'' \times F' \to \mathcal{A}'$ by

$$\langle \phi_1(\Lambda, \mu), a \rangle = \langle a \cdot \Lambda, \mu \rangle \qquad (\Lambda \in F'', \mu \in F', a \in \mathcal{A}).$$

Then ϕ_1 extends to a norm-decreasing map $F''\widehat{\otimes}F' \to \mathcal{A}'$. Similarly define $\phi_2 : F'\widehat{\otimes}F \to \mathcal{A}'$ by

$$\langle \phi_2(\mu \otimes x), a \rangle = \langle \mu, a \cdot x \rangle \qquad (\mu \otimes x \in F'\widehat{\otimes}F, a \in \mathcal{A}).$$

Lemma 2.6.1. *The maps ϕ_1 and ϕ_2 are $\mathcal{A}$-bimodule homomorphisms.*

Proof. For $a, b \in \mathcal{A}$ and $\mu \otimes x \in F'\widehat{\otimes}F$, we have

$$\begin{aligned}\langle a \cdot \phi_2(\mu \otimes x), b \rangle &= \langle \phi_2(\mu \otimes x), ba \rangle = \langle \mu, ba \cdot x \rangle \\ &= \langle \phi_2(\mu \otimes a \cdot x), b \rangle = \langle \phi_2(a \cdot (\mu \otimes x)), b \rangle.\end{aligned}$$

The other cases follow in a similar manner. □

Then $\phi_1' : \mathcal{A}'' \to \mathcal{B}(F'')$, with the action given by

$$\langle \phi_1'(\Phi)(\Lambda), \mu\rangle = \langle \Phi, \phi_1(\Lambda \otimes \mu)\rangle \qquad (\Phi \in \mathcal{A}'', \Lambda \in F'', \mu \in F').$$

Similarly, $\phi_2' : \mathcal{A}'' \to \mathcal{B}(F')$. We can also verify the following identities:

$$\Phi \cdot \phi_1(\Lambda \otimes \mu) = \phi_1(\phi_1'(\Phi)(\Lambda) \otimes \mu) \qquad (\Phi \in \mathcal{A}'', \Lambda \otimes \mu \in F''\widehat{\otimes}F'),$$

$$\phi_2(\mu \otimes x) \cdot \Phi = \phi_2(\phi_2'(\Phi)(\mu) \otimes x) \qquad (\Phi \in \mathcal{A}'', \mu \otimes x \in F'\widehat{\otimes}F).$$

Definition 2.6.2. For a Banach space E, we have the isometric map $\mathcal{B}(E) \to \mathcal{B}(E')$; $T \mapsto T'$. For a subset $X \subseteq \mathcal{B}(E)$ write

$$X^a = \{T' : T \in X\} \subseteq \mathcal{B}(E'),$$

so that, in particular, $\mathcal{B}(E)^a$ is a subalgebra of $\mathcal{B}(E')$. We can show that $\mathcal{B}(E)^a = \mathcal{B}(E')$ if and only if E is reflexive. For a Banach algebra $\mathcal{A}$ and $\psi \in \mathcal{B}(\mathcal{A}, \mathcal{B}(E))$, we define $\psi^a \in \mathcal{B}(\mathcal{A}, \mathcal{B}(E'))$ by $\psi^a(b) = \psi(b)'$ for $b \in \mathcal{A}$.

Let $\theta_1 = \phi_1'$ and $\theta_2 = (\phi_2')^a$.

Proposition 2.6.3. *The maps $\theta_1 : (\mathcal{A}'', \Box) \to \mathcal{B}(F'')$ and $\theta_2 : (\mathcal{A}'', \Diamond) \to \mathcal{B}(F'')$ are norm-decreasing homomorphisms. Thus θ_1 and θ_2 induce a module structure on F'' so that we can, respectively, view F'' as a Banach left $(\mathcal{A}'', \Box)$-module or a Banach left $(\mathcal{A}'', \Diamond)$-module.*

Proof. Let $\Phi, \Psi \in \mathcal{A}''$ and $\Lambda \otimes \mu \in F''\widehat{\otimes}F'$. Then we have

$$\begin{aligned}\langle \theta_1(\Phi\Box\Psi)(\Lambda), \mu\rangle &= \langle \Phi, \Psi \cdot \phi_1(\lambda \otimes \mu)\rangle = \langle \Phi, \phi_1(\phi_1'(\Psi)(\Lambda) \otimes \mu)\rangle \\ &= \langle \phi_1'(\Phi)(\phi_1'(\Psi)(\Lambda)), \mu\rangle = \langle \big(\theta_1(\Phi) \circ \theta_2(\Psi)\big)(\Lambda), \mu\rangle.\end{aligned}$$

We will show that ϕ_2' is an anti-homomorphism, so that θ_2 is a homomorphism. For $\Phi, \Psi \in \mathcal{A}''$ and $\mu \otimes x \in F'\widehat{\otimes}F$, we have

$$\begin{aligned}\langle \phi_2'(\Phi\Diamond\Psi)(\mu), x\rangle &= \langle \Psi, \phi_2(\mu \otimes x) \cdot \Phi\rangle = \langle \Psi, \phi_2(\phi_2'(\Phi)(\mu) \otimes x)\rangle \\ &= \langle \phi_2'(\Psi)(\phi_2'(\Phi)(\mu)), x\rangle.\end{aligned}$$

□

An alternative way to look at these maps is through the use of nets (or ultrafilters and ultraproducts, see Section 3.1). For $\Phi, \Psi \in \mathcal{A}''$, suppose that

$$\Phi = \lim_\alpha a_\alpha\ ,\ \Psi\ = \lim_\beta b_\beta,$$

where these are limits along nets, with convergence in the weak*-topology on $\mathcal{A}''$. We then have that

$$\theta_1(\Phi\Box\Psi)(\Lambda) = \lim_\alpha \lim_\beta a_\alpha b_\beta \cdot \Lambda \qquad (\Lambda \in F''),$$

where the limit is in the weak*-topology on F''. Similarly, we have

$$\phi_2'(\Phi\Diamond\Psi)(\mu) = \lim_\beta \lim_\alpha \mu \cdot (a_\alpha b_\beta) \qquad (\mu \in F'),$$

where the limit is in the weak*-topology on F'.

Note that the following diagram is commutative:

$$\begin{tikzcd}
F'\widehat{\otimes}F \arrow[r] \arrow[dr, "\phi_2"'] & F\widehat{\otimes}F' \arrow[r, "\kappa_F \otimes \mathrm{Id}_{F'}"] & F''\widehat{\otimes}F' \arrow[dl, "\phi_1"] \\
 & \mathcal{A}' &
\end{tikzcd}$$

We can then easily check that, for $\Phi \in \mathcal{A}''$, we have $\theta_1(\Phi) \circ \kappa_F = \theta_2(\Phi) \circ \kappa_F$.

For a general Banach algebra $\mathcal{A}$ and module F, the behaviour of θ_1 applied to $\Diamond$ (or θ_2 applied to $\Box$) has no simple description. However, in a large number of cases, we can say something.

Definition 2.6.4. For $a \in \mathcal{A}$, define $T_a \in \mathcal{B}(F)$ by $T_a(x) = a \cdot x$ for $x \in F$. We say that the action of $\mathcal{A}$ on F is *weakly-compact* if $T_a \in \mathcal{W}(F)$ for every $a \in \mathcal{A}$.

The following definitions appear in [Dales, Lau, 2004], but we give a more general treatment here; the use of these ideas appears to be "folklore" in that they are certainly known, but there is no definitive source for them (see, for example, [Grosser, 1987], [Grosser, 1984] or [Palmer, 1985], all of which deal with ideals of approximable operators). Let F be a Banach space, and for $T \in \mathcal{B}(F'')$, define $\eta(T) \in \mathcal{B}(F')$ and $\mathcal{Q}(T) \in \mathcal{B}(F'')$ by

$$\eta(T) = \kappa_F' \circ T' \circ \kappa_{F'} \quad , \quad \mathcal{Q}(T) = \eta(T)' = \kappa_{F'}' \circ T'' \circ \kappa_F'' = \kappa_{F'}' \circ (T \circ \kappa_F)''.$$

Then note that $\eta(T') = T$ for $T \in \mathcal{B}(F')$, so that $\mathcal{B}(F')^a$ is a one-complemented subspace of $\mathcal{B}(F'')$. Define a bilinear operation $\star$ on $\mathcal{B}(F'')$ by $T \star S = \mathcal{Q}(T) \circ S$ for $T, S \in \mathcal{B}(F'')$.

Proposition 2.6.5. *The operation $\star$ is a Banach algebra product on $\mathcal{B}(F'')$. When the action of $\mathcal{A}$ on F is weakly-compact, the map $\theta_1 : (\mathcal{A}'', \Diamond) \to (\mathcal{B}(F''), \star)$ is a homomorphism.*

Proof. We see immediately that $\star$ satisfies $\|T \star S\| \leq \|T\|\|S\|$, and that if suffices to show that $(T \star S) \star R = T \star (S \star R)$ for each $R, S, T \in \mathcal{B}(F'')$. We have

$$\eta(S) \circ \eta(T) = \kappa'_F \circ S' \circ \kappa_{F'} \circ \eta(T) = \kappa'_F \circ S' \circ \eta(T)'' \circ \kappa_{F'} = \eta(\mathcal{Q}(T) \circ S),$$

and thus

$$\begin{aligned}(T \star S) \star R = \mathcal{Q}(T \star S) \circ R = \mathcal{Q}(\mathcal{Q}(T) \circ S) \circ R = \eta(\mathcal{Q}(T) \circ S)' \circ R \\ = \eta(T)' \circ \eta(S)' \circ R = \mathcal{Q}(T) \circ \mathcal{Q}(S) \circ R = T \star (S \star R).\end{aligned}$$

For $a \in \mathcal{A}$ and $\Lambda \in F''$, we can verify that $a \cdot \Lambda = T''_a(\Lambda)$. As $T_a \in \mathcal{W}(F)$, by Theorem 2.2.8, we have $T''_a(\Lambda) \in \kappa_F(F)$. Thus let $\kappa_F(y) = a \cdot \Lambda$, so that for $\Phi \in \mathcal{A}''$ and $\Lambda \otimes \mu \in F''\widehat{\otimes}F'$, we have

$$\begin{aligned}\langle \phi_1(\Lambda \otimes \mu) \cdot \Phi, a \rangle &= \langle \Phi, \phi_1(a \cdot \Lambda \otimes \mu) \rangle = \langle \theta_1(\Phi)(a \cdot \Lambda), \mu \rangle = \langle \theta_1(\Phi)'\kappa_{F'}(\mu), a \cdot \Lambda \rangle \\ &= \langle \theta_1(\Phi)'\kappa_{F'}(\mu), \kappa_F(y) \rangle = \langle \eta(\theta_1(\Phi))(\mu), y \rangle = \langle \kappa_F(y), \eta(\theta_1(\Phi))(\mu) \rangle \\ &= \langle a \cdot \Lambda, \eta(\theta_1(\Phi))(\mu) \rangle = \langle \phi_1\big(\Lambda \otimes \eta(\theta_1(\Phi))(\mu)\big), a \rangle.\end{aligned}$$

Thus for $\Phi, \Psi \in \mathcal{A}''$ and $\Lambda \otimes \mu \in F''\widehat{\otimes}F'$, we have

$$\begin{aligned}\langle \theta_1(\Phi \Diamond \Psi)(\Lambda), \mu \rangle &= \langle \Psi, \phi_1(\Lambda \otimes \mu) \cdot \Phi \rangle = \langle \Psi, \phi_1\big(\Lambda \otimes \eta(\theta_1(\Phi))(\mu)\big) \rangle \\ &= \langle \theta_1(\Psi)(\Lambda), \eta(\theta_1(\Phi))(\mu) \rangle = \langle \mathcal{Q}(\theta_1(\Phi))\theta_1(\Psi)(\Lambda), \mu \rangle,\end{aligned}$$

so that $\theta_1(\Phi \Diamond \Psi) = \theta_1(\Phi) \star \theta_1(\Psi)$. $\square$

Suppose that F is reflexive, so that the action of $\mathcal{A}$ on F is certainly weakly-compact. Then $\star = \circ$ on $\mathcal{B}(F)$, so that θ_1 is a homomorphism $\mathcal{A}'' \to \mathcal{B}(F)$ for either Arens product. In particular, if θ_1 is injective, then $\mathcal{A}$ must be Arens regular. We shall use this idea later to show that $\mathcal{B}(E)$ is Arens regular for certain Banach spaces E. In fact, for *any* Arens regular Banach algebra, we can find a Banach left $\mathcal{A}$-module F which is reflexive and for which θ_1 is injective: see Theorem 3.3.13.

We can actually define ϕ_1 in a slightly more subtle manner. By linearity (or the tensorial property) ϕ_1 is a map $F'' \otimes F' \to \mathcal{A}'$. We can use this to define a semi-norm on $F'' \otimes F'$ by

$$\|u\|_0 = \|\phi_1(u)\| = \sup\{|\langle \phi_1(u), a \rangle| : a \in \mathcal{A}_{[1]}\}. \qquad (u \in F'' \otimes F').$$

Definition 2.6.6. Let $\mathcal{A}$ be a Banach algebra and F be a Banach left $\mathcal{A}$-module. Suppose that, for each $u = \sum_{i=1}^n \Lambda_i \otimes \mu_i \in F'' \otimes F'$, we have

$$\sup\Big\{\Big|\sum_{i=1}^n \langle a \cdot \Lambda_i, \mu_i \rangle\Big| : a \in \mathcal{A}_{[1]}\Big\} \geq \sup\Big\{\Big\|\sum_{i=1}^n \langle \Lambda_i, \lambda \rangle \mu_i\Big\| : \lambda \in F'_{[1]}\Big\} = \varepsilon(u, F'' \check{\otimes} F').$$

Then we say that $(\mathcal{A}, F)$ is *tensorial*.

The reason we make this definition is the following. Let $(\mathcal{A}, F)$ be tensorial. Then $\|\cdot\|_0$ is a norm on $F'' \otimes F'$, and clearly $\varepsilon(u, F'' \otimes F') \leq \|u\|_0 \leq \pi(u, F'' \otimes F')$ for each $u \in F'' \otimes F'$. Thus $\|\cdot\|_0$ is a reasonable crossnorm on $F'' \otimes F'$.

Now, we might wonder if $\|\cdot\|_0$ is a tensor norm. Of course, we have not defined $\|\cdot\|_0$ on all spaces; this is a minor issue, as there are ways to extend to all pairs of Banach spaces. For example, for a pair of Banach spaces D and E, we can set

$$\|u\|_0 = \sup\{\|(T \otimes S)(u)\|_0 : T \in \mathcal{B}(D, F'')_{[1]}, S \in \mathcal{B}(E, F')_{[1]}\} \qquad (u \in D \otimes E).$$

Then, for Banach spaces D_1 and E_1, $u \in D_1 \otimes E_1$, $A \in \mathcal{B}(D_1, D)$ and $B \in \mathcal{B}(E_1, E)$, we have

$$\begin{aligned}
\|(A \otimes B)(u)\|_0 &= \sup\{\|(T \otimes S)(A \otimes B)(u)\|_0 : T \in \mathcal{B}(D, F'')_{[1]}, S \in \mathcal{B}(E, F')_{[1]}\} \\
&= \sup\{\|(TA \otimes SB)(u)\|_0 : T \in \mathcal{B}(D, F'')_{[1]}, S \in \mathcal{B}(E, F')_{[1]}\} \\
&\leq \sup\{\|(T \otimes S)(u)\|_0 : T \in \mathcal{B}(D_1, F'')_{[\|A\|]}, S \in \mathcal{B}(E_1, F')_{[\|B\|]}\} \\
&= \|A\|\|B\|\|u\|_0.
\end{aligned}$$

Thus we have made $\|\cdot\|_0$ into a uniform crossnorm. It is not, however, finitely generated, at least in general.

Proposition 2.6.7. *Let E be a Banach space, α be a tensor norm, and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then $(\mathcal{A}, E)$ is tensorial. When α' is totally accessible, or α is accessible and E' has the metric approximation property, $\|\cdot\|_0$ is actually the nuclear norm $\|\cdot\|_{\mathcal{N}_{\alpha'}}$. When α is accessible and E' has the bounded approximation property, $\|\cdot\|_0$ is equivalent to $\|\cdot\|_{\mathcal{N}_{\alpha'}}$.*

Proof. As $\mathcal{A} = \mathcal{N}_\alpha(E)$ is a quotient of $E'\widehat{\otimes}_\alpha E$, we see that $\mathcal{A}'$ is, isometrically, a subspace of $\mathcal{B}_{\alpha'}(E')$, namely

$$\mathcal{A}' = (\ker J_\alpha)^\circ = \{T \in \mathcal{B}_{\alpha'}(E') : \langle T, v\rangle = 0 \ (v \in E'\widehat{\otimes}_\alpha E, J_\alpha(v) = 0)\},$$

by Theorem 1.4.11. By Proposition 2.3.2, for $u \in E'' \otimes E'$, we have

$$\|u\|_0 = \|u\|_{\alpha'} \leq \|u\|_{\mathcal{N}_{\alpha'}} \leq \alpha'(u, E'' \otimes E'),$$

where we identify u with the operator in $\mathcal{F}(E')$ it induces. Let $u = \sum_{i=1}^n \Lambda_i \otimes \mu_i$, and let $x \in E$ and $\mu \in E'$. Then we have

$$\Big|\sum_{i=1}^n \langle \Lambda_i, \mu\rangle\langle \mu_i, x\rangle\Big| = |\langle u, \mu \otimes x\rangle| \leq \|u\|_0 \|\mu \otimes x\|_{\mathcal{N}_\alpha} = \|u\|_0 \|\mu\|\|x\|,$$

so that $\|u\|_0 \geq \varepsilon(u, E' \otimes E)$, and thus that $(\mathcal{A}, E)$ is tensorial.

When α' is totally accessible or α is accessible and E' has the metric approximation property, by Propositions 2.3.20 and 2.3.21, we immediately have

$$\|u\|_0 = \|u\|_{\alpha'} = \|u\|_{\mathcal{N}_{\alpha'}} \qquad (u \in E'' \otimes E'),$$

so that $\| \cdot \|_0 = \| \cdot \|_{\mathcal{N}_{\alpha'}}$. Similarly, Proposition 2.3.22 completes the proof. □

Let α be a tensor norm and E be a Banach such that α' is totally accessible, or E' has the metric approximation property (we can generalise this to the bounded approximation property in a simple way). Then $\| \cdot \|_0 = \| \cdot \|_{\mathcal{N}_{\alpha'}}$, so that, by continuity, ϕ_1 extends to a map $E''\widehat{\otimes}_{\alpha'}E' \to \mathcal{N}_\alpha(E)'$, and we see that ϕ_1 agrees with the map $J_{\alpha'}$, so that ϕ_1 is a quotient operator. Thus, in particular, $\theta_1 : \mathcal{N}_\alpha(E)'' \to \mathcal{N}_{\alpha'}(E')' = (\ker J_{\alpha'})^\circ \subseteq \mathcal{B}_\alpha(E'')$ is an isometry.

When α is a general tensor norm and E is a general Banach space, we only have that $\| \cdot \|_0 \leq \| \cdot \|_{\mathcal{N}_{\alpha'}}$. However, we can still extend ϕ_1 by continuity to a map $\phi_1 : E''\widehat{\otimes}_{\alpha'}E' \to \mathcal{N}_\alpha(E)'$, but now ϕ_1 is only norm-decreasing. We can check that ϕ_1 still agrees with the map $J_{\alpha'}$; that is, for $u \in E''\widehat{\otimes}_{\alpha'}E'$, we have that $\phi_1(u)$ and $J_{\alpha'}(u)$ are the same operator in $\mathcal{B}_{\alpha'}(E')$, but the natural norms associated with these operators are different. Thus we also still have $\theta_1 : \mathcal{N}_\alpha(E)'' \to (\ker J_{\alpha'})^\circ \subseteq \mathcal{B}_\alpha(E'')$, but again, θ_1 is no longer an isometry, merely norm-decreasing.

Example 2.6.8. Let E be a Banach space such that E' has the bounded approximation property. Then $\phi_1 : E''\widehat{\otimes}E' \to \mathcal{A}(E)'$ is an isomorphism onto its range (if E' has the metric approximation property, then ϕ_1 is even an isometry). Thus $\theta_1 : \mathcal{A}(E)'' \to (\ker J_\pi)^\circ = \{0\}^\circ = \mathcal{B}(E'')$ is surjective. As $\mathcal{A}(E)$ clearly has weakly-compact action on E, we see that $\theta_1 : (\mathcal{A}(E)'', \Box) \to \mathcal{B}(E'')$ and $\theta_1 : (\mathcal{A}(E)'', \Diamond) \to (\mathcal{B}(E''), \mathcal{Q})$ are homomorphisms. In particular, let $\Xi \in \mathcal{A}(E)''$ be such that $\theta_1(\Xi) = \mathrm{Id}_{E''}$. Then we have

$$\theta_1(\Phi \Box \Xi) = \theta_1(\Phi) \quad , \quad \theta_1(\Xi \Diamond \Phi) = \mathcal{Q}(\mathrm{Id}_{E''}) \circ \theta_1(\Phi) = \theta_1(\Phi) \qquad (\Phi \in \mathcal{A}(E)'').$$

In fact, using Proposition 2.7.3, we can show that Ξ is a mixed identity (see Section 1.7) for $\mathcal{A}(E)''$. □

2.7 Topological centres of biduals of operator ideals

We will continue the study of topological centres of biduals of operators ideals which, in the case of the approximable operators, was started in [Dales, Lau, 2004]. This work will also allow us to say when some operator ideals are Arens regular. We note that some of the following work is similar to work done in [Grosser, 1987], where Grosser studies *multipliers* of algebras of approximable operators. As Grosser points out in this paper, many of these ideas and results have entered folklore (for example, the maps η and $\mathcal{Q}$). Grosser does not study topological centres, but presumably he could have drawn the conclusions that are found in [Dales, Lau, 2004], for example. We will instead develop the theory for general tensor norms, and study more general Banach spaces than those studied in [Grosser, 1987] or [Dales, Lau, 2004].

Let E be a reflexive Banach space with the metric approximation property (this is not much of a restriction, by Theorem 2.3.23). We shall see later, for example in Corollary 2.7.25, that $\mathcal{A}(E)'' = \mathcal{B}(E)$ both as a Banach space and algebraically, so that $\mathcal{A}(E)$ is Arens regular, and $\mathcal{A}(E)''$ has a mixed identity, so that $\mathcal{A}(E)$ has a bounded approximate identity (see Section 1.7). Actually, we can take a more direct (and less circular) route. In [Grønbæk, Willis, 1993], the question of when $\mathcal{A}(E)$ has a bounded approximate identity is investigated. It is worth noting that a lot of parallel development has occurred in this area; [Grønbæk, Willis, 1993] is the best summary of available results, but many results were first proved elsewhere.

Theorem 2.7.1. *Let E be a Banach space. Then the following are equivalent:*

1. *E' has the bounded approximation property;*
2. *$\mathcal{A}(E)$ has a bounded approximate identity;*
3. *$\mathcal{A}(E')$ has a bounded left approximate identity;*
4. *$\mathcal{A}(E)''$ has a mixed identity.*

Proof. The first three equivalences follow from [Grønbæk, Willis, 1993, Theorem 3.3]. The equivalence of (4) and (2) follow by standard results (see Proposition 1.7.3). Alternatively, these results follow from Example 2.6.8 and standard properties of nuclear and integral operators. □

We will now turn our attention to ideals of α-nuclear operators for tensor norms α. Eventually we will come a full circle and use the above theorem. Our basic tool will be the Grothendieck Composition theorem, which will allow us, under many circumstances, to study integral operators (which are the dual of approximable operators, which hints as to why the above theorem will become useful).

Definition 2.7.2. Let E be a Banach space and α be a tensor norm. We say that (E, α) is a *Grothendieck pair* if α is accessible or E has the bounded approximation property. In this case, $K(E, \alpha)$ is the constant arising from the Grothendieck Composition theorem, so that $K(E, \alpha) = 1$ when α is accessible, and otherwise E has the bounded approximation property with bounded $K(E, \alpha)$.

Let E be a Banach space and α be a tensor norm. As in Proposition 2.6.7, $\mathcal{N}_\alpha(E)'$ is a subspace of $\mathcal{B}_{\alpha'}(E')$, and we can view $\phi_1 : E''\widehat{\otimes}_{\alpha'}E' \to \mathcal{N}_\alpha(E)'$ as a norm-decreasing map, which agrees, algebraically, with $J_{\alpha'}$.

Proposition 2.7.3. *Let E be a Banach space and α be a tensor norm. Let $\mathcal{A} = \mathcal{N}_\alpha(E)$, the α-nuclear operators on E, so that $\mathcal{A}'$ is a subspace of $\mathcal{B}_{\alpha'}(E')$. Then we have*

$$S \cdot R = R' \circ S \quad , \quad R \cdot S = S \circ R' \qquad (R \in \mathcal{A}, S \in \mathcal{A}'),$$

$$\Phi \cdot S = \eta(\theta_1(\Phi) \circ S') \;,\; S \cdot \Phi = \eta(\theta_1(\Phi)) \circ S \quad (S \in \mathcal{A}', \Phi \in \mathcal{A}'').$$

Furthermore, we have that $\theta_1 : \mathcal{A}'' \to \mathcal{B}_\alpha(E'')$ is a norm-decreasing map.

When (E', α) is a Grothendieck pair, we also have

$$\Phi \cdot S \in \mathcal{I}(E'), \|\Phi \cdot S\|_\pi \le K(E', \alpha)\|S\|\|\Phi\|.$$

Similarly, when (E'', α) is a Grothendieck pair, we have

$$S \cdot \Phi \in \mathcal{I}(E'), \|S \cdot \Phi\|_\pi \le K(E'', \alpha)\|S\|\|\Phi\|.$$

Proof. The first part is a simple calculation. For $\Phi \in \mathcal{A}'', S \in \mathcal{A}'$ and $R = \mu \otimes x \in \mathcal{A}$, we have

$$\begin{aligned}
\langle \Phi \cdot S, R\rangle &= \langle \Phi, \phi_1(R' \circ S)\rangle = \langle \Phi, \phi_1(S'(\kappa_E(x)) \otimes \mu)\rangle \\
&= \langle \theta_1(\Phi)(S'(\kappa_E(x))), \mu\rangle = \langle \eta(\theta_1(\Phi) \circ S'), R\rangle, \\
\langle S \cdot \Phi, R\rangle &= \langle \Phi, \phi_1(S \circ R')\rangle = \langle \Phi, \phi_1(\kappa_E(x) \otimes S(\mu))\rangle \\
&= \langle \theta_1(\Phi)(\kappa_E(x)), S(\mu)\rangle = \langle \eta(\theta_1(\Phi)) \circ S, R\rangle.
\end{aligned}$$

Thus we get the second part by linearity and continuity. That $\theta_1 : \mathcal{A}'' \to \mathcal{B}_\alpha(E'')$ is norm decreasing follows by the discussion after Proposition 2.6.7.

For $\Phi \in \mathcal{A}''$, we have that $\theta_1(\Phi)' \in \mathcal{B}_{\alpha^t}(E''')$ and so $\eta(\theta_1(\Phi)) = \kappa'_E \circ \theta_1(\Phi)' \circ \kappa_{E'} \in \mathcal{B}_{\alpha^t}(E')$ with $\|\eta(\theta_1(\Phi))\|_{\alpha^t} \le \|\theta_1(\Phi)\|_\alpha$. Then the Grothendieck Composition theorem says that, when (E', α) is a Grothendieck pair, for $S \in \mathcal{A}'$ and $\Phi \in \mathcal{A}''$, we have $S \cdot \Phi = \eta(\theta_1(\Phi)) \circ S \in \mathcal{I}(E')$, and

$$\|S \cdot \Phi\|_\pi \le K(E', \alpha)\|S\|_{\alpha'}\|\eta(\theta_1(\Phi))\|_{\alpha^t} \le K(E', \alpha)\|S\|_{\alpha'}\|\Phi\|.$$

Similarly, when (E'', α) is a Grothendieck pair, $\theta_1(\Phi) \in \mathcal{B}_\alpha(E'')$ and $S' \in \mathcal{B}_{\check{\alpha}}(E'')$, so that $\theta_1(\Phi) \circ S' \in \mathcal{I}(E'')$ and $\|\theta_1(\Phi) \circ S'\|_\pi \leq K(E'', \alpha)\|\Phi\|\|S\|_{\alpha'}$. Hence

$$\Phi \cdot S = \kappa'_E \circ S'' \circ \theta_1(\Phi)' \circ \kappa_{E'} \in \mathcal{I}(E'),$$

and $\|\Phi.S\|_\pi \leq K(E'', \alpha)\|S\|_{\alpha'}\|\Phi\|$. □

For a tensor norm α, we turn $E'''\widehat{\otimes}_\alpha E''$ into a Banach algebra in the obvious way, by extending the multiplication on $\mathcal{F}(E'')$. Thus, for $u, v \in E'''\widehat{\otimes}_\alpha E''$, we have

$$u \circ v = (\mathrm{Id}_{E'''} \otimes J_\alpha(u))(v).$$

In particular, $J_\alpha : E'''\widehat{\otimes} E'' \to \mathcal{N}_\alpha(E'')$ becomes a homomorphism. We can also define $\star$ as a Banach algebra multiplication on $E'''\widehat{\otimes}_\alpha E''$ by setting

$$u \star v = (\mathrm{Id}_{E'''} \otimes \mathcal{Q}(J_\alpha(u)))(v) \qquad (u, v \in E'''\widehat{\otimes}_\alpha E'').$$

Theorem 2.7.4. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. There exist norm-decreasing homomorphisms*

$$\psi_1 : (E'''\widehat{\otimes}_\alpha E'', \circ) \to (\mathcal{A}'', \Box) \quad , \quad \psi_2 : (E'''\widehat{\otimes}_\alpha E'', \star) \to (\mathcal{A}'', \Diamond),$$

such that $\theta_1 \circ \psi_1 = J_\alpha$ and $\theta_1 \circ \psi_2 = \mathcal{Q} \circ J_\alpha$. For $i = 1, 2$ and $T \in \mathcal{A}$, if $u \in E'\widehat{\otimes}_\alpha E$ is such that $T = J_\alpha(u)$, then we have $\psi_i(u'') = \psi_i((\kappa_{E'} \otimes \kappa_E)(u)) = \kappa_{\mathcal{A}}(T)$.

Proof. For $T \in \mathcal{F}(E'')$ and $S \in \mathcal{A}' \subseteq \mathcal{B}_{\alpha'}(E')$, we have that $\eta(T \circ S'), \eta(S' \circ T) \in \mathcal{F}(E')$. Thus we can define

$$\langle \psi_1(T), S\rangle = \mathrm{Tr}(\eta(T \circ S')) \quad , \quad \langle \psi_2(T), S\rangle = \mathrm{Tr}(\eta(S' \circ T)) \qquad (T \in \mathcal{F}(E''), S \in \mathcal{A}').$$

We then have, recalling that $\mathcal{F}(E'') = E''' \otimes E''$, and that $\eta(S' \circ T) = \eta(T) \circ S$,

$$\begin{aligned}
|\langle \psi_1(T), S\rangle| &= |\mathrm{Tr}(\kappa'_E \circ S'' \circ T' \circ \kappa_{E'})| = |\mathrm{Tr}(\kappa_{E'} \circ \kappa'_E \circ S'' \circ T')| \\
&= |\langle \kappa_{E'} \circ \kappa'_E \circ S'', T\rangle| \leq \alpha(T, E''' \otimes E'')\|\kappa_{E'} \circ \kappa'_E \circ S''\|_{\alpha'} \\
&\leq \alpha(T, E''' \otimes E'')\|S''\|_{\alpha'} = \alpha(T, E''' \otimes E'')\|S\|_{\alpha'}, \\
|\langle \psi_2(T), S\rangle| &= |\mathrm{Tr}(\eta(T) \circ S)| = |\mathrm{Tr}(\kappa'_E \circ T' \circ \kappa_{E'} \circ S)| \\
&= |\mathrm{Tr}(\kappa_{E'} \circ S \circ \kappa'_E \circ T')| \leq \alpha(T, E''' \otimes E'')\|\kappa_{E'} \circ S \circ \kappa'_E\|_{\alpha'} \\
&\leq \alpha(T, E''' \otimes E'')\|S\|_{\alpha'}.
\end{aligned}$$

Consequently, for $i = 1, 2$, $\|\psi_i(T)\| \leq \alpha(T)$, so that ψ_i extends by continuity to a norm-decreasing map $E'''\widehat{\otimes}_\alpha E'' \to \mathcal{A}''$.

For $\Lambda \in E''$, $\mu \in E'$ and $T \in \mathcal{N}_\alpha(E'')$, we have

$$\langle \theta_1(\psi_1(T))(\Lambda), \mu \rangle = \langle \psi_1(T), \phi_1(\Lambda \otimes \mu) \rangle = \mathrm{Tr}(\eta(T \circ (\kappa_{E'}(\mu) \otimes \Lambda))) = \langle T(\Lambda), \mu \rangle,$$
$$\langle \theta_1(\psi_2(T))(\Lambda), \mu \rangle = \langle \psi_2(T), \phi_1(\Lambda \otimes \mu) \rangle = \mathrm{Tr}(\eta(T) \circ \phi_1(\Lambda \otimes \mu)) = \langle \Lambda, \eta(T)(\mu) \rangle.$$

Thus we see that $\theta_1 \circ \psi_1 = J_\alpha$ and $\theta_1 \circ \psi_2 = \mathcal{Q} \circ J_\alpha$.

For $T = \mu \otimes x \in \mathcal{A}$ and $S \in \mathcal{A}'$, we have

$$\langle \psi_1(T''), S \rangle = \mathrm{Tr}(\eta(T'' \circ S')) = \mathrm{Tr}(S \circ T') = \langle S, T \rangle = \langle \kappa_{\mathcal{A}}(T), S \rangle,$$
$$\langle \psi_2(T''), S \rangle = \mathrm{Tr}(\eta(T'') \circ S) = \mathrm{Tr}(T' \circ S) = \mathrm{Tr}(S \circ T') = \langle \kappa_{\mathcal{A}}(T), S \rangle.$$

By linearity, for $i = 1, 2$, we have $\psi_i(T'') = \kappa_{\mathcal{A}}(T)$ for $T \in E' \otimes E$. Thus, for $T = J_\alpha(u) \in \mathcal{N}_\alpha(E)$, suppose that (u_n) is a sequence in $E' \otimes E$ with $\alpha(u_n - u) \to 0$. For $i = 1, 2$, we have

$$\psi_i(u'') = \lim_{n\to\infty} \psi_i(u_n'') = \lim_{n\to\infty} \kappa_{\mathcal{A}}(u_n) = \kappa_{\mathcal{A}}(T),$$

as required.

We defer a calculation to Lemma 2.7.6 to follow. We claim that, for $T_1, T_2 \in \mathcal{B}(E'')$ and $S \in \mathcal{B}(E')$, we have $\eta(T_1 \circ T_2 \circ S') = \eta(T_1 \circ \mathcal{Q}(T_2 \circ S'))$. Then, for $T_1, T_2 \in \mathcal{F}(E'')$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}\langle \psi_1(T_1) \Box \psi_1(T_2), S \rangle &= \langle \psi_1(T_1), \eta(\theta_1(\psi_1(T_2)) \circ S') \rangle = \langle \psi_1(T_1), \eta(T_2 \circ S') \rangle \\ &= \mathrm{Tr}(\eta(T_1 \circ \mathcal{Q}(T_2 \circ S')) = \mathrm{Tr}(\eta(T_1 \circ T_2 \circ S')) = \langle \psi_1(T_1 \circ T_2), S \rangle.\end{aligned}$$

We see that $\psi_1 : (E'''\widehat{\otimes}_\alpha E'', \circ) \to (\mathcal{A}'', \Box)$ is a homomorphism.

Similarly, for $T_1, T_2 \in \mathcal{F}(E'')$ and $S \in \mathcal{A}'$, we have $\eta(\mathcal{Q}(T_1) \circ T_2) = \eta(\eta(T_1)' \circ T_2) = \eta(T_2) \circ \eta(T_1)$, so that

$$\begin{aligned}\langle \psi_2(T_1) \Diamond \psi_2(T_2), S \rangle &= \langle \psi_2(T_2), \eta(\theta_1(\psi_2(T_1))) \circ S \rangle = \langle \psi_2(T_2), \eta(T_1) \circ S \rangle \\ &= \mathrm{Tr}(\eta(T_2) \circ \eta(T_1) \circ S) = \mathrm{Tr}(\eta(\mathcal{Q}(T_1) \circ T_2) \circ S) \\ &= \mathrm{Tr}(\eta(T_1 \star T_2) \circ S) = \langle \psi_2(T_1 \star T_2), S \rangle.\end{aligned}$$

We see that $\psi_2 : (E'''\widehat{\otimes} E'', \star) \to (\mathcal{A}'', \Diamond)$ is a homomorphism. □

It would have been more natural to define the above maps from $\mathcal{N}_\alpha(E'')$. However, in general we cannot do this, as the next example shows.

Example 2.7.5. Let E be a Banach space with the approximation property such that E' does not have the approximation property. For example, let $E = l^2 \widehat{\otimes} l^2$, so that $E' =$

$\mathcal{B}(l^2)$ does not have the approximation property by [Szankowski, 1981], but E does by [Ryan, 2002, Section 4.3]. Then let $\mathcal{A} = \mathcal{N}(E) = E'\widehat{\otimes}E$, so that $\mathcal{A}' = \mathcal{B}(E')$. Thus, if we had defined $\psi_1 : \mathcal{N}(E'') \to \mathcal{B}(E')'$, then we would have defined a trace on $\mathcal{N}(E')$, by

$$\mathrm{Tr}(T) = \mathrm{Tr}(\eta(T')) = \langle \psi_1(T'), \mathrm{Id}_{E'} \rangle \qquad (T \in \mathcal{N}(E')).$$

This is impossible, as $\mathcal{N}(E') \neq E''\widehat{\otimes}E'$, so that $\mathcal{N}(E')' \subsetneq \mathcal{B}(E'')$ and thus $\mathrm{Id}'_{E'} = \mathrm{Id}_{E''} \notin \mathcal{N}(E')'$. □

Lemma 2.7.6. *Let E be a Banach space. For $T_1, T_2 \in \mathcal{B}(E'')$ and $S \in \mathcal{B}(E')$, we have $\eta(T_1 \circ T_2 \circ S') = \eta(T_1 \circ \mathcal{Q}(T_2 \circ S'))$.*

Proof. For $T_1, T_2 \in \mathcal{B}(E'')$, $S \in \mathcal{B}(E')$, $x \in E$ and $\mu \in E'$, we have

$$\begin{aligned} \langle (T_2 \circ S' \circ \kappa_E)(x), \mu \rangle &= \langle \kappa_{E'}(\mu), (T_2 \circ S' \circ \kappa_E)(x) \rangle = \langle (\kappa'_E \circ S'' \circ T'_2 \circ \kappa_{E'})(\mu), x \rangle \\ &= \langle \kappa_E(x), \eta(T_2 \circ S')(\mu) \rangle = \langle (\mathcal{Q}(T_2 \circ S') \circ \kappa_E)(x), \mu \rangle. \end{aligned}$$

We hence see that $T_2 \circ S' \circ \kappa_E = \mathcal{Q}(T_2 \circ S') \circ \kappa_E$. Thus we have

$$\begin{aligned} \eta(T_1 \circ \mathcal{Q}(T_2 \circ S')) &= \kappa'_E \circ \mathcal{Q}(T_2 \circ S')' \circ T'_1 \circ \kappa_{E'} = (T_2 \circ S' \circ \kappa_E)' \circ T'_1 \circ \kappa_{E'} \\ &= \kappa'_E \circ S'' \circ T'_2 \circ T'_1 \circ \kappa_{E'} = \eta(T_1 \circ T_2 \circ S'), \end{aligned}$$

as required. □

The maps ψ_1 and ψ_2 allow us to study the topological centres of $\mathcal{N}_\alpha(E)''$. Recall that the topological centres of $\mathcal{A}''$, for a Banach algebra $\mathcal{A}$, are defined to be

$$\begin{aligned} \mathfrak{Z}_t^{(1)}(\mathcal{A}'') &= \{\Phi \in \mathcal{A}'' : \Phi \Box \Psi = \Phi \Diamond \Psi \ (\Psi \in \mathcal{A}'')\}, \\ \mathfrak{Z}_t^{(2)}(\mathcal{A}'') &= \{\Phi \in \mathcal{A}'' : \Psi \Box \Phi = \Psi \Diamond \Phi \ (\Psi \in \mathcal{A}'')\}. \end{aligned}$$

Lemma 2.7.7. *Let E be a Banach space and $T \in \mathcal{B}(E'')$. Then the following are equivalent:*

1. $\mathcal{Q}(T) = T$;
2. $T \in \mathcal{B}(E')^a$;
3. $\mathcal{Q}(T) \circ R = T \circ R$ *for each* $R \in \mathcal{F}(E')^a$.

The following are also equivalent:

1. $T(E'') \subseteq \kappa_E(E)$;
2. $\mathcal{Q}(R) \circ T = R \circ T$ *for each* $R \in \mathcal{B}(E'')$;

3. *$\mathcal{Q}(R) \circ T = R \circ T$ for each $R \in \mathcal{F}(E'')$.*

Proof. For the first equivalence, (1)⇔(2) is clear. Then, setting $R = \kappa_{E'}(\mu) \otimes \Lambda \in \mathcal{F}(E')^a$, we have $\mathcal{Q}(T) \circ R = \kappa_{E'}(\mu) \otimes \mathcal{Q}(T)(\Lambda)$ and $T \circ R = \kappa_{E'}(\mu) \otimes T(\Lambda)$, so that we clearly have (1)⇔(3).

For the second equivalence, we clearly have (2)⇒(3). If (1) holds, then we can find $T_0 \in \mathcal{B}(E'', E)$ with $\kappa_E \circ T_0 = T$. For $M \in E'''$ and $\Lambda \in E''$, we have

$$\begin{aligned}\langle \kappa_E''(T(\Lambda)), M\rangle &= \langle \kappa_E(T_0(\Lambda)), \kappa_E'(M)\rangle = \langle \kappa_E'(M), T_0(\Lambda)\rangle = \langle M, \kappa_E(T_0(\Lambda))\rangle \\ &= \langle M, T(\Lambda)\rangle = \langle \kappa_{E''}(T(\Lambda)), M\rangle,\end{aligned}$$

so that $\kappa_E'' \circ T = \kappa_{E''} \circ T$. Thus, for $R \in \mathcal{B}(E'')$, we have

$$\mathcal{Q}(R) \circ T = \kappa_{E'}' \circ R'' \circ \kappa_E'' \circ T = \kappa_{E'}' \circ R'' \circ \kappa_{E''} \circ T = \kappa_{E'}' \circ \kappa_{E''} \circ R \circ T = R \circ T.$$

Hence (1)⇒(2). Finally, if (3) holds but (1) does not, then for some $\Lambda \in E''$, we have $T(\Lambda) \not\in \kappa_E(E)$. Thus we can find $M \in \kappa_E(E)^\circ \subseteq E'''$ with, say, $\langle M, T(\Lambda)\rangle = 1$. Let $R = M \otimes \Lambda \in \mathcal{F}(E'')$, so that $\eta(R) = \Lambda \otimes \kappa_E'(M) = 0$, as $M \in \kappa_E(E)^\circ$. Thus $\mathcal{Q}(R) \circ T = 0$, but $R(T(\Lambda)) = \Lambda\langle M, T(\Lambda)\rangle = \Lambda \neq 0$. This contradiction shows that (3)⇒(1). ☐

For a Banach space E and a tensor norm α, define the following subsets of $\mathcal{B}_\alpha(E'')$:

$$\begin{aligned}Z_1^0(E,\alpha) &= \{T' : T \in \mathcal{B}_{\alpha^t}(E'), T \circ \kappa_E' \circ S'' = \kappa_E' \circ T'' \circ S'' \ (S \in \mathcal{N}_\alpha(E)')\},\\ Z_2^0(E,\alpha) &= \{T \in \mathcal{B}_\alpha(E'') : T(E'') \subseteq \kappa_E(E), T \circ S' \in \mathcal{W}(E)^{aa} \ (S \in \mathcal{N}_\alpha(E)')\}.\end{aligned}$$

Proposition 2.7.8. *Let E be a Banach space, let α be a tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then*

$$\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{A}'')) \subseteq Z_1^0(E,\alpha) \quad , \quad \theta_1(\mathfrak{Z}_t^{(2)}(\mathcal{A}'')) \subseteq Z_2^0(E,\alpha).$$

Furthermore,

$$\begin{aligned}\psi_2(T) &\in \mathfrak{Z}_t^{(1)}(\mathcal{A}'') \qquad (T \in \mathcal{F}(E'') \cap Z_1^0(E,\alpha)),\\ \psi_1(T) &\in \mathfrak{Z}_t^{(2)}(\mathcal{A}'') \qquad (T \in \mathcal{F}(E'') \cap Z_2^0(E,\alpha)).\end{aligned}$$

Proof. By the homomorphism properties of θ_1, we see that

$$\begin{aligned}\theta_1(\Phi) \circ \theta_1(\Psi) &= \mathcal{Q}(\theta_1(\Phi)) \circ \theta_1(\Psi) \qquad (\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}''), \Psi \in \mathcal{A}''),\\ \theta_1(\Psi) \circ \theta_1(\Phi) &= \mathcal{Q}(\theta_1(\Psi)) \circ \theta_1(\Phi) \qquad (\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}''), \Psi \in \mathcal{A}'').\end{aligned}$$

Then, as $\theta_1 \circ \psi_1$ is the identity on $\mathcal{F}(E'')$, setting $\Psi = \psi_1(R)$ for $R \in \mathcal{F}(E'')$, we have

$$\theta_1(\Phi) \circ R = \mathcal{Q}(\theta_1(\Phi)) \circ R \qquad (\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}''), R \in \mathcal{F}(E'')),$$
$$R \circ \theta_1(\Phi) = \mathcal{Q}(R) \circ \theta_1(\Phi) \qquad (\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}''), R \in \mathcal{F}(E'')).$$

So Lemma 2.7.7 immediately gives us

$$\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{A}'')) \subseteq \mathcal{B}(E')^a \quad , \quad \theta_1(\mathfrak{Z}_t^{(2)}(\mathcal{A}'')) \subseteq \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E)\}.$$

Recall that $\theta_1(\mathcal{A}'') \subseteq \mathcal{B}_\alpha(E'')$, so that, for example, $\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{A}'')) \subseteq \mathcal{B}_{\alpha^t}(E')^a$.

Furthermore, for $R = M \otimes \Lambda \in E''' \otimes E''$, $S \in \mathcal{A}'$ and $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$, let $T = \eta(\theta_1(\Phi))$ so that $\theta_1(\Phi) = T'$, so that

$$\begin{aligned}
\langle \Phi \Box \psi_1(R), S\rangle &= \langle \Phi, \eta(R \circ S')\rangle = \langle \Phi, \Lambda \otimes \kappa_E'(S''(M))\rangle = \langle \theta_1(\Phi)(\Lambda), \kappa_E'(S''(M))\rangle \\
&= \langle (T'' \circ \kappa_{E'} \circ \kappa_E' \circ S'')(M), \Lambda\rangle = \langle (\kappa_{E'} \circ T \circ \kappa_E' \circ S'')(M), \Lambda\rangle \\
&= \langle \Lambda, (T \circ \kappa_E' \circ S'')(M)\rangle \\
\langle \Phi \Diamond \psi_1(R), S\rangle &= \langle \psi_1(R), \eta(\theta_1(\Phi)) \circ S\rangle = \langle \psi_1(R), T \circ S\rangle = \mathrm{Tr}(\eta(R \circ S' \circ T')) \\
&= \mathrm{Tr}\big(\eta\big(T''(S''(M)) \otimes \Lambda\big)\big) = \langle \Lambda, (\kappa_E' \circ T'' \circ S'')(M)\rangle.
\end{aligned}$$

Thus we have $T \circ \kappa_E' \circ S'' = \kappa_E' \circ T'' \circ S''$, so that $\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{A}'')) \subseteq Z_1^0(E, \alpha)$.

For $S \in \mathcal{A}'$ and $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, letting $T \in \mathcal{B}(E'', E)$ be such that $\kappa_E \circ T = \theta_1(\Phi)$, we have

$$\eta(\theta_1(\Phi) \circ S') = \kappa_E' \circ S'' \circ (\kappa_E \circ T)' \circ \kappa_{E'} = \kappa_E' \circ S'' \circ T'.$$

For $R = \Lambda \otimes \mu \in E'' \otimes E'$, we hence have

$$\begin{aligned}
\langle \psi_1(R') \Box \Phi, S\rangle &= \langle \psi_1(R'), \eta(\theta_1(\Phi) \circ S')\rangle = \langle \psi_1(R'), \kappa_E' \circ S'' \circ T'\rangle \\
&= \mathrm{Tr}(\kappa_E' \circ S'' \circ T' \circ R) = \langle \Lambda, (\kappa_E' \circ S'' \circ T')(\mu)\rangle \\
\langle \psi_1(R') \Diamond \Phi, S\rangle &= \langle \Phi, R \circ S\rangle = \langle \Phi, S'(\Lambda) \otimes \mu\rangle = \langle (\kappa_E \circ T \circ S')(\Lambda), \mu\rangle \\
&= \langle \mu, (T \circ S')(\Lambda)\rangle = \langle (S'' \circ T')(\mu), \Lambda\rangle.
\end{aligned}$$

Thus we have $\kappa_{E'} \circ \kappa_E' \circ S'' \circ T' = S'' \circ T'$. By Lemma 2.7.9 below, this is if and only if $\kappa_E \circ T \circ S' \in \mathcal{B}(E')^a$. By Lemma 2.7.10 below, we have

$$\mathcal{B}(E')^a \cap \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E)\} = \mathcal{W}(E)^{aa},$$

which implies that $\theta_1(\mathfrak{Z}_t^{(2)}(\mathcal{A}'')) \subseteq Z_2^0(E, \alpha)$.

Suppose that $R = \Lambda \otimes \mu \in \mathcal{F}(E')$ is such that $R' \in Z_1^0(E, \alpha)$, so that for $S \in \mathcal{A}'$, we have $R \circ \kappa_E' \circ S'' = \kappa_E' \circ R'' \circ S''$; that is

$$(S''' \circ \kappa_E'')(\Lambda) \otimes \mu = (S''' \circ \kappa_{E''})(\Lambda) \otimes \mu.$$

Thus, for $\Phi \in \mathcal{A}''$, we have

$$\begin{aligned}\langle \psi_2(R')\Box\Phi, S\rangle &= \langle \psi_2(R'), \eta(\theta_1(\Phi)\circ S')\rangle = \mathrm{Tr}(R\circ\eta(\theta_1(\Phi)\circ S'))\\ &= \langle \Lambda, \eta(\theta_1(\Phi)\circ S')(\mu)\rangle = \langle \Lambda, (\kappa_E'\circ S''\circ\theta_1(\Phi)'\circ\kappa_{E'})(\mu)\rangle\\ &= \langle (S'''\circ\kappa_E'')(\Lambda), (\theta_1(\Phi)'\circ\kappa_{E'})(\mu)\rangle\\ &= \langle (S'''\circ\kappa_{E''})(\Lambda), (\theta_1(\Phi)'\circ\kappa_{E'})(\mu)\rangle\\ &= \langle (S''\circ\theta_1(\Phi)'\circ\kappa_{E'})(\mu), \Lambda\rangle = \langle (\theta_1(\Phi)\circ S')(\Lambda), \mu\rangle\\ &= \langle \Phi, S'(\Lambda)\otimes\mu\rangle = \langle \Phi, R\circ S\rangle = \langle \psi_2(R')\Diamond\Phi, S\rangle.\end{aligned}$$

Thus $\psi_2(Z_1^0(E,\alpha)\cap\mathcal{F}(E''))\subseteq \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$.

Similarly, for $R = M\otimes\kappa_E(x)\in\mathcal{F}(E'')\cap Z_2^0(E,\alpha)$ and $S\in\mathcal{A}'$, we have $R\circ S'\in\mathcal{W}(E)^{aa}$, which is if and only if $S''(M)\otimes\kappa_E(x)\in\mathcal{W}(E)^{aa}$. This is if and only if $S''(M)=\kappa_{E'}(\mu)$ for some $\mu\in E'$. Then, for $\Phi\in\mathcal{A}''$, we have

$$\begin{aligned}\langle \Phi\Diamond\psi_1(R), S\rangle &= \langle \psi_1(R), \eta(\theta_1(\Phi))\circ S\rangle = \mathrm{Tr}\big(\eta\big(R\circ S'\circ\mathcal{Q}(\theta_1(\Phi))\big)\big)\\ &= \mathrm{Tr}\big(\kappa_E'\circ\eta(\theta_1(\Phi))''\circ(\kappa_{E'}(\mu)\otimes\kappa_E(x))'\circ\kappa_{E'}\big)\\ &= \mathrm{Tr}\big(\kappa_E'\circ\eta(\theta_1(\Phi))''\circ(\kappa_E(x)\otimes\kappa_{E'}(\mu))\big)\\ &= \langle \kappa_E(x), (\kappa_E'\circ\eta(\theta_1(\Phi))''\circ\kappa_{E'})(\mu)\rangle\\ &= \langle \eta(\theta_1(\Phi))(\mu), x\rangle = \langle (\kappa_E'\circ\theta_1(\Phi)'\circ\kappa_{E'})(\mu), x\rangle\\ &= \langle \theta_1(\Phi)(\kappa_E(x)), \mu\rangle = \langle \Phi, \kappa_E(x)\otimes\mu\rangle = \langle \Phi, \eta(S''(M)\otimes\kappa_E(x))\rangle\\ &= \langle \Phi, \eta(R\circ S')\rangle = \langle \Phi\Box\psi_1(R), S\rangle.\end{aligned}$$

Thus $\psi_1(\mathcal{F}(E'')\cap Z_2^0(E,\alpha))\subseteq\mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. $\Box$

Lemma 2.7.9. *Let $S\in\mathcal{B}(E')$ and $T\in\mathcal{B}(E'',E)$. Then $\kappa_{E'}\circ\kappa_E'\circ S''\circ T' = S''\circ T'$ if and only if $\kappa_E\circ T\circ S'\in\mathcal{B}(E')^a$.*

Proof. We have that $\kappa_E\circ T\circ S'\in\mathcal{B}(E')^a$ if and only if $\mathcal{Q}(\kappa_E\circ T\circ S') = \kappa_E\circ T\circ S'$. Now, for $\Lambda\in E''$ and $\mu\in E'$, we have

$$\begin{aligned}\langle \mathcal{Q}(\kappa_E\circ T\circ S')(\Lambda), \mu\rangle &= \langle \Lambda, \eta(\kappa_E\circ T\circ S')(\mu)\rangle = \langle \Lambda, (\kappa_E'\circ S''\circ T'\circ\kappa_E'\circ\kappa_{E'})(\mu)\rangle\\ &= \langle \Lambda, (\kappa_E'\circ S''\circ T')(\mu)\rangle = \langle (\kappa_{E'}\circ\kappa_E'\circ S''\circ T')(\mu), \Lambda\rangle,\end{aligned}$$

and also

$$\langle (\kappa_E\circ T\circ S')(\Lambda), \mu\rangle = \langle \mu, (T\circ S')(\Lambda)\rangle = \langle (S''\circ T')(\mu), \Lambda\rangle.$$

Thus $\kappa_E\circ T\circ S'\in\mathcal{B}(E')^a$ if and only if $S''\circ T' = \kappa_{E'}\circ\kappa_E'\circ S''\circ T'$, as required. $\Box$

Note that the above proof (and the lemma below) shows that

$$Z_2^0(E,\alpha) = \{T \in \mathcal{B}_\alpha(E'') : T(E'') \subseteq \kappa_E(E), T \circ S' \in \mathcal{B}(E')^a \ (S \in \mathcal{N}_\alpha(E)')\}.$$

Lemma 2.7.10. *For a Banach space E and a tensor norm α, we have*

$$Z_1^0(E,\alpha) \cap Z_2^0(E,\alpha) = (W(E) \cap \mathcal{B}_\alpha(E))^{aa},$$
$$\mathcal{B}(E')^a \cap \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E)\} = \mathcal{W}(E)^{aa}.$$

Proof. Firstly, for $T \in \mathcal{B}(E')$, suppose that $T'(E'') \subseteq \kappa_E(E)$. Then we can find $T_0 \in \mathcal{B}(E'', E)$ with $\kappa_E \circ T_0 = T'$. Then, for $x \in E$ and $\mu \in E'$, we have

$$\langle \mu, T_0(\kappa_E(x)) \rangle = \langle T'(\kappa_E(x)), \mu \rangle = \langle T(\mu), x \rangle,$$

so that $(T_0 \circ \kappa_E)' = T$. Furthermore, $(T_0 \circ \kappa_E)''(E'') = T'(E'') \subseteq \kappa_E(E)$, so that by Theorem 2.2.8, $(T_0 \circ \kappa_E) \in \mathcal{W}(E)$. Thus we have the second equality.

Now suppose that $T' \in Z_1^0(E,\alpha) \cap Z_2^0(E,\alpha)$, so that we immediately have $T = R'$ for some $R \in \mathcal{W}(E)$. Then $R'' \in \mathcal{B}_\alpha(E'')$, so that $R \in \mathcal{B}_\alpha(E)$, by Proposition 2.2.6.

Conversely, let $R \in \mathcal{W}(E) \cap \mathcal{B}_\alpha(E)$. Then, for $S \in \mathcal{B}(E')$, we have

$$R' \circ \kappa_E' \circ S'' = (\kappa_E \circ R)' \circ S'' = (R'' \circ \kappa_E)' \circ S'' = \kappa_E' \circ R''' \circ S'',$$

so that $R' \in Z_1^0(E,\alpha)$. We clearly have that $R'' \in Z_2^0(E,\alpha)$, completing the proof. $\square$

In some special cases, we can say more than the above proposition.

Theorem 2.7.11. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Suppose that $\mathcal{A}' \subseteq \mathcal{W}(E')$. Then*

$$Z_1^0(E,\alpha) = \mathcal{B}_{\alpha^t}(E')^a \quad , \quad Z_2^0(E,\alpha) = \kappa_E \circ \mathcal{B}_\alpha(E'', E),$$

where $\kappa_E \circ \mathcal{B}(E'', E) = \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E)\}$.

Consequently, we have

$$\psi_2(T') \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'') \qquad (T \in \mathcal{F}(E')),$$
$$\psi_1(\kappa_E \circ T) \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'') \qquad (T \in \mathcal{F}(E'', E)).$$

When E is not reflexive, the two topological centres of $\mathcal{A}''$ are distinct, neither contains the other, and both strictly contain $\kappa_\mathcal{A}(\mathcal{A})$.

Proof. For $S \in \mathcal{A}'$, as $S \in \mathcal{W}(E')$, $S''(E''') \subseteq \kappa_{E'}(E')$. Hence we have $\kappa_{E'} \circ \kappa_E' \circ S'' = S''$, as κ_E' is a projection of E''' onto E'. We immediately have $Z_1^0(E,\alpha) = \mathcal{B}_{\alpha^t}(E')^a$.

Similarly, for $S \in \mathcal{A}'$ and $T \in \kappa_E \circ \mathcal{B}(E'', E)$, we have that $T = \kappa_E \circ T_0$ for some $T_0 \in \mathcal{B}(E'', E)$. As $\kappa_{E'} \circ \kappa'_E \circ S'' = S''$, for $\mu \in E'$ and $\Lambda \in E''$, we have

$$\begin{aligned}\langle (T_0 \circ S' \circ \kappa_E)''(\Lambda), \mu \rangle &= \langle \Lambda, (\kappa'_E \circ S'' \circ T'_0)(\mu) \rangle = \langle (S'' \circ T'_0)(\mu), \Lambda \rangle \\ &= \langle \mu, (T_0 \circ S')(\Lambda) \rangle = \langle (\kappa_E \circ T_0 \circ S')(\Lambda), \mu \rangle.\end{aligned}$$

Thus $\kappa_E \circ T_0 \circ S' = (T_0 \circ S' \circ \kappa_E)'' \in \mathcal{W}(E)^{aa}$, so that $T \in Z_2^0(E, \alpha)$. We conclude that $Z_2^0(E, \alpha) = \kappa_E \circ \mathcal{B}_\alpha(E'', E)$.

Suppose that E is not reflexive, so that $\mathcal{A}$ is not Arens regular. Let $\Lambda \in E''$ and $\mu \in E'$ be non-zero, and let $T_1 = \Lambda \otimes \mu \in \mathcal{F}(E')$, so that $\psi_2(T'_1) \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$. Suppose that $\psi_2(T'_1) = \kappa_{\mathcal{A}}(T)$ for some $T \in \mathcal{A}$, so that $T'_1 = \theta_1(\psi_2(T'_1)) = T''$, which is a contradiction. Also, $\theta_1(\psi_2(T'_1)) = T'_1 \notin Z_2^0(E, \alpha)$, so that $\psi_2(T'_1) \notin \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. Thus the first topological centre strictly contains $\kappa_{\mathcal{A}}(\mathcal{A})$ and is not contained in the second topological centre.

Similarly, let $M \in \kappa_E(E)^\circ \subseteq E'''$ and $x \in E$ be non-zero, and let $T_2 = M \otimes x \in \mathcal{F}(E'', E)$, so that $\psi_1(\kappa_E \circ T_2) \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. Again, we see that $\psi_1(\kappa_E \circ T_2) \notin \kappa_{\mathcal{A}}(\mathcal{A})$, and that $\psi_1(\kappa_E \circ T_2) \notin \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$, so that the second topological centre strictly contains $\kappa_{\mathcal{A}}(\mathcal{A})$ and is not contained in the first topological centre. $\square$

The above certainly applies when $\alpha = \varepsilon$, as then $\mathcal{A}' = \mathcal{A}(E)' = \mathcal{I}(E') \subseteq \mathcal{W}(E')$ (by Corollary 2.3.7). However, it does not apply when $\alpha = \pi$ in the interesting case of when E is not reflexive, for when E has the approximation property, $\mathcal{A} = \mathcal{N}(E) = E'\widehat{\otimes}E$, and so $\mathcal{A}' = \mathcal{B}(E') \neq \mathcal{W}(E')$. We shall see later (Corollary 2.7.25) that this is a real problem, and not just an artifact of the method of proof.

The key to extending the above theorem is to look at the map θ_1.

Proposition 2.7.12. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Let*

$$I_1 = \ker\theta_1 \subseteq \mathcal{A}'' \quad , \quad I_2 = \ker(\mathcal{Q} \circ \theta_1) \subseteq \mathcal{A}''.$$

Then I_1 is a closed ideal for either Arens product, and I_2 is a closed ideal in $(\mathcal{A}'', \Diamond)$. Furthermore, we have

$$\mathcal{A}'' \Box I_1 = I_1 \Diamond \mathcal{A}'' = I_2 \Diamond \mathcal{A}'' = \{0\}.$$

In particular, $I_1 \Box I_1 = I_1 \Diamond I_1 = I_2 \Diamond I_2 = \{0\}$. For $i = 1, 2$, we have

$$I_1 \Box \psi_i(E'''\widehat{\otimes}_\alpha E'') = \psi_i(E'''\widehat{\otimes}_\alpha E'') \Diamond I_1 = I_2 \Box \psi_i(\kappa_{E'}(E')\widehat{\otimes}_\alpha \kappa_E(E)) = \{0\}.$$

Proof. By the homomorphism properties of θ_1, we see that I_1 is a closed ideal in $\mathcal{A}''$, with respect to either Arens product. By (the proof of) Proposition 2.6.5, for $T, S \in \mathcal{B}(E'')$, we have $\mathcal{Q}(T) \circ \mathcal{Q}(S) = \mathcal{Q}(\mathcal{Q}(T) \circ S)$, and so, for $\Phi \in I_2$ and $\Psi \in \mathcal{A}''$, we have

$$\mathcal{Q}(\theta_1(\Phi\Diamond\Psi)) = \mathcal{Q}(\mathcal{Q}(\theta_1(\Phi)) \circ \theta_1(\Psi)) = 0,$$
$$\mathcal{Q}(\theta_1(\Psi\Diamond\Phi)) = \mathcal{Q}(\mathcal{Q}(\theta_1(\Psi)) \circ \theta_1(\Phi)) = \mathcal{Q}(\theta_1(\Psi)) \circ \mathcal{Q}(\theta_1(\Phi)) = 0,$$

so that I_2 is a closed ideal in $(\mathcal{A}'', \Diamond)$.

For $S \in \mathcal{A}'$, $\Psi \in \mathcal{A}''$ and $\Phi \in I_1$, we have

$$\langle \Psi\Box\Phi, S\rangle = \langle \Psi, \eta(\theta_1(\Phi) \circ S')\rangle = 0 \quad , \quad \langle \Phi\Diamond\Psi, S\rangle = \langle \Psi, \eta(\theta_1(\Phi)) \circ S\rangle = 0.$$

Thus we see that $\Psi\Box\Phi = \Phi\Diamond\Psi = 0$. Similarly, for $\Phi \in I_2$ and $\Psi \in \mathcal{A}''$, we have

$$\langle \Phi\Diamond\Psi, S\rangle = \langle \Psi, \eta(\theta_1(\Phi)) \circ S\rangle = 0 \qquad (S \in \mathcal{A}'),$$

so that $\Phi\Diamond\Psi = 0$ for each $\Psi \in \mathcal{A}''$.

For $u \in E''' \otimes E''$ and $S \in \mathcal{A}'$, we have that $\psi_i(u) \cdot S$ and $S \cdot \psi_i(u)$ are in $E'' \otimes E'$, for $i = 1, 2$. Thus, for $\Phi \in I_1$ and $i = 1, 2$, we have

$$\langle \Phi\Box\psi_i(u), S\rangle = \langle \Phi, \psi_i(u) \cdot S\rangle = 0 = \langle \Phi, S \cdot \psi_i(u)\rangle = \langle \psi_i(u)\Diamond\Phi, S\rangle.$$

Similarly, for $u = \kappa_{E'}(\mu) \otimes \kappa_E(x) \in E''' \otimes E''$, $S \in \mathcal{A}'$, $\Phi \in I_2$ and $i = 1, 2$, we have

$$\begin{aligned}\langle \Phi\Box\psi_i(u), S\rangle &= \langle \Phi, S \circ (\kappa_E(x) \otimes \mu)\rangle = \langle (\theta_1(\Phi)' \circ \kappa_{E'} \circ S)(\mu), \kappa_E(x)\rangle \\ &= \langle (\eta(\theta_1(\Phi)) \circ S)(\mu), x\rangle = 0.\end{aligned}$$

$\square$

Theorem 2.7.13. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Suppose that (E', α) and (E'', α) are both a Grothendieck pair (in particular, this holds if α is accessible). Then the two topological centres of $\mathcal{A}''$ strictly contain $\kappa_{\mathcal{A}}(\mathcal{A})$. Suppose that the two sets*

$$\overline{\mathrm{lin}}\{S \cdot \Phi : S \in \mathcal{A}', \Phi \in \mathcal{A}''\} \quad , \quad \overline{\mathrm{lin}}\{\Phi \cdot S : S \in \mathcal{A}', \Phi \in \mathcal{A}''\}$$

are distinct, and neither contains the other. Then the topological centres are distinct and neither contains the other.

Proof. By the Theorem 2.7.11, we may suppose that $\mathcal{A}' \not\subseteq \mathcal{W}(E')$. Furthermore, by continuity, we may suppose that $\mathcal{W}(E') \cap \mathcal{A}'$ is not dense in $\mathcal{A}'$. By Proposition 2.7.3, we have

$$\{S \cdot \Phi : S \in \mathcal{A}', \Phi \in \mathcal{A}''\} + \{\Phi \cdot S : S \in \mathcal{A}', \Phi \in \mathcal{A}''\} \subseteq \mathcal{I}(E') \subseteq \mathcal{W}(E').$$

Note also that $\phi_1(E'' \otimes E') = \mathcal{F}(E') \subseteq \mathcal{W}(E')$. Consequently, by the Hahn-Banach theorem, we can find a non-zero $\Phi \in \mathcal{A}''$ so that

$$\langle \Phi, \phi_1(u) \rangle = 0 \qquad (u \in E'' \otimes E'),$$
$$\langle \Phi, S \cdot \Psi \rangle = \langle \Phi, \Psi \cdot S \rangle = 0 \qquad (\Psi \in \mathcal{A}'', S \in \mathcal{A}').$$

Then $\theta_1(\Phi) = 0$ so that $\Phi \in I_1$ (and hence $\Phi \notin \kappa_{\mathcal{A}}(\mathcal{A})$), and thus, for $\Psi \in \mathcal{A}''$ and $S \in \mathcal{A}'$, we have

$$\langle \Phi \Box \Psi, S \rangle = \langle \Phi, \Psi \cdot S \rangle = 0 = \langle \Phi \Diamond \Psi, S \rangle,$$

as $\Phi \Diamond \mathcal{A}'' = \{0\}$. Hence $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$. Similarly, we have

$$\langle \Psi \Diamond \Phi, S \rangle = \langle \Phi, S \cdot \Psi \rangle = 0 = \langle \Psi \Box \Phi, S \rangle,$$

as $\mathcal{A}'' \Box \Phi = \{0\}$, so that $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$.

Define

$$X_1 = \overline{\mathrm{lin}}\{S \cdot \Phi : S \in \mathcal{A}', \Phi \in \mathcal{A}''\} \quad , \quad X_2 = \overline{\mathrm{lin}}\{\Phi \cdot S : S \in \mathcal{A}', \Phi \in \mathcal{A}''\}.$$

When $X_1 \not\subseteq X_2$, we can find a non-zero $\Phi \in \mathcal{A}''$ with $\theta_1(\Phi) = 0$, $\langle \Phi, \lambda \rangle = 0$ for each $\lambda \in X_2$, and $\langle \Phi, S_0 \cdot \Phi_0 \rangle \neq 0$ for some $S_0 \in \mathcal{A}'$ and $\Phi_0 \in \mathcal{A}''$. As above, we see that $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, but we have $\langle \Phi_0 \Diamond \Phi, S_0 \rangle = \langle \Phi, S_0 \cdot \Phi_0 \rangle \neq 0$, while $\langle \Phi_0 \Box \Phi, S_0 \rangle = 0$ as $\mathcal{A}'' \Box \Phi = \{0\}$. Thus $\Phi \notin \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$. Similarly, when $X_2 \not\subseteq X_1$, we have $\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \not\subseteq \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. $\square$

We will show later, in Corollary 2.7.25, that we cannot hope to completely remove the second condition in the above theorem.

To conclude, in slightly less than full generality, we have the following.

Theorem 2.7.14. *Let E be a Banach space which is not reflexive, let α be an accessible tensor norm and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. The topological centres of $\mathcal{A}''$ both strictly contain $\kappa_{\mathcal{A}}(\mathcal{A})$, and are both strictly contained in $\mathcal{A}''$. When $\mathcal{A}' \subseteq \mathcal{W}(E')$, the topological centres are distinct and neither contains the other.* $\square$

2.7.1 When the dual space has the bounded approximation property

To say more about the topological centres of $\mathcal{N}_\alpha(E)''$, we need to impose some conditions on the Banach space E. Following Grosser, we shall now study the case where E' has the bounded approximation property. It turns out that this is an important special case which makes up, in some sense, for the fact that E is not assumed to be reflexive. For example, in [Grosser, 1987], the concept of *Arens semi-regularity* is studied.

Definition 2.7.15. Let $\mathcal{A}$ be a Banach algebra. A *multiplier* on $\mathcal{A}$ is a pair (L, R) of maps in $\mathcal{B}(\mathcal{A})$ such that

$$L(ab) = L(a)b \quad , \quad R(ab) = aR(b) \quad , \quad aL(b) = R(a)b \qquad (a, b \in \mathcal{A}).$$

The collection of multipliers on $\mathcal{A}$ is denoted by $\mathcal{M}(\mathcal{A})$.

Definition 2.7.16. Let $\mathcal{A}$ be a Banach algebra with a bounded approximate identity, let $\Xi \in \mathcal{A}''$ be a mixed identity and let

$$D(\Xi) = \{L''(\Xi) : (L, R) \in \mathcal{M}(\mathcal{A})\}.$$

Then $\mathcal{A}$ is *Arens semi-regular* if and only if the Arens products coincide on $D(\Xi)$, for each mixed identity Ξ (see [Grosser, 1984]).

In [Grosser, 1987], Grosser shows that $\mathcal{A}(E)$ (when E' has the bounded approximation property) is Arens semi-regular when $\mathcal{I}(E') = \mathcal{N}(E')$. He also demonstrates (see [Grosser, 1987, Section 4]) that when E' has the bounded approximation property, we have $\mathcal{A}(E)'' = \mathcal{B}(E'') \oplus \ker \theta_1$.

This last property can be generalised to the α-nuclear case, and we shall see that, when E' has the bounded approximation property, we can completely identify the topological centres of $\mathcal{N}_\alpha(E)''$, at least when α is accessible.

Throughout this section, E will be a Banach space such that E' has the bounded approximation property. For a tensor norm α, as E has the bounded approximation property as well, $\mathcal{A} = \mathcal{N}_\alpha(E) = E'\widehat{\otimes}_\alpha E$ and so $\mathcal{A}' = \mathcal{B}_{\alpha'}(E')$. Suppose that α is accessible (so that α' is also accessible). As in Proposition 2.6.7, we see that $\phi_1 : \mathcal{N}_{\alpha'}(E') = E''\widehat{\otimes}_{\alpha'} E' \to \mathcal{A}'$ is an isomorphism onto its range, and so $\theta_1 : \mathcal{A}'' \to \mathcal{B}_\alpha(E'')$ is surjective. When E' has the metric approximation property, or α' is totally accessible, ϕ_1 is actually an isometry onto its range, and so θ_1 is a quotient operator.

Theorem 2.7.17. *Let E be a Banach space such that E' has the bounded approximation property. Let α be an accessible tensor norm, and $\mathcal{A} = \mathcal{N}_\alpha(E)$. There exists a homomorphism, which is also an isomorphism onto its range, $\psi_1 : (\mathcal{B}_\alpha(E''), \circ) \to (\mathcal{A}'', \Box)$ such that $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{B}_\alpha(E'')}$. There also exists a bounded homomorphism $\psi_2 : (\mathcal{B}_\alpha(E''), \star) \to (\mathcal{A}'', \Diamond)$ such that $\theta_1 \circ \psi_2 = \mathcal{Q}$. For $i = 1, 2$ and $T \in \mathcal{A}$, we have $\psi_i(T'') = \kappa_{\mathcal{A}}(T)$. When E' has the metric approximation property, ψ_1 can be chosen to be an isometry and ψ_2 can be chosen to be norm-decreasing. Furthermore, these maps extend the maps defined in Theorem 2.7.4, when they are restricted to $\mathcal{F}(E'')$.*

Proof. As in Example 2.6.8, as E' has the bounded approximation property, we can find $\Xi \in \mathcal{I}(E')' = \mathcal{A}(E)''$ so that $\theta_1(\Xi) = \mathrm{Id}_{E''}$. As α is accessible, by the Grothendieck Composition theorem, for $T \in \mathcal{B}_\alpha(E'')$ and $S \in \mathcal{B}_{\alpha'}(E') = \mathcal{A}'$, we have that $S' \in \mathcal{B}_{\breve{\alpha}}(E'')$, so that $T \circ S' \in \mathcal{I}(E'')$, and hence $\eta(T \circ S') \in \mathcal{I}(E')$. Similarly, $\eta(T) \in \mathcal{B}_\alpha(E')$, and so $\eta(T) \circ S = \eta(S' \circ T) \in \mathcal{I}(E')$. Define, for $i = 1, 2$, $\psi_i : \mathcal{B}_\alpha(E'') \to \mathcal{A}''$ by

$$\langle \psi_1(T), S\rangle = \langle \Xi, \eta(T \circ S')\rangle \quad , \quad \langle \psi_2(T), S\rangle = \langle \Xi, \eta(T) \circ S\rangle$$

for $T \in \mathcal{B}_\alpha(E'')$ and $S \in \mathcal{B}_{\alpha'}(E')$. Then we have

$$|\langle \psi_1(T), S\rangle| \leq \|\Xi\| \|\eta(T \circ S')\|_\pi \leq \|\Xi\| \|T \circ S'\|_\pi \leq \|\Xi\| \|T\|_\alpha \|S\|_{\alpha'},$$

so that $\|\psi_1\| \leq \|\Xi\|$. Similarly, $\|\psi_2\| \leq \|\Xi\|$. As we form Ξ from a bounded approximate identity for $\mathcal{A}(E)$, by results in [Grønbæk, Willis, 1993], we see that the smallest we can make $\|\Xi\|$ is the bound for which E' has the bounded approximation property. In particular, if E' has the metric approximation property, then ψ_1 and ψ_2 can be constructed to be norm-decreasing.

For $T \in \mathcal{F}(E'')$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}\langle \psi_1(T), S\rangle &= \langle \Xi, \eta(T \circ S')\rangle = \mathrm{Tr}(\eta(T \circ S')),\\ \langle \psi_2(T), S\rangle &= \langle \Xi, \eta(T) \circ S\rangle = \mathrm{Tr}(\eta(T) \circ S),\end{aligned}$$

so that the maps ψ_i extend those defined in Theorem 2.7.4.

For $\Lambda \in E''$, $\mu \in E'$ and $T \in \mathcal{B}_\alpha(E'')$, we have $\eta(T \circ \phi_1(\Lambda \otimes \mu)') = \eta(\kappa_{E'}(\mu) \circ T(\Lambda)) = \phi_1(T(\Lambda) \otimes \mu)$. Thus we have

$$\langle \theta_1(\psi_1(T))(\Lambda), \mu\rangle = \langle \Xi, \eta(T \circ \phi_1(\Lambda \otimes \mu)')\rangle = \langle \Xi, \phi_1(T(\Lambda) \otimes \mu)\rangle = \langle T(\Lambda), \mu\rangle,$$

as $\theta_1(\Xi) = \mathrm{Id}_{E''}$. Thus $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{B}_\alpha(E'')}$. As θ_1 is norm-decreasing, we see that ψ_1 is an isomorphism onto its range, and an isometry when E' has the metric approximation property (for a suitably chosen Ξ). Similarly, we have

$$\langle \theta_1(\psi_2(T))(\Lambda), \mu\rangle = \langle \Xi, \eta(T) \circ \phi_1(\Lambda \otimes \mu)\rangle = \langle \Lambda, \eta(T)(\mu)\rangle,$$

so that $\theta_1 \circ \psi_2 = \mathcal{Q}$.

For $T \in \mathcal{A} = E'\widehat{\otimes}_\alpha E$, we have $T \in \mathcal{B}_\alpha(E)$, so that $T'' \in \mathcal{B}_\alpha(E'')$. Suppose that

$T = \mu \otimes x$. Then, for $S \in \mathcal{B}_{\alpha'}(E')$, we have

$$\begin{aligned}\langle \psi_1(T''), S\rangle &= \langle \Xi, \eta(T'' \circ S')\rangle = \langle \Xi, S \circ T'\rangle = \langle \Xi, \kappa_E(x) \otimes S(\mu)\rangle \\ &= \langle \kappa_E(x), S(\mu)\rangle = \langle S, T\rangle = \langle \kappa_{\mathcal{A}}(T), S\rangle, \\ \langle \psi_2(T''), S\rangle &= \langle \Xi, T' \circ S\rangle = \langle \Xi, S'(\kappa_E(x)) \otimes \mu\rangle \\ &= \langle S'(\kappa_E(x)), \mu\rangle = \langle S(\mu), x\rangle = \langle S, T\rangle = \langle \kappa_{\mathcal{A}}(T), S\rangle.\end{aligned}$$

By linearity and continuity, we see that $\psi_i(T'') = \kappa_{\mathcal{A}}(T)$ for $T \in \mathcal{A}$ and $i = 1, 2$.

By Lemma 2.7.6, for $T_1, T_2 \in \mathcal{B}(E'')$ and $S \in \mathcal{B}(E')$, we have $\eta(T_1 \circ T_2 \circ S') = \eta(T_1 \circ \mathcal{Q}(T_2 \circ S'))$. Then, for $T_1, T_2 \in \mathcal{B}_\alpha(E'')$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}\langle \psi_1(T_1)\Box\psi_1(T_2), S\rangle &= \langle \psi_1(T_1), \eta(\theta_1(\psi_1(T_2)) \circ S')\rangle = \langle \psi_1(T_1), \eta(T_2 \circ S')\rangle \\ &= \langle \Xi, \eta(T_1 \circ \mathcal{Q}(T_2 \circ S'))\rangle = \langle \Xi, \eta(T_1 \circ T_2 \circ S')\rangle \\ &= \langle \psi_1(T_1 \circ T_2), S\rangle.\end{aligned}$$

We see that $\psi_1 : (\mathcal{B}_\alpha(E''), \circ) \to (\mathcal{A}'', \Box)$ is a homomorphism. Similarly, we have

$$\begin{aligned}\langle \psi_2(T_1 \star T_2), S\rangle &= \langle \psi_2(\mathcal{Q}(T_1) \circ T_2), S\rangle = \langle \Xi, \eta(\mathcal{Q}(T_1) \circ T_2) \circ S\rangle \\ &= \langle \Xi, \eta(S' \circ \mathcal{Q}(T_1) \circ T_2)\rangle = \langle \Xi, \kappa_E' \circ T_2' \circ \eta(T_1)'' \circ S'' \circ \kappa_{E'}\rangle \\ &= \langle \Xi, \kappa_E' \circ T_2' \circ \kappa_{E'} \circ \eta(T_1) \circ S\rangle = \langle \Xi, \eta(T_2) \circ \eta(T_1) \circ S\rangle \\ &= \langle \Xi, \eta(T_2) \circ \eta(S' \circ T_1)\rangle = \langle \psi_2(T_2), \eta(S' \circ T_1)\rangle \\ &= \langle \psi_2(T_2), \eta(T_1) \circ S\rangle = \langle \psi_2(T_1)\Diamond\psi_2(T_2), S\rangle.\end{aligned}$$

We see that $\psi_2 : (\mathcal{B}_\alpha(E''), \star) \to (\mathcal{A}'', \Diamond)$ is a homomorphism. $\Box$

For $T \in \mathcal{B}_{\alpha^t}(E')$, $\theta_1(\psi_2(T')) = \mathcal{Q}(T') = T'$, so that ψ_2 restricted to $\mathcal{B}_\alpha(E'') \cap \mathcal{B}(E')^a$ is an isomorphism onto its range (and an isometry when E' has the metric approximation property).

There is evidently some choice in the construction of ψ_1 and ψ_2, as we are free to choose a mixed identity $\Xi \in \mathcal{A}(E)''$. However, below we shall see that this is unimportant as far as the study of topological centres go.

As $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{B}_\alpha(E'')}$, we have that $\psi_1 \circ \theta_1$ is a projection of $\mathcal{A}''$ onto $\psi_1(\mathcal{B}_\alpha(E''))$. Thus we can write

$$\mathcal{A}'' = \mathcal{B}_\alpha(E'') \oplus \ker \theta_1 = \mathcal{B}_\alpha(E'') \oplus I_1,$$

with reference to Proposition 2.7.12.

For a Banach space E (such that E' has the bounded approximation property) and an accessible tensor norm α, define

$$\begin{aligned}
Z_1(E,\alpha) &= \{T' : T \in \mathcal{B}_{\alpha^t}(E'), T\circ S \in \mathcal{N}_{\alpha'}(E'), \\
&\qquad \kappa_{E'} \circ T \circ \kappa_E' \circ S'' = T'' \circ S''\ (S \in \mathcal{B}_{\alpha'}(E'))\}, \\
Z_2(E,\alpha) &= \{T \in \mathcal{B}_\alpha(E'') : T(E'') \subseteq \kappa_E(E), T\circ S' \in \mathcal{N}_{\alpha'}(E')^a\ (S \in \mathcal{B}_{\alpha'}(E'))\}, \\
X_1(E,\alpha) &= \overline{\mathrm{lin}}\{\eta(T\circ S') : S \in \mathcal{B}_{\alpha'}(E'), T \in \mathcal{B}_\alpha(E'')\} \subseteq \mathcal{B}_{\alpha'}(E'), \\
X_2(E,\alpha) &= \overline{\mathrm{lin}}\{T\circ S : S \in \mathcal{B}_{\alpha'}(E'), T \in \mathcal{B}_{\alpha^t}(E')\} \subseteq \mathcal{B}_{\alpha'}(E').
\end{aligned}$$

By the Grothendieck Composition theorem, we see that X_1 and X_2 are subsets of $\overline{\mathcal{I}(E')}$, where the closure is taken with respect to $\mathcal{B}_{\alpha'}(E')$.

Theorem 2.7.18. *Let E be a Banach space such that E' has the bounded approximation property, let α be an accessible tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then*

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \{\psi_2(T) + \Phi : T \in Z_1(E,\alpha), \Phi \in X_1(E,\alpha)^\circ\}.$$

Proof. Let $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$, so that we can write $\Phi = \Phi_0 + \psi_1(T')$ for some $\Phi_0 \in I_1$ and $T \in Z_1^0(E,\alpha)$, by Proposition 2.7.8, and the discussion above. Similarly, for $\Psi \in \mathcal{A}''$, let $\Psi = \Psi_0 + \psi_1(R)$, for some $\Psi_0 \in I_1$ and $R \in \mathcal{B}_\alpha(E'')$. Then we have, with reference to Proposition 2.7.12,

$$\begin{aligned}
\Phi\Box\Psi &= \Phi\Box\psi_1(R) = \Phi_0\Box\psi_1(R) + \psi_1(T'\circ R), \\
\Phi\Diamond\Psi &= \psi_1(T')\Diamond\Psi = \psi_1(T')\Diamond\Psi_0 + \psi_1(T')\Diamond\psi_1(R).
\end{aligned}$$

Setting $\Psi_0 = 0$, we have

$$\Phi_0\Box\psi_1(R) + \psi_1(T'\circ R) = \psi_1(T')\Diamond\psi_1(R) \qquad (R \in \mathcal{B}_\alpha(E'')), \tag{2.1}$$

and so we also have

$$\psi_1(T')\Diamond\Psi_0 = 0 \qquad (\Psi_0 \in I_1).$$

For $S \in \mathcal{B}_{\alpha'}(E') = \mathcal{A}'$, we thus have $\langle \Psi_0, S\cdot\psi_1(T')\rangle = \langle \Psi_0, T\circ S\rangle = 0$ for each $\Psi_0 \in I_1$. By the Hahn-Banach theorem, this holds if and only if

$$T\circ S \in \phi_1(E''\widehat{\otimes}_{\alpha'}E') = \mathcal{N}_{\alpha'}(E') \qquad (S \in \mathcal{B}_{\alpha'}(E')),$$

as $\phi_1(E''\widehat{\otimes}_{\alpha'}E') = \mathcal{N}_{\alpha'}(E')$ is a closed subspace of $\mathcal{A}'$, by Proposition 2.3.22, given that E' has the bounded approximation property. As $\mathcal{N}_{\alpha'}(E') \subseteq \mathcal{W}(E')$, we thus have that $\kappa_{E'} \circ \kappa_E' \circ T'' \circ S'' = T'' \circ S''$, and so we see that

$$Z_1^0(E,\alpha) \cap \{T' : T \in \mathcal{B}_{\alpha^t}(E'), T\circ S \in \mathcal{N}_{\alpha'}(E')\ (S \in \mathcal{B}_{\alpha'}(E')\} = Z_1(E,\alpha).$$

Then, for $S \in \mathcal{B}_{\alpha'}(E') = \mathcal{A}'$ and $R \in \mathcal{B}_\alpha(E'')$, we have

$$\begin{aligned}\langle \Phi_0 \Box \psi_1(R) + \psi_1(T' \circ R), S\rangle &= \langle \Phi_0, \eta(R \circ S')\rangle + \langle \Xi, \eta(T' \circ R \circ S')\rangle \\ &= \langle \Phi_0, \eta(R \circ S')\rangle + \langle \psi_1(T'), \eta(R \circ S')\rangle,\end{aligned}$$

as $\eta(T' \circ R \circ S') = \eta(R \circ S') \circ T = \eta(\eta(R \circ S')') \circ T = \eta(T' \circ \eta(R \circ S')')$. We also know that $T \circ \kappa'_E \circ S'' = \kappa'_E \circ T'' \circ S''$, so that

$$\begin{aligned}\langle \psi_1(T') \Diamond \psi_1(R), S\rangle &= \langle \psi_1(R), T \circ S\rangle = \langle \Xi, \eta(R \circ S' \circ T')\rangle \\ &= \langle \Xi, \kappa'_E \circ T'' \circ S'' \circ R' \circ \kappa_{E'}\rangle = \langle \Xi, T \circ \kappa'_E \circ S'' \circ R' \circ \kappa_{E'}\rangle \\ &= \langle \Xi, T \circ \eta(R \circ S')\rangle = \langle \psi_2(T'), \eta(R \circ S')\rangle.\end{aligned}$$

By equation 2.1, we see that

$$\langle \Phi_0, \eta(R \circ S')\rangle = \langle \psi_2(T') - \psi_1(T'), \eta(R \circ S')\rangle \qquad (R \in \mathcal{B}_\alpha(E''), S \in \mathcal{B}_{\alpha'}(E')).$$

Thus, for $S \in X_1(E, \alpha)$, we have

$$\langle \Phi, S\rangle = \langle \psi_1(T') + \Phi_0, S\rangle = \langle \psi_2(T'), S\rangle,$$

and so $\Phi - \psi_2(T') \in X(E, \alpha)^\circ$. Hence $\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \subseteq \psi_2(Z_1(E, \alpha)) + X_1(E, \alpha)^\circ$.

Conversely, for $T' \in Z_1(E, \alpha)$ and $\Phi \in X_1(E, \alpha)^\circ$, for $\Psi_0 \in I_1$, $R \in \mathcal{B}_\alpha(E'')$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}\langle (\Phi + \psi_2(T')) \Box (\Psi_0 + \psi_1(R)), S\rangle &= \langle (\Phi + \psi_2(T')) \Box \psi_1(R), S\rangle \\ &= \langle \Phi + \psi_2(T'), \psi_1(R).S\rangle = \langle \Phi + \psi_2(T'), \eta(R \circ S')\rangle \\ &= \langle \psi_2(T'), \eta(R \circ S')\rangle = \langle \Xi, T \circ \eta(R \circ S')\rangle.\end{aligned}$$

As $\mathcal{N}_{\alpha'}(E') \subseteq X_1(E, \alpha)$, we have that $\Phi \in I_1$, and as $T' \in Z_1(E, \alpha)$, we have $T \circ S \in \mathcal{N}_{\alpha'}(E')$, so that

$$\begin{aligned}\langle (\Phi + \psi_2(T')) \Diamond (\Psi_0 + \psi_1(R)), S\rangle &= \langle \psi_2(T') \Diamond (\Psi_0 + \psi_1(R)), S\rangle \\ &= \langle \Psi_0 + \psi_1(R), T \circ S\rangle = \langle \psi_1(R), T \circ S\rangle \\ &= \langle \Xi, \eta(R \circ S' \circ T')\rangle = \langle \Xi, T \circ \eta(R \circ S')\rangle,\end{aligned}$$

again using the fact that $T \in Z_1(E, \alpha)$. Hence $\Phi + \psi_2(T') \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$, and we have

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \psi_2(Z_1(E, \alpha)) + X(E, \alpha)^\circ,$$

as required. $\square$

Theorem 2.7.19. *Let E be a Banach space such that E' has the bounded approximation property, let α be an accessible tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then*

$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \{\psi_1(T) + \Phi : T \in Z_2(E,\alpha), \Phi \in X_2(E,\alpha)^\circ\}.$$

Proof. Let $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$. With reference to Proposition 2.7.8, we can write $\Phi = \Phi_0 + \psi_1(T)$ for some $\Phi_0 \in I_1$ and $T \in Z_2^0(E,\alpha)$. Similarly, for $\Psi \in \mathcal{A}''$, let $\Psi = \Psi_0 + \psi_1(R)$, for some $\Psi_0 \in I_1$ and $R \in \mathcal{B}_\alpha(E'')$. Then we have, with reference to Proposition 2.7.12,

$$\Psi \Box \Phi = (\Psi_0 + \psi_1(R)) \Box \psi_1(T) = \Psi_0 \Box \psi_1(T) + \psi_1(R \circ T),$$
$$\Psi \Diamond \Phi = \psi_1(R) \Diamond (\Phi_0 + \psi_1(T)) = \psi_1(R) \Diamond \Phi_0 + \psi_1(R) \Diamond \psi_1(T).$$

Setting $\Psi_0 = 0$ gives us

$$\psi_1(R \circ T) = \psi_1(R) \Diamond \Phi_0 + \psi_1(R) \Diamond \psi_1(T) \qquad (R \in \mathcal{B}_\alpha(E'')), \tag{2.2}$$

and thus also that $\Psi_0 \Box \psi_1(T) = 0$ for each $\Psi_0 \in I_1$. Again, this holds if and only if, for each $S \in \mathcal{B}_{\alpha'}(E')$, we have $\psi_1(T) \cdot S = \eta(T \circ S') \in \mathcal{N}_{\alpha'}(E')$.

As $T \in Z_2^0(E,\alpha)$, for $S \in \mathcal{B}_{\alpha'}(E')$, we have $T \circ S' \in \mathcal{B}(E')^a$, so that

$$\langle \psi_1(R) \Diamond \psi_1(T), S \rangle = \langle \Xi, \eta(T \circ S' \circ \mathcal{Q}(R)) \rangle = \langle \Xi, \eta(R) \circ \eta(T \circ S') \rangle.$$

Then, as $T = \kappa_E \circ T_0$ for some $T_0 \in \mathcal{B}(E'', E)$, we have

$$\begin{aligned}\eta(R \circ T \circ S') &= \kappa_E' \circ S'' \circ T' \circ R' \circ \kappa_{E'} = \kappa_E' \circ S'' \circ T_0' \circ \kappa_E' \circ R' \circ \kappa_{E'} \\ &= \kappa_E' \circ S'' \circ T_0' \circ \kappa_E' \circ \kappa_{E'} \circ \eta(R) = \eta(T \circ S') \circ \eta(R).\end{aligned}$$

Thus we get

$$\langle \psi_1(R \circ T), S \rangle = \langle \Xi, \eta(R \circ T \circ S') \rangle = \langle \Xi, \eta(T \circ S') \circ \eta(R) \rangle.$$

Now, for $S \in \mathcal{B}_{\alpha'}(E')$, we have $\eta(T \circ S') = \phi_1(u)$ for some $u \in E'' \widehat{\otimes}_{\alpha'} E'$. By equation 2.2, we have

$$\begin{aligned}\langle \psi_1(R) \Diamond \Phi_0, S \rangle = \langle \Phi_0, \eta(R) \circ S \rangle &= \langle \psi_1(R \circ T) - \psi_1(R) \Diamond \psi_1(T), S \rangle \\ &= \langle \Xi, \eta(T \circ S') \circ \eta(R) - \eta(R) \circ \eta(T \circ S') \rangle \\ &= \langle \Xi, \phi_1(u) \circ \eta(R) - \eta(R) \circ \phi_1(u) \rangle = 0.\end{aligned}$$

Thus $\Phi_0 \in X_2(E,\alpha)^\circ$, and we see that $\mathfrak{Z}_t^{(2)}(\mathcal{A}'') \subseteq \psi_1(Z_2(E,\alpha)) + X_2(E,\alpha)^\circ$.

Conversely, for $T \in Z_2(E,\alpha)$ and $S \in \mathcal{B}_{\alpha'}(E')$, we have $\eta(T \circ S') = \phi_1(u)$ for some $u \in E'' \widehat{\otimes}_{\alpha'} E'$, and that $T \circ S' = \eta(T \circ S')'$. Thus, for $\Phi_0 \in X_2(E,\alpha)^\circ$, $\Psi_0 \in I_1$ and

$R \in \mathcal{B}_\alpha(E'')$, we have

$$\begin{aligned}\langle(\Psi_0 + \psi_1(R))\Box(\Phi_0 + \psi_1(T)), S\rangle &= \langle\Psi_0\Box\psi_1(T), S\rangle + \langle\psi_1(R \circ T), S\rangle \\ &= \langle\Psi_0, \eta(T \circ S')\rangle + \langle\Xi, \eta(R \circ T \circ S')\rangle \\ &= \langle\Psi_0, \phi_1(u)\rangle + \langle\Xi, \eta(T \circ S') \circ \eta(R)\rangle \\ &= \langle\Xi, \phi_1(u) \circ \eta(R)\rangle = \mathrm{Tr}(\phi_1(u) \circ \eta(R)),\end{aligned}$$

by using the same calculation as above, given that $T(E'') \subseteq \kappa_E(E)$. Similarly, as $\Phi_0 \in X_2(E, \alpha)^\circ$, we have

$$\begin{aligned}\langle(\Psi_0 + \psi_1(R))\Diamond(\Phi_0 + \psi_1(T)), S\rangle &= \langle\psi_1(R)\Diamond\Phi_0, S\rangle + \langle\psi_1(R)\Diamond\psi_1(T), S\rangle \\ &= \langle\Phi_0, \eta(R) \circ S\rangle + \langle\psi_1(T), \eta(R) \circ S\rangle \\ &= \langle\Xi, \eta(T \circ S' \circ \mathcal{Q}(R))\rangle = \langle\Xi, \eta(\phi_1(u)' \circ \mathcal{Q}(R))\rangle \\ &= \langle\Xi, \eta(R) \circ \phi_1(u)\rangle = \mathrm{Tr}(\phi_1(u) \circ \eta(R)).\end{aligned}$$

Consequently, $\Phi_0 + \psi_1(T) \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, and so we conclude that

$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \psi_1(Z_2(E, \alpha)) + X_2(E, \alpha)^\circ,$$

as required. $\square$

These results (that is, the definitions of $Z_i(E, \alpha)$ and $X_i(E, \alpha)$, for $i = 1, 2$) might seem overly complicated. However, the next couple of corollaries will show that, in the general case, we cannot remove any of the conditions.

Corollary 2.7.20. *Let E be a Banach space such that E' has the bounded approximation property, and let $\mathcal{A} = \mathcal{A}(E)$. Then we have*

$$\begin{aligned}\mathfrak{Z}_t^{(1)}(\mathcal{A}'') &= \{\psi_2(T') : T \in \mathcal{B}(E'), T \circ S \in \mathcal{N}(E')\ (S \in \mathcal{I}(E'))\}, \\ \mathfrak{Z}_t^{(2)}(\mathcal{A}'') &= \{\psi_1(T) : T \in \mathcal{B}(E''), T(E'') \subseteq \kappa_E(E), T \circ S' \in \mathcal{N}(E'')\ (S \in \mathcal{I}(E'))\}, \\ \mathfrak{Z}_t^{(1)}(\mathcal{A}'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}'') &= \psi_1(\mathcal{W}(E)^{aa}) = \psi_2(\mathcal{W}(E)^{aa}).\end{aligned}$$

Proof. We apply the above theorems with $\alpha = \varepsilon$. In particular, $\mathcal{A}'' = \mathcal{B}(E'') \oplus I_1$, $\mathcal{A}' = \mathcal{B}_\pi(E') = \mathcal{I}(E')$, and $\mathcal{B}_\alpha(E'') = \mathcal{B}(E'')$. It is then clear that $X_1(E, \varepsilon) = X_2(E, \varepsilon) = \mathcal{I}(E')$, and so $X_1(E, \varepsilon)^\circ = X_2(E, \varepsilon)^\circ = \{0\}$.

Then, for $S \in \mathcal{I}(E')$, as $\mathcal{I}(E') \subseteq \mathcal{W}(E')$, we have that $\kappa_{E'} \circ \kappa_E' \circ S'' = S''$, and so, for $T \in \mathcal{B}(E'')$, we have $\kappa_{E'} \circ T \circ \kappa_E' \circ S'' = T'' \circ \kappa_{E'} \circ \kappa_E' \circ S'' = T'' \circ S''$. Thus

$$Z_1(E, \varepsilon) = \{T' : T \in \mathcal{B}(E'), T \circ S \in \mathcal{N}(E')\ (S \in \mathcal{I}(E'))\},$$

which gives the result for $\mathfrak{Z}_t^{(1)}(\mathcal{A}'')$.

Similarly, for $T \in \mathcal{B}(E'')$ with $T = \kappa_E \circ T_0$ for some $T_0 \in \mathcal{B}(E'', E)$, and $S \in \mathcal{I}(E')$, $\mu \in E'$ and $\Lambda \in E''$, we have

$$\begin{aligned}\langle \Lambda, (\kappa_E' \circ S'' \circ T' \circ \kappa_{E'})(\mu)\rangle &= \langle (\kappa_{E'} \circ \kappa_E' \circ S'' \circ T_0' \circ \kappa_E' \circ \kappa_{E'})(\mu), \Lambda\rangle \\ &= \langle (S'' \circ T_0')(\mu), \Lambda\rangle = \langle \mu, (T_0 \circ S')(\Lambda)\rangle = \langle (T \circ S')(\Lambda), \mu\rangle.\end{aligned}$$

Thus $\eta(T \circ S')' = T \circ S'$, and so we have

$$Z_2(E, \varepsilon) = \{T \in \mathcal{B}(E'') : T(E'') \subseteq \kappa_E(E), T \circ S' \in \mathcal{N}(E'') \ (S \in \mathcal{I}(E'))\},$$

as required.

We apply Lemma 2.7.10 and Theorem 2.3.24 to see that

$$\{T' : T \in \mathcal{B}(E'), T'(E'') \subseteq \kappa_E(E), T \circ S, S \circ T \in \mathcal{N}(E') \ (S \in \mathcal{I}(E'))\} = \mathcal{W}(E)^{aa}.$$

To complete the proof, we need to show that $\psi_1(T'') = \psi_2(T'')$ for $T \in \mathcal{W}(E)$. For $T \in \mathcal{W}(E)$ and $S \in \mathcal{I}(E')$, we have

$$\langle \psi_1(T'') - \psi_2(T''), S\rangle = \langle \Xi, S \circ T' - T' \circ S\rangle = \langle \mathrm{Id}_{E''}, S \circ T' - T' \circ S\rangle,$$

as Theorem 2.3.24 tells us that $S \circ T'$ and $T' \circ S$ are nuclear. The proof is complete with an application of the next lemma. $\square$

The following was known to Grosser (see [Grosser, 1989]) although he seems to have been unaware of Theorem 2.2.9, and so does not use the following simple factorisation argument.

Lemma 2.7.21. *Let E be Banach space so that E' has the approximation property. Let $T \in \mathcal{W}(E')$ and $S \in \mathcal{I}(E')$. Then $T \circ S, S \circ T \in \mathcal{N}(E') = E''\widehat{\otimes}E'$ and $\langle \mathrm{Id}_{E''}, T \circ S\rangle = \langle \mathrm{Id}_{E''}, S \circ T\rangle$.*

Proof. We follow the proof of Theorem 2.3.24, and again use Theorem 2.2.9. As T is weakly-compact, we can find a reflexive Banach space F, $T_1 \in \mathcal{B}(E', F)$ and $T_2 \in \mathcal{B}(F, E')$ so that $T = T_2 \circ T_1$. Then, as F is reflexive, $\mathcal{N}(E', F) = \mathcal{I}(E', F)$, so that $T_1 \circ S \in \mathcal{N}(E', F)$. Similarly, $T_2' \circ S' \in \mathcal{N}(E'', F')$, so that $S'' \circ T_2'' \in \mathcal{N}(F'', E''')$. As F is reflexive, we identify F'' with F, and we have that $T_2'' = \kappa_{E'} \circ T_2$. Thus $S'' \circ T_2'' = S'' \circ \kappa_{E'} \circ T_2 = \kappa_{E'} \circ S \circ T_2 \in \mathcal{N}(F, E''')$. As $S \in \mathcal{W}(E')$, we have that $S \circ T_2 = \kappa_E' \circ \kappa_{E'} \circ S \circ T_2 \in \mathcal{N}(F, E')$.

Immediately, we have $T \circ S = T_2 \circ T_1 \circ S \in \mathcal{N}(E')$ and $S \circ T = S \circ T_2 \circ T_1 \in \mathcal{N}(E')$. We also have, as $S \circ T_2 \in E''\widehat{\otimes}E'$ and $T_1 \circ S \in E''\widehat{\otimes}E'$,

$$\langle \mathrm{Id}_{E''}, S \circ T\rangle = \mathrm{Tr}(S \circ T_2 \circ T_1) = \mathrm{Tr}(T_1 \circ S \circ T_2) = \mathrm{Tr}(T_2 \circ T_1 \circ S) = \langle \mathrm{Id}_{E''}, T \circ S\rangle,$$

as required. □

Note that when $\mathcal{N}(E') \neq \mathcal{I}(E')$, we have $Z_1(E,\varepsilon) \neq \mathcal{B}(E')^a$. When $E = F'$ for some Banach space F, we have that $\kappa_E \circ \kappa'_F \in Z_2(E,\varepsilon)$ if and only if $\kappa_E \circ \kappa'_F \circ S' \in \mathcal{N}(E'')$ for each $S \in \mathcal{I}(E')$. This is if and only if

$$\eta(\kappa_E \circ \kappa'_F \circ S') = \kappa'_E \circ S'' \circ \kappa''_F \circ \kappa'_E \circ \kappa_{E'} = \kappa'_{F'} \circ S'' \circ \kappa''_F \in \mathcal{N}(E')$$

for each $S \in \mathcal{I}(E')$. In particular,

$$\kappa'_{F'} \circ S''' \circ \kappa''_F = (S'' \circ \kappa_{F'})' \circ \kappa''_F = (\kappa_{F'} \circ S)' \circ \kappa''_F = S' \in \mathcal{N}(F'') \qquad (S \in \mathcal{I}(F')).$$

Thus we have that $\kappa_E \circ \kappa'_F \in Z_2(E,\varepsilon)$ implies that $S = \eta(S') \in \mathcal{N}(F')$ for each $S \in \mathcal{I}(F')$, that is, $\mathcal{I}(F') = \mathcal{N}(F')$. We see that we cannot, in general, remove any of the conditions which define $Z_1(E,\varepsilon)$ and $Z_2(E,\varepsilon)$.

Example 2.7.22. The above considerations all apply to l^1 whose dual, l^∞, has the metric approximation property, but not the Radon-Nikodým property. By Example 2.4.5 we have that $\mathcal{N}(l^\infty) \neq \mathcal{I}(l^\infty)$, and so $Z_1(l^1,\varepsilon)$ and $Z_2(l^1,\varepsilon)$ are non-trivial, in the sense just described. However, as we shall show in Theorem 5.1.8, $\mathcal{A}(l^1)$ is the unique, closed, two-sided ideal in $\mathcal{B}(l^1)$, so that $\mathcal{A}(l^1) = \mathcal{W}(l^1)$, and hence $\mathcal{A} = \mathcal{A}(l^1)$ satisfies

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \kappa_{\mathcal{A}}(\mathcal{A}).$$

This is an example of a Banach space not dealt with in [Dales, Lau, 2004]. □

Corollary 2.7.23. *Let E be a Banach space such that E' has the bounded approximation property, and let $\mathcal{A} = \mathcal{N}(E) = E'\widehat{\otimes}E$. Then we have*

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \{\psi_2(T'') + \Phi : T \in \mathcal{I}(E) \cap \mathcal{A}(E), \Phi \in \mathcal{I}(E')^\circ\},$$
$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \{\psi_1(T'') + \Phi : T \in \mathcal{I}(E) \cap \mathcal{A}(E), \Phi \in \mathcal{I}(E')^\circ\}.$$

Proof. We have $\alpha = \pi$ so that $\mathcal{A}'' = \mathcal{B}_\pi(E'') \oplus I_1 = \mathcal{I}(E'') \oplus I_1$. We also have $\mathcal{A}' = \mathcal{B}(E')$

and $\mathcal{N}_{\alpha'}(E') = \mathcal{N}_\varepsilon(E') = \mathcal{A}(E')$. Thus we have

$$\begin{aligned}
X_1(E,\pi) &= \overline{\operatorname{lin}}\{\eta(T\circ S') : S\in\mathcal{B}(E'), T\in\mathcal{I}(E'')\} = \overline{\mathcal{I}(E')},\\
X_2(E,\pi) &= \overline{\operatorname{lin}}\{T\circ S : S\in\mathcal{B}(E'), T\in\mathcal{I}(E')\} = \overline{\mathcal{I}(E')},\\
Z_1(E,\pi) &= \{T' : T\in\mathcal{I}(E'), T\circ S\in\mathcal{A}(E'),\\
&\qquad\qquad\qquad \kappa_{E'}\circ T\circ\kappa_E'\circ S'' = T''\circ S''\ (S\in\mathcal{B}(E'))\},\\
Z_2(E,\pi) &= \{T\in\mathcal{I}(E'') : T(E'')\subseteq\kappa_E(E), T\circ S'\in\mathcal{A}(E')^a\ (S\in\mathcal{B}(E'))\}.
\end{aligned}$$

Note that $\overline{\mathcal{I}(E')}$ is closure with respect to the topology on $\mathcal{B}(E')$.

Letting $S = \mathrm{Id}_{E'}$ in the expression for $Z_1(E,\pi)$ above yields $Z_1(E,\pi)\subseteq\mathcal{I}(E')\cap\mathcal{A}(E')$ and that $T\in Z_1(E,\pi)$ implies that $\kappa_{E'}\circ T\circ\kappa_E' = T''$. For $M\in\kappa_E(E)^\circ$, we have $\kappa_E'(M) = 0$, so that $T''(M) = 0$. Thus a Hahn-Banach argument tells us that $T'(E'')\subseteq\kappa_E(E)$. As in the proof of Lemma 2.7.10, we have $\mathcal{B}(E')^a\cap(\kappa_E\circ\mathcal{B}(E'',E)) = \mathcal{B}(E)^{aa}$, so that $T\in\mathcal{I}(E)^a$. Thus we have $Z_1(E,\pi) = \{T'' : T\in\mathcal{I}(E), T'\in\mathcal{A}(E')\}$, noting that for $T\in\mathcal{I}(E)$, we have $\kappa_{E'}\circ T'\circ\kappa_{E'} = T'''$. Now, by [Ryan, 2002, Proposition 5.55], we know that $T'\in\mathcal{A}(E')$ if and only if $T\in\mathcal{A}(E)$. Thus

$$Z_1(E,\pi) = \{T'' : T\in\mathcal{I}(E)\cap\mathcal{A}(E)\},$$

as required.

For $T\in Z_2(E,\pi)$, we similarly see that $T\in\mathcal{A}(E')^a$ and that $T\in\mathcal{W}(E)^{aa}$, as before. Thus we can again conclude that

$$Z_2(E,\pi) = \{T'' : T\in\mathcal{I}(E)\cap\mathcal{A}(E)\},$$

as required. □

The space $\mathcal{I}(E)\cap\mathcal{A}(E)$ is easily seen to a closed subspace of $\mathcal{I}(E)$; indeed, let (T_n) be a sequence in $\mathcal{I}(E)\cap\mathcal{A}(E)$ with $\|T_n - T\|_\pi\to 0$ for some $T\in\mathcal{I}(E)$. Then $\|T_n - T\|\leq\|T_n - T\|_\pi\to 0$, so that $T\in\mathcal{A}(E)$.

Example 2.7.24. As in Section 2.4, set $E = C([0,1])$, so that we can find $T\in\mathcal{I}(E)\cap\mathcal{A}(E)$ with $T\notin\mathcal{N}(E)$, and so that E' has the bounded approximation property. Hence the conditions in the above theorem are not vacuous, as we do not have $\mathcal{N}(E) = \mathcal{I}(E)\cap\mathcal{A}(E)$. □

Note that, when E is not reflexive, $\mathcal{I}(E')\subseteq\mathcal{W}(E')\subsetneq\mathcal{B}(E')$, so that $X_1(E,\pi) = X_2(E,\pi)$ is a non-trivial subspace of $\mathcal{N}(E)'$.

Corollary 2.7.25. *Let E be a Banach space such that E' has the bounded approximation property and $\mathcal{N}(E') = \mathcal{I}(E')$. Then we can identify $\mathcal{A}(E)''$ with $\mathcal{B}(E'')$, and we have*

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}(E)'') = \mathcal{B}(E')^a \quad , \quad \mathfrak{Z}_t^{(2)}(\mathcal{A}(E)'') = \kappa_E \circ \mathcal{B}(E'', E).$$

Furthermore, we have $\mathcal{N}(E)'' = \mathcal{B}(E')'$ and

$$\mathfrak{Z}_t^{(1)}(\mathcal{N}(E)'') = \mathfrak{Z}_t^{(2)}(\mathcal{N}(E)'') = \kappa_{\mathcal{N}(E)}(\mathcal{N}(E)) + \ker\theta_1.$$

Proof. For $\mathcal{A}(E)''$, the result was first shown in [Dales, Lau, 2004], and follows immediately from Corollary 2.7.20, given that $\mathcal{N}(E') = \mathcal{I}(E') = E''\widehat{\otimes}E'$.

For $\mathcal{N}(E)''$, we have that $\mathcal{N}(E) = E'\widehat{\otimes}E$ and $\mathcal{N}(E)' = \mathcal{B}(E')$. Then $\overline{\mathcal{I}(E')} = \overline{\mathcal{N}(E')} = \mathcal{A}(E') = E''\widehat{\otimes}_\varepsilon E'$ in $\mathcal{B}(E')$. These agree with the image of ϕ_1, so that $\overline{\mathcal{I}(E')}^\circ = \ker\theta_1 = \mathcal{A}(E')^\circ$. For $T \in \mathcal{I}(E) \cap \mathcal{A}(E)$, we have $T' \in \mathcal{N}(E')$, so that by Proposition 2.3.5, $T \in \mathcal{N}(E) \subseteq \mathcal{A}(E)$. Hence $\mathcal{I}(E) \cap \mathcal{A}(E) = \mathcal{N}(E)$. Clearly ψ_1 and ψ_2 agree on $\mathcal{N}(E)$, so we are done. □

Example 2.7.26. Following [Dales, Lau, 2004], consider c_0, so that $c_0' = l^1$, as a separable dual space, has the Radon-Nikodým property, and thus we have $\mathcal{I}(l^1) = \mathcal{N}(l^1) = l^\infty\widehat{\otimes}l^1$, as l^∞ has the metric approximation property. Thus the above corollary holds, and we have $\mathcal{A}(c_0)'' = \mathcal{B}(l^\infty)$. By Corollary 2.7.20, we have that $\mathfrak{Z}_t^{(1)}(\mathcal{A}(c_0)'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}(c_0)'') = \mathcal{W}(c_0)^{aa}$. We shall see later in Theorem 5.1.8 that $\mathcal{B}(c_0)$ contains only one proper, closed two-sided ideal, namely $\mathcal{A}(c_0)$. In particular, $\mathcal{A}(c_0) = \mathcal{W}(c_0)$, so (as in the l^1 case) we again have, for $\mathcal{A} = \mathcal{A}(c_0)$, that $\mathfrak{Z}_t^{(1)}(\mathcal{A}'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \kappa_{\mathcal{A}}(\mathcal{A})$.

We can also apply the above corollary to $\mathcal{N}(c_0) = l^1\widehat{\otimes}c_0$ to see that $\mathfrak{Z}_t^{(1)}(\mathcal{N}(c_0)'') = \mathfrak{Z}_t^{(2)}(\mathcal{N}(c_0)'')$. We have that $\phi_1 : E''\check{\otimes}E' \to \mathcal{B}(l^1) = \mathcal{N}(c_0)'$ is an isometry onto its range, which is $\mathcal{A}(l^1)$, so that

$$\ker\theta_1 = \{\Phi \in \mathcal{N}(c_0)'' : \langle \Phi, S\rangle = 0 \ (S \in \mathcal{A}(E'))\}.$$

□

Example 2.7.27. Let P be Pisier's space, as constructed in [Pisier, 1983], so that $\mathcal{A}(P) = \mathcal{N}(P)$. Applying Theorem 2.7.14, we see that the topological centres of $\mathcal{A}(P)''$ are distinct and neither contains the other. Hence this also holds for $\mathcal{N}(P)''$, and we conclude that, in general, we cannot say that the topological centres of the bidual of the nuclear operators are equal. □

Example 2.7.28. Again, following [Dales, Lau, 2004], consider J, the James space, which was defined in [James, 1951]. Let c_{00} be the vector space of sequences of complex numbers which are eventually zero, and for $x = (x_n) \in c_{00}$, let

$$\|x\|_J = \sup\left\{ \Big(\sum_{i=1}^{n} |x_{r_i} - x_{r_{i+1}}|^2 + |x_{r_{n+1}} - x_{r_1}|^2 \Big)^{1/2} \right\},$$

where the supremum is taken over all integers n and increasing sequences of integers $(r_i)_{i=1}^{n+1}$. We can show that $\|\cdot\|_J$ is a norm; let J be the completion of $(c_{00}, \|\cdot\|_J)$. We can show that J is $\{x \in c_0 : \|x\|_J < \infty\}$. Then, as shown in [James, 1951], J is isometric with J'', but $J''/\kappa_J(J)$ is isomorphic to $\mathbb{C}$. The standard unit vector basis (e_n) is a basis for J.

So, for some $\Lambda_0 \in J''$, the map $J \oplus \mathbb{C} \to J''; (x, \alpha) \mapsto \kappa_J(x) + \alpha\Lambda_0$ is an isomorphism. Let $M_0 \in J'''$ be such that $M_0 \in \kappa_J(J)^\circ$ and $\langle \Lambda_0, M_0 \rangle = 1$. Then we have

$$P(\Lambda) := \Lambda - \langle M_0, \Lambda \rangle \Lambda_0 \in \kappa_J(J) \qquad (\Lambda \in J''),$$

and $P \circ \kappa_J = \kappa_J$. We can verify that

$$M - \langle M, \Lambda_0 \rangle M_0 = \kappa_{J'}(\kappa_J'(M)) \qquad (M \in J''').$$

In particular, $\kappa_J''(\Lambda_0) = 0$. Let $P_1 = \kappa_J^{-1} \circ P : J'' \to J$, so that $P_1' : J' \to J'''$, and we can verify that

$$P_1' = \kappa_{J'} - \Lambda_0 \otimes M_0.$$

For $S \in \mathcal{B}(J')$, let $T = P_1 \circ S' \circ \kappa_J \in \mathcal{B}(J)$ and $\mu = \kappa_J'(S''(M_0)) \in J'$. Then we have

$$\begin{aligned} T' + \Lambda_0 \otimes \mu &= \kappa_J' \circ \Big(S'' \circ P_1' + \Lambda_0 \otimes S''(M_0) \Big) \\ &= \kappa_J' \circ \Big(S'' \circ \kappa_{J'} - \Lambda_0 \otimes S''(M_0) + \Lambda_0 \otimes S''(M_0) \Big) = \kappa_J' \circ S'' \circ \kappa_{J'} = S. \end{aligned}$$

We can consequently see that the map

$$\mathcal{B}(J) \oplus J' \to \mathcal{B}(J'); (T, \mu) \mapsto T' + \Lambda_0 \otimes \mu$$

is an isomorphism.

Similarly, for $S \in \mathcal{B}(J'')$ such that $S(J'') \subseteq \kappa_J(J)$, let $T = P_1 \circ S \circ \kappa_J \in \mathcal{B}(J)$, let $x = (P_1 \circ S)(\Lambda_0)$ and let $\hat{S} = P \circ T'' + M_0 \otimes \kappa_J(x)$. Then we have

$$T' = \kappa_J' \circ S' \circ P_1' = \eta(S) - \Lambda_0 \otimes \kappa_J'(S'(M_0)) = \eta(S),$$

as $S'(M_0) = 0$. As $\kappa_J''(\Lambda_0) = 0$, we have $T''(\Lambda_0) = \mathcal{Q}(S)(\Lambda_0) = 0$, so that $\hat{S}(\Lambda_0) = \kappa_J(x) = S(\Lambda_0)$. For $y \in J$, we have

$$\begin{aligned} \hat{S}(\kappa_J(y)) &= \kappa_J(T(y)) - \langle M_0, \kappa_J(T(y)) \rangle \Lambda_0 + \langle M_0, \kappa_J(y)) \rangle \kappa_J(x) \\ &= \kappa_J(T(y)) = S(\kappa_J(y)), \end{aligned}$$

so that $S = \hat{S}$. Thus we can see that the map

$$\mathcal{B}(J) \oplus J \to \kappa_J \circ \mathcal{B}(J'', J); (T, x) \mapsto P \circ T'' + M_0 \otimes \kappa_J(x)$$

is an isomorphism.

By [Diestel, Uhl, 1977, Chapter VII], J' has the Radon-Nikodým property, so that $\mathcal{N}(J') = \mathcal{I}(J')$. As J has a basis, it has the bounded approximation property (and thus J'' has the bounded approximation property, so that J' also does). We can thus again apply the above corollaries, and so we have $\mathcal{A}(J)'' = \mathcal{B}(J'')$. Thus we have, given the above,

$$\begin{aligned} \mathfrak{Z}_t^{(1)}(\mathcal{A}(J)'') &= \mathcal{B}(J')^a = \mathcal{B}(J) \oplus J', \\ \mathfrak{Z}_t^{(2)}(\mathcal{A}(J)'') &= \kappa_J \circ \mathcal{B}(J'', J) = \mathcal{B}(J) \oplus J, \\ \mathfrak{Z}_t^{(1)}(\mathcal{A}(J)'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}(J)'') &= \mathcal{W}(J)^{aa}. \end{aligned}$$

It is reasonably simple to show that $\mathcal{W}(J)$ is a maximal closed ideal in $\mathcal{B}(J)$ (in fact, it is the unique maximal closed ideal in $\mathcal{B}(J)$, as shown by Laustsen in [Laustsen, 2002]) and that $\mathcal{W}(J)$ has co-dimension one in $\mathcal{B}(J)$. As summarised in [Laustsen, Loy, 2003, Section 3], $\mathcal{A}(J) = \mathcal{K}(J)$ is not equal to $\mathcal{W}(J)$, so that

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}(J)'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{A}(J)'') \neq \kappa_{\mathcal{A}(J)}(\mathcal{A}(J)).$$

We can apply the above to study $\mathcal{N}(J') = J''\widehat{\otimes}J'$. We have $\mathcal{N}(J')' = \mathcal{B}(J'')$ and so $\ker\theta_1 = \mathcal{A}(J'')^\circ$, and

$$\mathfrak{Z}_t^{(1)}(\mathcal{N}(J')'') = \mathfrak{Z}_t^{(2)}(\mathcal{N}(J')'') = \kappa_{\mathcal{N}(J')}(\mathcal{N}(J')) + \mathcal{A}(J'')^\circ.$$

Now, we have $\mathcal{A}(J)' = \mathcal{N}(J')$ and $\mathcal{A}(J)'' = \mathcal{B}(J'')$, so that $\kappa'_{\mathcal{A}(J)} : \mathcal{N}(J')'' \to \mathcal{N}(J')$ is an projection. Hence we can write

$$\begin{aligned} \mathcal{N}(J')'' = \mathcal{B}(J'')' &= \kappa_{\mathcal{N}(J')}(\mathcal{N}(J')) \oplus \ker\kappa'_{\mathcal{A}(J)} = \kappa_{\mathcal{N}(J')}(\mathcal{N}(J')) \oplus \kappa_{\mathcal{A}(J)}(\mathcal{A}(J))^\circ \\ &= \kappa_{\mathcal{N}(J')}(\mathcal{N}(J')) \oplus (\mathcal{A}(J)^{aa})^\circ. \end{aligned}$$

Notice that as $\kappa_{\mathcal{A}(J)}(\mathcal{A}(J)) = \mathcal{A}(J)^{aa} \subseteq \mathcal{A}(J'')$, we have $\mathcal{A}(J'')^\circ \subseteq (\mathcal{A}(J)^{aa})^\circ$, and so we have

$$\mathcal{N}(J')''/\mathfrak{Z}_t^{(1)}(\mathcal{N}(J')'') = (\mathcal{A}(J)^{aa})^\circ/\mathcal{A}(J'')^\circ.$$

□

2.7.2 When the integral and nuclear operators coincide

We now drop the requirement that E' have the bounded approximation property. Motivated by the fact that, for many Banach spaces E, we have $\mathcal{A}(E)' = \mathcal{I}(E') = \mathcal{N}(E')$, we might consider studying the case when $\mathcal{N}_\alpha(E)' = \mathcal{N}_{\alpha'}(E')$. However, this seems too strong a condition (for example, it seems unlikely that it is ever true for $\alpha = \pi$). This said, we can again use the Grothendieck Composition theorem to show that, when $\mathcal{A} = \mathcal{N}_\alpha(E)$ for an accessible α, we have $\mathcal{A}'' \cdot \mathcal{A}' + \mathcal{A}' \cdot \mathcal{A}'' \subseteq \mathcal{I}(E')$. Thus the case when $\mathcal{N}(E') = \mathcal{I}(E')$ should be interesting to study, and it is certainly not a vacuous condition to impose upon E, by the following lemma.

For a Banach space E, recall that $E^{[n]}$ is the nth iterated dual of E, so that $E^{[1]} = E'$ etc.

Lemma 2.7.29. *Let E be a Banach space such that $E^{[n]}$ has the Radon-Nikodým property for some $n \in \mathbb{N}$. Then $\mathcal{I}(E^{[m]}) = \mathcal{N}(E^{[m]})$ for each $1 \leq m \leq n$.*

Proof. By Theorem 2.3.11, if E' has the Radon-Nikodým property, then $\mathcal{N}(E') = \mathcal{I}(E')$.

Suppose that F is a Banach space such that $\mathcal{N}(F'') = \mathcal{I}(F'')$. For $T \in \mathcal{I}(F')$, we have $T' \in \mathcal{I}(F'') = \mathcal{N}(F'')$, and so $T = \eta(T') = \kappa_F' \circ T' \circ \kappa_{F'} \in \mathcal{N}(F')$. Thus, by induction, if $\mathcal{N}(E^{[n+1]}) = \mathcal{I}(E^{[n+1]})$ for some $n \in \mathbb{N}$, then $\mathcal{N}(E^{[m]}) = \mathcal{I}(E^{[m]})$ for each $1 \leq m \leq n+1$. We are thus done by another application of Theorem 2.3.11. ☐

Example 2.7.30. Let JT be the James Tree Space (defined in [James, 1974]), so that each even dual of JT has the Radon-Nikodým property, but each odd dual does not (see [Diestel, Uhl, 1977, Chapter VII, Section 5]). Thus, by the above lemma, $\mathcal{I}(JT') = \mathcal{N}(JT')$ while JT' does not have the Radon-Nikodým property. ☐

Let E be a Banach space and α be a tensor norm. With reference to Proposition 2.6.7, we treat ϕ_1 as a map $E''\widehat{\otimes}_{\alpha'}E' \to \mathcal{N}_{\alpha'}(E') \subseteq \mathcal{N}_\alpha(E)' \subseteq \mathcal{B}_{\alpha'}(E')$. Then $\theta_1 : \mathcal{N}_\alpha(E)'' \to \mathcal{B}_\alpha(E'')$ actually maps into

$$\mathcal{N}_{\alpha'}(E')' = (\ker J_{\alpha'})^\circ = \{T \in \mathcal{B}_\alpha(E'') : \langle T, u\rangle = 0 \ (u \in E''\widehat{\otimes}_{\alpha'}E', J_{\alpha'}(u) = 0)\}.$$

The following lemma tells us that, in this case, $(\ker J_{\alpha'})^\circ$ is a right ideal in $(\mathcal{B}_\alpha(E''), \circ)$ and a left ideal in $(\mathcal{B}_\alpha(E''), \star)$.

Lemma 2.7.31. *Let E be a Banach space and α be a tensor norm. Then $(\ker J_{\alpha'})^\circ$ is a right ideal in $(\mathcal{B}_\alpha(E''), \circ)$. Furthermore, for $T \in (\ker J_{\alpha'})^\circ$ and $S \in \mathcal{B}_{\alpha^t}(E')^a$, we have $S \circ T \in (\ker J_{\alpha'})^\circ$.*

Proof. Let $T \in (\ker J_{\alpha'})^\circ$ and $u \in \ker J_{\alpha'}$. Let (u_n) be a sequence in $\mathcal{F}(E')$ such that $\sum_{n=1}^\infty u_n = u$ in $E''\widehat{\otimes}_{\alpha'}E'$. Let $S \in \mathcal{B}_\alpha(E'')$, and let $v = (S \otimes \mathrm{Id}_{E'})(u)$. Then, for $\mu \in E'$

and $\Lambda \in E''$, we have

$$\begin{aligned}\langle \Lambda, J_{\alpha'}(v)(\mu)\rangle &= \sum_{n=1}^{\infty} \langle \Lambda, J_{\alpha'}((S\otimes \mathrm{Id}_{E'})(u_n))(\mu)\rangle = \sum_{n=1}^{\infty} \langle J_{\alpha'}((S\otimes \mathrm{Id}_{E'})(u_n))'(\Lambda), \mu\rangle \\ &= \sum_{n=1}^{\infty} \langle S'(\kappa_{E'}(\mu)), J_{\alpha'}(u_n)'(\Lambda)\rangle = \langle S'(\kappa_{E'}(\mu)), J_{\alpha'}(u)'(\Lambda)\rangle = 0,\end{aligned}$$

as $J_{\alpha'}(u) = 0$. Thus $v \in \ker J_{\alpha'}$. We then have

$$\begin{aligned}\langle T\circ S, u\rangle &= \sum_{n=1}^{\infty} \langle T\circ S, u_n\rangle = \sum_{n=1}^{\infty} \mathrm{Tr}(T\circ S\circ u_n') = \sum_{n=1}^{\infty} \mathrm{Tr}\Big(T\circ((S\otimes \mathrm{Id}_{E'})(u_n))'\Big) \\ &= \sum_{n=1}^{\infty} \langle T, (S\otimes \mathrm{Id}_{E'})(u_n)\rangle = \langle T, v\rangle = 0,\end{aligned}$$

as $T \in (\ker J_{\alpha'})^\circ$. Thus $T\circ S \in (\ker J_{\alpha'})^\circ$.

Similarly, for $T \in (\ker J_{\alpha'})^\circ$, $S \in \mathcal{B}_{\alpha^t}(E')$ and $u \in \ker J_{\alpha'}$, let $v = (\mathrm{Id}_{E''}\otimes S)(u)$. We can show that $v \in \ker J_{\alpha'}$, and similarly that

$$\langle S'\circ T, u\rangle = \langle T, (\mathrm{Id}_{E''}\otimes S)(u)\rangle = \langle T, v\rangle = 0,$$

so that $S'\circ T \in (\ker J_{\alpha'})^\circ$. $\square$

For a Banach space E and a tensor norm α, recall the following definitions:

$$\begin{aligned}Z_1^0(E,\alpha) &= \{T' : T \in \mathcal{B}_{\alpha^t}(E'), T\circ\kappa_E'\circ S'' = \kappa_E'\circ T''\circ S''\ (S \in \mathcal{N}_\alpha(E)')\}, \\ Z_2^0(E,\alpha) &= \{T \in \mathcal{B}_\alpha(E'') : T(E'') \subseteq \kappa_E(E), T\circ S' \in \mathcal{W}(E)^{aa}\ (S \in \mathcal{N}_\alpha(E)')\}.\end{aligned}$$

Theorem 2.7.32. *Let E be a Banach space such that $\mathcal{N}(E') = \mathcal{I}(E')$, let α be an accessible tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then, for $i = 1, 2$, we have*

$$\mathfrak{Z}_t^{(i)}(\mathcal{A}'') = \theta_1^{-1}(Z_i^0(E,\alpha)).$$

Proof. For $\Phi \in \mathcal{A}''$ and $S \in \mathcal{A}' \subseteq \mathcal{B}_{\alpha'}(E')$, by Proposition 2.7.3, we have

$$\Phi\cdot S = \eta(\phi_1(\Phi)\circ S') \in \mathcal{N}(E') \quad , \quad S\cdot\Phi = \eta(\phi_1(\Phi))\circ S \in \mathcal{N}(E'),$$

as $\mathcal{N}(E') = \mathcal{I}(E')$. Then, as $\alpha' \le \pi$ on $E''\otimes E'$, we clearly have that $\Phi\cdot S$ and $S\cdot\Phi$ are in $\mathcal{N}_{\alpha'}(E') \subseteq \mathcal{A}'$.

Then, for $\Phi, \Psi \in \mathcal{A}''$ and $S \in \mathcal{A}'$, we have

$$\begin{aligned}\langle \Phi\Box\Psi, S\rangle &= \langle \Phi, \eta(\theta_1(\Psi)\circ S')\rangle = \mathrm{Tr}\big(\theta_1(\Phi)\circ\mathcal{Q}(\theta_1(\Psi)\circ S')\big), \\ \langle \Phi\Diamond\Psi, S\rangle &= \langle \Psi, \eta(\theta_1(\Phi))\circ S\rangle = \mathrm{Tr}\big(\theta_1(\Psi)\circ S'\circ\mathcal{Q}(\theta_1(\Phi))\big),\end{aligned}$$

as, for example, $\eta(\theta_1(\Psi) \circ S \in \mathcal{N}_{\alpha'}(E') = \phi_1(E''\widehat{\otimes}_{\alpha'}E')$. We thus see that $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$ if and only if

$$\mathrm{Tr}\big(\theta_1(\Phi) \circ \mathcal{Q}(\theta_1(\Psi) \circ S')\big) = \mathrm{Tr}\big(\theta_1(\Psi) \circ S' \circ \mathcal{Q}(\theta_1(\Phi))\big) \qquad (S \in \mathcal{A}', \Psi \in \mathcal{A}''),$$

and that $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$ if and only if

$$\mathrm{Tr}\big(\theta_1(\Psi) \circ \mathcal{Q}(\theta_1(\Phi) \circ S')\big) = \mathrm{Tr}\big(\theta_1(\Phi) \circ S' \circ \mathcal{Q}(\theta_1(\Psi))\big) \qquad (S \in \mathcal{A}', \Psi \in \mathcal{A}'').$$

Suppose that $\theta_1(\Phi) \in Z_1^0(E, \alpha)$, so that $\theta_1(\Phi) = \mathcal{Q}(\theta_1(\Phi))$. Taking adjoints, we also have

$$S''' \circ \kappa_E'' \circ \theta_1(\Phi) = S''' \circ \theta_1(\Phi)'' \circ \kappa_E'' \qquad (S \in \mathcal{A}').$$

Thus, for $S \in \mathcal{A}'$ and $\Psi \in \mathcal{A}''$, we have

$$\begin{aligned}\mathrm{Tr}\big(\theta_1(\Phi) \circ \mathcal{Q}(\theta_1(\Psi) \circ S')\big) &= \mathrm{Tr}\big(\kappa_{E'}' \circ \theta_1(\Psi)'' \circ S''' \circ \kappa_E'' \circ \theta_1(\Phi)\big) \\ &= \mathrm{Tr}\big(\kappa_{E'}' \circ \theta_1(\Psi)'' \circ S''' \circ \theta_1(\Phi)'' \circ \kappa_E''\big),\end{aligned}$$

noting that $\eta(\theta_1(\Psi) \circ S') \in \mathcal{N}_{\alpha'}(E')$, a fact which allows us to alter the order of maps inside the trace. As $\eta(\theta_1(\Phi)) \circ S \in \mathcal{N}_{\alpha'}(E') \subseteq \mathcal{K}(E') \subseteq \mathcal{W}(E')$, we have $\kappa_{E'} \circ \kappa_E' \circ \eta(\theta_1(\Phi))'' = \eta(\theta_1(\Phi))''$. Thus we have

$$\begin{aligned}\mathrm{Tr}\big(\kappa_{E'}' \circ \theta_1(\Psi)''\circ S''' \circ \theta_1(\Phi)'' \circ \kappa_E''\big) &= \mathrm{Tr}\big(\theta_1(\Psi)'' \circ S''' \circ \eta(\theta_1(\Phi))''' \circ \kappa_E'' \circ \kappa_{E'}'\big) \\ = \mathrm{Tr}\big(\theta_1(\Psi)'' \circ S''' \circ \eta(\theta_1(\Phi))'''\big) &= \mathrm{Tr}\big(\theta_1(\Psi) \circ S' \circ \mathcal{Q}(\theta_1(\Phi))\big).\end{aligned}$$

Hence $\Phi \in \mathfrak{Z}_t^{(1)}(\mathcal{A}'')$. Applying Proposition 2.7.8 allows us to conclude that

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \theta_1^{-1}(Z_1^0(E, \alpha)).$$

Similarly, suppose that $\theta_1(\Phi) \in Z_2^0(E, \alpha)$. Then $\theta_1(\Phi)(E'') \subseteq \kappa_E(E)$ and

$$\theta_1(\Phi) \circ S' \in \mathcal{B}(E')^a \qquad (S \in \mathcal{A}').$$

Let $T \in \mathcal{B}(E'', E)$ be such that $\kappa_E \circ T = \theta_1(\Phi)$. Then, for $S \in \mathcal{A}'$, we have $\kappa_E \circ T \circ S' = R_S'$ for some $R_S \in \mathcal{B}(E')$. As $R_S'(E'') \subseteq \kappa_E(E)$, by the argument used in Lemma 2.7.10, $R_S = R'$ where $R = T \circ S' \circ \kappa_E \in \mathcal{W}(E)$. Then $R_S = \eta(R_S') = \eta(\theta_1(\Phi) \circ S') \in \mathcal{N}_{\alpha'}(E')$. In particular, $R_S \in \mathcal{W}(E')$ and so $\kappa_{E'} \circ \kappa_E' \circ R_S'' = R_S''$, and so, for $\Psi \in \mathcal{A}''$, we have

$$\begin{aligned}&\mathrm{Tr}\big(\theta_1(\Psi) \circ \mathcal{Q}(\theta_1(\Phi) \circ S')\big) = \mathrm{Tr}\big(\theta_1(\Psi) \circ \theta_1(\Phi) \circ S'\big) = \mathrm{Tr}\big(\theta_1(\Psi) \circ R_S'\big) \\ &\quad = \mathrm{Tr}\big(R_S'' \circ \theta_1(\Psi)'\big) = \mathrm{Tr}\big(\kappa_{E'} \circ \kappa_E' \circ R_S'' \circ \theta_1(\Psi)'\big) = \mathrm{Tr}\big(\kappa_E' \circ R_S'' \circ \theta_1(\Psi)' \circ \kappa_{E'}\big) \\ &\quad = \mathrm{Tr}\big(\kappa_E' \circ S'' \circ T' \circ \kappa_E' \circ \theta_1(\Psi)' \circ \kappa_{E'}\big) = \mathrm{Tr}\big(R' \circ \eta(\theta_1(\Psi))\big) \\ &\quad = \mathrm{Tr}\big(R_S \circ \eta(\theta_1(\Psi))\big) = \mathrm{Tr}\big(\eta(\theta_1(\Psi)) \circ R_S\big) = \mathrm{Tr}\big(\theta_1(\Phi) \circ S' \circ \mathcal{Q}(\theta_1(\Psi))\big).\end{aligned}$$

Hence $\Phi \in \mathfrak{Z}_t^{(2)}(\mathcal{A}'')$, and another application of Proposition 2.7.8 allows us to conclude that

$$\mathfrak{Z}_t^{(2)}(\mathcal{A}'') = \theta_1^{-1}(Z_2^0(E,\alpha)). \qquad \square$$

Theorem 2.7.33. *Let E be a Banach space such that $\mathcal{N}(E') = \mathcal{I}(E')$, let α be a tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Suppose that E'' has the bounded approximation property. Then, for $i = 1, 2$, we have*

$$\mathfrak{Z}_t^{(i)}(\mathcal{A}'') = \theta_1^{-1}(Z_i^0(E,\alpha)).$$

Proof. As E'' has the bounded approximation property, so does E'. Thus, in the language of Proposition 2.7.3, (E', α) and (E'', α) are Grothendieck pairs. The rest of the proof runs exactly as above. $\square$

We can then apply the same sort of arguments used in, for example, Theorem 2.7.11, to state some corollaries. Rather than do this, we state the most interesting case.

Corollary 2.7.34. *Let E be a Banach space such that $\mathcal{N}(E') = \mathcal{I}(E')$. Let $\mathcal{A} = \mathcal{A}(E)$, and let*

$$X = (\ker J_\pi)^\circ = \{T \in \mathcal{B}(E'') : \langle T, u\rangle = 0 \ (u \in E''\widehat{\otimes}E', J_\pi(u) = 0)\}.$$

Then $\theta_1 : \mathcal{A}'' \to X$ is an isometry, and, when we identify $\mathcal{A}''$ with X, we have

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = X \cap \mathcal{B}(E')^a \quad , \quad \mathfrak{Z}_t^{(2)}(\mathcal{A}'') = X \cap (\kappa_E \circ \mathcal{B}(E'', E)).$$

Proof. We have $\mathcal{A}' = \mathcal{I}(E') = \mathcal{N}(E')$ so that $\phi_1 : E''\widehat{\otimes}E' \to \mathcal{A}'$ is a quotient map, and thus θ_1 is an isometry. The results now follow from the calculations done in the proof of Theorem 2.7.11, and the results of the above theorem. $\square$

In the case of the nuclear operators, we cannot say much more than the above theorem gives, as, in general, we have no good description of $\mathcal{N}(E)'$ (see Example 2.7.27). However, the next example shows that in special cases we can say more than we could before.

Example 2.7.35. By [Figiel, Johnson, 1973] and Proposition 2.3.18, we can find a Banach space E_0 with the approximation property, such that $\mathcal{N}(E_0') = \mathcal{I}(E_0')$, and such that there exists $T_0 \in \mathcal{I}(E_0) \setminus \mathcal{N}(E_0)$ with $T_0' \in \mathcal{N}(E_0')$. Then let $\mathcal{A} = \mathcal{N}(E_0) = E_0'\widehat{\otimes}E_0$, so that $\mathcal{A}' = \mathcal{B}(E_0')$, and we have

$$Z_1^0(E_0,\pi) = \{T' : T \in \mathcal{I}(E_0'), T \circ \kappa'_{E_0} = \kappa'_{E_0} \circ T''\},$$
$$Z_2^0(E_0,\pi) = \{T \in \mathcal{I}(E_0'') : T(E_0'') \subseteq \kappa_{E_0}(E_0), T \in \mathcal{W}(E_0)^{aa}\} = \mathcal{I}(E_0)^{aa}.$$

As argued before, for $T' \in Z_1^0(E_0,\pi)$, we have $T''(M) = 0$ for each $M \in \kappa_{E_0}(E_0)^\circ$, so that $T'(E_0'') \subseteq \kappa_{E_0}(E_0)$, and thus $T' \in \mathcal{W}(E_0)^{aa}$. Thus we conclude

$$Z_1^0(E_0,\pi) = \mathcal{I}(E_0)^{aa} = Z_2^0(E_0,\pi) \ , \ \mathfrak{Z}_t^{(1)}(\mathcal{N}(E_0)'') = \mathfrak{Z}_t^{(1)}(\mathcal{N}(E_0)'') = \theta_1^{-1}(\mathcal{I}(E)^{aa}).$$

Examining the proof of Proposition 2.3.18, we see that $\mathcal{N}(E_0) \neq \mathcal{I}(E_0)$, so that we directly verify that $\mathcal{A}$ is not strongly Arens irregular. Of course, this fact also follows from Theorem 2.7.14. Finally, we note that $\phi_1 : E_0''\widehat{\otimes}_\varepsilon E_0' = \mathcal{A}(E_0') \rightarrow \mathcal{B}(E_0') = \mathcal{A}'$ certainly does not have dense range, so that θ_1 is not injective (thereby giving yet another way to show that $\mathcal{A}$ is not strongly Arens irregular). ☐

2.7.3 Arens regularity of ideals of nuclear operators

We have so far not discussed when $\mathcal{N}_\alpha(E)$ is Arens regular. This is because we needed the above work to build up the necessary machinery.

Theorem 2.7.36. *Let E be a reflexive Banach space, let α be a tensor norm, and let $\mathcal{A} = \mathcal{N}_\alpha(E)$. Suppose that either α is accessible, or that E has the approximation property. Then $\mathcal{A}$ is Arens regular.*

Proof. The case when $\alpha = \varepsilon$ is well known: see, for example, [Young, 1976, Theorem 3]. The case when $\alpha = \pi$ is [Dales, 2000, Theorem 2.6.23], where the result is attributed to A. Ülger.

Suppose α is accessible. Then we simply apply Theorem 2.7.32. As E is reflexive, E' has the Radon-Nikodým property, and so $\mathcal{N}(E') = \mathcal{I}(E')$. Then, identifying E with E'', we have $\mathcal{W}(E) = \mathcal{B}(E)$, and so

$$Z_1^0(E,\alpha) = \mathcal{B}_\alpha(E) = Z_2^0(E,\alpha).$$

As the image of θ_1 is contained in $\mathcal{B}_\alpha(E)$, we immediately see that

$$\mathfrak{Z}_t^{(1)}(\mathcal{A}'') = \theta^{-1}(\mathcal{B}_\alpha(E)) = \mathcal{A}'',$$

so that $\mathcal{A}$ is Arens regular.

When E has the approximation property, E and E' have the metric approximation property by Theorem 2.3.23. We then simply apply Theorem 2.7.33. ☐

2.8 Radicals of biduals of operator ideals

We now study the radical of $\mathcal{N}_\alpha(E)''$ for either Arens product. This is quite simple, essentially because of Proposition 2.7.12. Recall the definition of the radical of a Banach algebra $\mathcal{A}$,

$$\operatorname{rad}\mathcal{A} = \{a \in \mathcal{A} : e_{\mathcal{A}^\sharp} - ba \in \operatorname{Inv}\mathcal{A}^\sharp \ (b \in \mathcal{A}^\sharp)\}.$$

Lemma 2.8.1. *Let $\mathcal{A}$ be a non-unital Banach algebra. Then the following are equivalent:*

1. $a \in \operatorname{rad}\mathcal{A}$*;*
2. *for each $b \in \mathcal{A}$ and $\beta \in \mathbb{C}$, we have $c - bac - \beta ac = c - cba - \beta ca = ba + \beta a$ for some $c \in \mathcal{A}$;*
3. *for each $b \in \mathcal{A}$ and $\beta \in \mathbb{C}$, we have $c - abc - \beta ac = c - cab - \beta ca = ab + \beta a$ for some $c \in \mathcal{A}$.*

Proof. An arbitrary element $b_0 \in \mathcal{A}^\sharp$ can be uniquely written as $b_0 = b + \beta e_{\mathcal{A}^\sharp}$ for some $b \in \mathcal{A}$ and $\beta \in \mathbb{C}$. Similarly, let $c_0 = c + \gamma e_{\mathcal{A}^\sharp} \in \mathcal{A}^\sharp$, so that

$$(e_{\mathcal{A}^\sharp} - b_0 a)c_0 = c + \gamma e_{\mathcal{A}^\sharp} - bac - \gamma ba - \beta ac - \beta\gamma a,$$
$$c_0(e_{\mathcal{A}^\sharp} - b_0 a) = c + \gamma e_{\mathcal{A}^\sharp} - cba - \gamma ba - \beta ca - \beta\gamma a.$$

Thus $e_{\mathcal{A}^\sharp} - b_0 a \in \operatorname{Inv}\mathcal{A}^\sharp$ if and only if, for some $c \in \mathcal{A}$,

$$c - bac - ba - \beta ac - \beta a = 0 = c - cba - ba - \beta ca - \beta a.$$

The equivalence of (1) and (3) follows in an entirely analogous manner. □

Recall the maps ψ_1 and ψ_2 defined in Theorem 2.7.4, and the sets I_1 and I_2 defined in Proposition 2.7.12.

Theorem 2.8.2. *Let E be a Banach space, α be a tensor norm and $\mathcal{A} = \mathcal{N}_\alpha(E)$. Then*

$$\operatorname{rad}(\mathcal{A}'', \Box) = I_1 \quad , \quad \operatorname{rad}(\mathcal{A}'', \Diamond) = I_2.$$

In particular, when E is not reflexive, $\operatorname{rad}(\mathcal{A}'', \Box) \subsetneq \operatorname{rad}(\mathcal{A}'', \Diamond)$.

Proof. Suppose that $\Phi \notin I_1$, so that $\theta_1(\Phi) \neq 0$. Then, for some $\Lambda \in E''$ and $M \in E'''$, we have $\langle M, \theta_1(\Phi)(\Lambda)\rangle = 1$. Let $R = M \otimes \Lambda \in \mathcal{F}(E'')$, and suppose that $\Phi \in \operatorname{rad}(\mathcal{A}'', \Box)$. Then, for some $\Psi \in \mathcal{A}''$, we have

$$\psi_1(R)\Box\Phi = \Psi - \Psi\Box\psi_1(R)\Box\Phi.$$

Applying θ_1, we have

$$R(\theta_1(\Phi)(\Lambda)) = \theta_1(\Psi)(\Lambda) - (\theta_1(\Psi) \circ R \circ \theta_1(\Phi))(\Lambda),$$

which is $\Lambda = \theta_1(\Psi)(\Lambda) - \theta_1(\Psi)(\Lambda)$, a contradiction, as $\Lambda \neq 0$.

Conversely, suppose that $\Phi \in I_1$. Fix $\beta \in \mathbb{C}$, and let $\Upsilon = \beta\Phi$. By Proposition 2.7.12, we have $\mathcal{A}''\Box\Phi = 0$, so for $\Psi \in \mathcal{A}''$, we have

$$\begin{aligned}
\Upsilon - \Psi\Box\Phi\Box\Upsilon - \beta\Phi\Box\Upsilon &= \beta\Phi - \beta^2\Phi\Box\Phi = \beta\Phi, \\
\Upsilon - \Upsilon\Box\Psi\Box\Phi - \beta\Upsilon\Box\Phi &= \beta\Phi, \\
\Psi\Box\Phi + \beta\Phi &= \beta\Phi,
\end{aligned}$$

which verifies condition (2) in the above lemma. Thus $\Phi \in \operatorname{rad}(\mathcal{A}'', \Box)$.

Similarly, suppose that, for $\Phi \in \mathcal{A}''$, we have $\mathcal{Q}(\theta_1(\Phi)) \neq 0$. Then, for some $\Lambda \in E''$, we have $\Lambda_0 := \mathcal{Q}(\theta_1(\Phi))(\Lambda) \neq 0$. Let $\mu \in E'$ be such that $\langle \Lambda_0, \mu \rangle = 1$, and set $R = \Lambda \otimes \mu \in \mathcal{F}(E')$. Suppose that $\Phi \in \operatorname{rad}(\mathcal{A}'', \Diamond)$, so that for some $\Psi \in \mathcal{A}''$, we have

$$\Phi\Diamond\psi_2(R') = \Psi - \Phi\Diamond\psi_2(R')\Diamond\Psi.$$

Applying θ_1, we have

$$\mathcal{Q}(\theta_1(\Phi)) \circ R' = \theta_1(\Psi) - \mathcal{Q}(\theta_1(\Phi)) \circ R' \circ \theta_1(\Psi),$$

where $\mathcal{Q}(\theta_1(\Phi)) \circ R' = \kappa_{E'}(\mu) \otimes \Lambda_0$, so that applying the above to Λ_0, we get

$$\langle \Lambda_0, \mu \rangle \Lambda_0 = \theta_1(\Psi)(\Lambda_0) - \langle \theta_1(\Psi)(\Lambda_0), \mu \rangle \Lambda_0.$$

Applying μ to this gives us, as $\langle \Lambda_0, \mu \rangle = 1$,

$$1 = \langle \theta_1(\Psi)(\Lambda_0), \mu \rangle - \langle \theta_1(\Psi)(\Lambda_0), \mu \rangle = 0,$$

a contradiction.

Conversely, suppose that $\Phi \in I_2$, so that $\mathcal{Q}(\theta_1(\Phi)) = 0$, and $\Phi\Diamond\mathcal{A}'' = \{0\}$. Then, for $\Psi \in \mathcal{A}''$ and $\beta \in \mathbb{C}$, let $\Upsilon = \beta\Phi$, so that we have

$$\begin{aligned}
\Upsilon - \Phi\Diamond\Psi\Diamond\Upsilon - \beta\Phi\Diamond\Upsilon &= \Upsilon = \beta\Phi, \\
\Upsilon - \Upsilon\Diamond\Phi\Diamond\Psi - \beta\Upsilon\Diamond\Phi &= \Upsilon - \beta^2\Phi\Diamond\Phi = \beta\Phi, \\
\Phi\Diamond\Psi + \beta\Phi &= \beta\Phi,
\end{aligned}$$

which verifies condition (3) in the above lemma. Thus $\Phi \in \operatorname{rad}(\mathcal{A}'', \Diamond)$. □

Corollary 2.8.3. *Let E be an infinite-dimensional Banach space with the approximation property. Then $\mathcal{N}(E)''$, with either Arens product, is not semi-simple.*

Proof. We need to show that I_1 is not zero, as I_2 contains I_1. That is, we wish to show that $\theta_1 : \mathcal{N}(E)'' \to \mathcal{I}(E'')$ is not injective; that is, $\phi_1 : \mathcal{A}(E') \to \mathcal{N}(E)'$ does not have dense range, which in this case is equivalent to $\mathcal{N}(E)' = \mathcal{A}(E')$. As E has the approximation property, $\mathcal{N}(E)' = (E'\widehat{\otimes}E)' = \mathcal{B}(E')$, so we are done. $\square$

Corollary 2.8.4. *Let E be a Banach space with $\mathcal{N}(E') = \mathcal{I}(E')$. Then $(\mathcal{A}(E)'', \Box)$ is semi-simple while $(\mathcal{A}(E)'', \Diamond)$ is not semi-simple.* $\square$

2.9 Ideals of compact operators

We have not dealt yet with $\mathcal{K}(E)$, except when E has the approximation property, in which case $\mathcal{K}(E) = \mathcal{A}(E)$. In particular, we shall now generalise Theorem 2.7.14 and Theorem 2.7.36.

We need a description of $\mathcal{K}(E)'$ for which we use an idea from [Feder, Saphar, 1975]. Let E be a Banach space and let $I \subseteq E'_{[1]}$ be a *norming* subset, that is

$$\|x\| = \sup\{|\langle \mu, x\rangle| : \mu \in I\} \qquad (x \in E).$$

For example, when E is separable, we can take I to be countable. Then let $\iota : E \to l^\infty(I)$ be the map

$$\iota(x) = \big(\langle \mu, x\rangle\big)_{\mu\in I} \in l^\infty(I),$$

so that ι is an isometry. Let $J : \mathcal{K}(E) \to K(E, l^\infty(I))$ be given by $J(T) = \iota \circ T$ for $T \in \mathcal{K}(E)$, so that J is an isometry. As $l^\infty(I)'$ has the metric approximation property, we have

$$\mathcal{K}(E, l^\infty(I)) = \mathcal{A}(E, l^\infty(I)) = E'\check{\otimes} l^\infty(I),$$

so that $\mathcal{K}(E, l^\infty(I))' = \mathcal{I}(E', l^\infty(I)')$. Thus $J' : \mathcal{I}(E', l^\infty(I)') \to \mathcal{K}(E)'$ is a quotient operator, and $J'' : \mathcal{K}(E)'' \to \mathcal{I}(E', l^\infty(I)')'$ is an isometry onto its range.

We now collect together some properties of these maps.

Lemma 2.9.1. *Let E be a Banach space, and I, ι and J be as above. Then we have:*

1. *$\iota'' : E'' \to l^\infty(I)''$ is an isometry onto $\iota(E)^{\circ\circ}$;*
2. *for each $\lambda \in \mathcal{K}(E)'$, there exists $S \in \mathcal{I}(E', l^\infty(I)')$ with $\|S\|_\pi = \|\lambda\|$ and $J'(S) = \lambda$;*
3. *for $S \in \mathcal{I}(E', l^\infty(I)')$, we have that $J'(S) = 0$ implies that $S(E') \subseteq \iota(E)^\circ$;*

4. *for* $S \in \mathcal{I}(E', l^\infty(I)')$ *and* $R \in \mathcal{K}(E)$, *let* $S_1, S_2 \in \mathcal{I}(E', l^\infty(I)')$ *be such that* $J'(S) \cdot R = J'(S_1)$ *and* $R \cdot J'(S) = J'(S_2)$. *Then* $\iota' \circ S_1 = R' \circ \iota' \circ S$ *and* $\iota' \circ S_2 = \iota' \circ S \circ R'$;

5. *for* $S \in \mathcal{I}(E', l^\infty(I)')$, *we have* $\kappa_{E'} \circ \kappa_E' \circ \iota''' \circ S'' = \iota''' \circ S''$.

Proof. For (1), by Theorem 1.4.11(4), we see that $\iota' : l^\infty(I)'/\iota(E)^\circ \to E'$ is an isometry. Then, by Theorem 1.4.11(5), we have that $\iota'' : E'' \to (\ker \iota')^\circ = \iota(E)^{\circ\circ}$ is an isometry, as required.

For (2), as J isometrically identifies $\mathcal{K}(E)$ with a subspace of $\mathcal{A}(E, l^\infty(I))$, for $\lambda \in \mathcal{K}(E)'$, we can extend λ to a member of $\mathcal{A}(E, l^\infty(I))'$ by the Hahn-Banach theorem. This gives us the required $S \in \mathcal{I}(E', l^\infty(I)')$.

For (3), let $S \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S) = 0$. In particular, for $T = \mu \otimes x \in \mathcal{K}(E)$, we have $0 = \langle J'(S), T\rangle = \langle S, \mu \otimes \iota(x)\rangle = \langle S(\mu), \iota(x)\rangle$. We hence see that $S(E') \subseteq \iota(E)^\circ$.

For (4), suppose we have R, S, S_1 and S_2 as stated. Then, for $T = \mu \otimes x \in \mathcal{A}(E) \subseteq \mathcal{K}(E)$, we have

$$\langle S_1(\mu), \iota(x)\rangle = \langle J'(S_1), T\rangle = \langle J'(S) \cdot R, T\rangle = \langle J'(S), \mu \otimes R(x)\rangle = \langle S(\mu), \iota(R(x))\rangle,$$
$$\langle S_2(\mu), \iota(x)\rangle = \langle J'(S_2), T\rangle = \langle R \cdot J'(S), T\rangle = \langle J'(S), R'(\mu) \otimes x\rangle = \langle S(R'(\mu)), \iota(x)\rangle.$$

Thus we have $\iota' \circ S_1 = R' \circ \iota' \circ S$ and $\iota' \circ S_2 = \iota' \circ S \circ R'$.

For (5), for $M \in E'''$, as S is weakly-compact, we have that $S''(M) = \kappa_{l^\infty(I)'}(\lambda)$ for some $\lambda \in l^\infty(I)'$. Then, for $x \in E$, we have

$$\langle (\kappa_E' \circ \iota''' \circ S'')(M), x\rangle = \langle \iota''(\kappa_E(x)), \lambda\rangle = \langle \iota'(\lambda), x\rangle,$$

so that for $\Lambda \in E''$, we have

$$\langle \Lambda, (\kappa_E' \circ \iota''' \circ S'')(M)\rangle = \langle \iota''(\Lambda), \lambda\rangle = \langle S''(M), \iota''(\Lambda)\rangle = \langle (\iota''' \circ S'')(M), \Lambda\rangle,$$

so we see that $\kappa_{E'} \circ \kappa_E' \circ \iota''' \circ S'' = \iota''' \circ S''$, as required. $\square$

Proposition 2.9.2. *Let E be a Banach space. Then there is an isometry $\psi_1 : \mathcal{A}(E'') \to \mathcal{K}(E)''$ and a norm-decreasing map $\psi_2 : \mathcal{A}(E'') \to \mathcal{K}(E)''$ such that $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{A}(E'')}$ and $\theta_1 \circ \psi_2 = \mathcal{Q}$. Furthermore, $\psi_1 : \mathcal{A}(E'') \to (\mathcal{K}(E)'', \Box)$ and $\psi_2 : (\mathcal{A}(E''), \star) \to (\mathcal{K}(E)'', \Diamond)$ are homomorphisms.*

Proof. For $S \in \mathcal{I}(E', l^\infty(I)')$, we have that $S'' \in \mathcal{I}(E''', l^\infty(I)''') = \mathcal{A}(E'', l^\infty(I)'')'$, so that we may define

$$\langle \psi_1(T), J'(S) \rangle = \langle S'', \iota'' \circ T \rangle = \operatorname{Tr}(S' \circ \iota'' \circ T) \qquad (T \in \mathcal{F}(E'')),$$
$$\langle \psi_2(T), J'(S) \rangle = \langle S'', \iota'' \circ \mathcal{Q}(T) \rangle = \operatorname{Tr}(\eta(T) \circ \iota' \circ S) \qquad (T \in \mathcal{F}(E'')).$$

This is well-defined, for if $J'(S) = 0$ then $S(E') \subseteq \iota(E)^\circ$ so that $\{0\} = S'(\iota(E)^{\circ\circ}) = S'(\iota''(E''))$. Thus $S' \circ \iota'' = 0$ and hence $\langle \psi_1(T), J'(S) \rangle = 0 = \langle \psi_2(T), J'(S) \rangle$.

Then, for $\lambda \in \mathcal{K}(E)'$, let $S \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S) = \lambda$ and $\|S\|_\pi = \|\lambda\|$. For $T \in \mathcal{F}(E'')$, we thus have

$$|\langle \psi_1(T), \lambda \rangle| = |\langle S'', \iota'' \circ T \rangle| \leq \|S''\|_\pi \|\iota'' \circ T\| = \|S\|_\pi \|T\| = \|\lambda\| \|T\|,$$

so we see that ψ_1 is norm-decreasing, and hence that ψ_1 extends by continuity to $\mathcal{A}(E'')$. Similarly ψ_2 is norm-decreasing and extends to $\mathcal{A}(E'')$.

For $\Lambda \in E''$ and $\mu \in E'$, let $\mu_0 \in l^\infty(I)'$ be such that $\iota'(\mu_0) = \mu$ and $\|\mu\| = \|\mu_0\|$. Then, for $R \in \mathcal{K}(E)$, we have

$$\langle J'(\Lambda \otimes \mu_0), R \rangle = \langle \Lambda \otimes \mu_0, \iota \circ R \rangle = \langle \Lambda, R'(\iota'(\mu_0)) \rangle = \langle \Lambda, R'(\mu) \rangle = \langle \phi_1(\Lambda \otimes \mu), R \rangle,$$

so that $J(\Lambda \otimes \mu_0) = \phi_1(\Lambda \otimes \mu)$. Then, for $T \in \mathcal{A}(E'')$, we have

$$\begin{aligned} \langle \theta_1(\psi_1(T))(\Lambda), \mu \rangle &= \langle \psi_1(T), \phi_1(\Lambda \otimes \mu) \rangle = \langle \psi_1(T), J'(\Lambda \otimes \mu_0) \rangle = \langle (\Lambda \otimes \mu_0)'', \iota'' \circ T \rangle \\ &= \langle \iota''(T(\Lambda)), \mu_0 \rangle = \langle T(\Lambda), \iota'(\mu_0) \rangle = \langle T(\Lambda), \mu \rangle. \end{aligned}$$

Thus we have $\theta_1 \circ \psi_1 = \mathrm{Id}_{\mathcal{A}(E'')}$, and as θ_1 is norm-decreasing, we have that ψ_1 is an isometry onto its range. Similarly, we have

$$\begin{aligned} \langle \theta_1(\psi_2(T))(\Lambda), \mu \rangle &= \langle \psi_2(T), \phi_1(\Lambda \otimes \mu) \rangle = \langle \psi_2(T), J'(\Lambda \otimes \mu_0) \rangle \\ &= \langle (\Lambda \otimes \mu_0)'', \iota'' \circ \mathcal{Q}(T) \rangle = \langle \mathcal{Q}(T)(\Lambda), \iota'(\mu_0) \rangle = \langle \mathcal{Q}(T)(\Lambda), \mu \rangle, \end{aligned}$$

so that $\theta_1 \circ \psi_2 = \mathcal{Q}$ on $\mathcal{A}(E'')$.

For $R \in \mathcal{K}(E)$ and $S \in \mathcal{I}(E', l^\infty(I)')$, let $S_1 \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S_1) = J'(S) \cdot R \in \mathcal{K}(E)'$. Let $T = M \otimes \Lambda \in \mathcal{F}(E'')$, so that as $R \in \mathcal{W}(E)$, $R''(\Lambda) \in \kappa_E(E)$. Then, by the above lemma, we have

$$\begin{aligned} \langle \psi_1(T) \cdot J'(S), R \rangle &= \langle \psi_1(T), J'(S_1) \rangle = \langle M, S_1'(\iota''(\Lambda)) \rangle = \langle M, (S' \circ \iota'' \circ R'')(\Lambda) \rangle \\ &= \langle R''(\Lambda), (\kappa_E' \circ \iota''' \circ S'')(M) \rangle = \langle \Lambda \otimes (\kappa_E' \circ \iota''' \circ S'')(M), R \rangle. \end{aligned}$$

Thus we have

$$\psi_1(M \otimes \Lambda) \cdot J'(S) = \phi_1\Big(\Lambda \otimes (\kappa_E' \circ \iota''' \circ S'')(M)\Big) \qquad (M \otimes \Lambda \in \mathcal{F}(E'')).$$

Let $T_i = M_i \otimes \Lambda_i$ for $i = 1, 2$, so that by (5) in the above lemma, we have

$$\begin{aligned}\langle \psi_1(T_1)\Box\psi_1(T_2), J'(S)\rangle &= \langle (\theta_1 \circ \psi_1)(T_1), \Lambda_2 \otimes (\kappa_E' \circ \iota''' \circ S'')(M_2)\rangle \\ &= \langle T_1(\Lambda_2), (\kappa_E' \circ \iota''' \circ S'')(M_2)\rangle = \langle M_1, \Lambda_2\rangle\langle \Lambda_1, (\kappa_E' \circ \iota''' \circ S'')(M_2)\rangle \\ &= \langle M_1, \Lambda_2\rangle\langle (\iota''' \circ S'')(M_2), \Lambda_1\rangle = \langle M_1, \Lambda_2\rangle\langle M_2, S'(\iota''(\Lambda_1))\rangle \\ &= \langle M_1, \Lambda_2\rangle\langle \psi_1(M_2 \otimes \Lambda_1), J'(S)\rangle = \langle \psi_1(T_1 \circ T_2), J'(S)\rangle.\end{aligned}$$

By linearity and continuity, we see that $\psi_1 : \mathcal{A}(E'') \to (\mathcal{K}(E)'', \Box)$ is a homomorphism.

Similarly, let $R \in \mathcal{K}(E)$ and $S, S_2 \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S_2) = R \cdot J'(S)$. Let $T = M \otimes \Lambda$, so that with reference to the above lemma, we have

$$\begin{aligned}\langle J'(S) \cdot \psi_2(T), R\rangle &= \langle \psi_2(T), J'(S_2)\rangle = \mathrm{Tr}(\eta(T) \circ \iota' \circ S_2) = \langle \Lambda, (\iota' \circ S_2 \circ \kappa_E')(M)\rangle \\ &= \langle \Lambda, (\iota' \circ S \circ R' \circ \kappa_E')(M)\rangle = \langle (S' \circ \iota'')(\Lambda) \otimes \kappa_E'(M), R\rangle.\end{aligned}$$

Let $T_i = M_i \otimes \Lambda_i$ for $i = 1, 2$, so that we have

$$\begin{aligned}\langle \psi_2(T_1)\Diamond\psi_2(T_2), J'(S)\rangle &= \langle (\theta_1 \circ \psi_2)(T_2), (S' \circ \iota'')(\Lambda_1) \otimes \kappa_E'(M_1)\rangle \\ &= \langle (S' \circ \iota'')(\Lambda_1), (\eta(T_2) \circ \kappa_E')(M_1)\rangle = \langle \Lambda_2, \kappa_E'(M_1)\rangle\langle (S' \circ \iota'')(\Lambda_1), \kappa_E'(M_2)\rangle \\ &= \langle \Lambda_2, \kappa_E'(M_1)\rangle \mathrm{Tr}((\Lambda_1 \otimes \kappa_E'(M_2)) \circ \iota' \circ S) \\ &= \langle \Lambda_2, \kappa_E'(M_1)\rangle\langle \psi_2(M_2 \otimes \Lambda_1), J'(S)\rangle = \langle \psi_2(T_1 \star T_2), J'(S)\rangle.\end{aligned}$$

By linearity and continuity, we see that $\psi_2 : (\mathcal{A}(E''), \star) \to (\mathcal{K}(E)'', \Diamond)$ is a homomorphism. $\square$

Lemma 2.9.3. *Let E be a Banach space and I, ι, J be as above. For $\lambda \in \mathcal{K}(E)'$, let $S \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S) = \lambda$. For $\Phi \in \mathcal{K}(E)''$, let $\Phi \cdot \lambda = J'(S_1)$ and $\lambda \cdot \Phi = J'(S_2)$ for some $S_1, S_2 \in \mathcal{I}(E', l^\infty(I)')$. Then we have*

$$\iota' \circ S_1 = \iota' \circ \kappa_{l^\infty(I)}' \circ S'' \circ \theta_1(\Phi)' \circ \kappa_{E'} \quad , \quad \iota' \circ S_2 = \eta(\theta_1(\Phi)) \circ \iota' \circ S.$$

Proof. Let $R = \mu \otimes x \in \mathcal{K}(E)$ so that, for $T \in \mathcal{K}(E)$, we have

$$\begin{aligned}\langle \lambda \cdot R, T\rangle &= \langle J'(S), R \circ T\rangle = \langle S, T'(\mu) \otimes \iota(x)\rangle = \langle (\kappa_{l^\infty(I)} \circ \iota)(x), (S \circ T')(\mu)\rangle \\ &= \langle \phi_1\big((S' \circ \kappa_{l^\infty(I)} \circ \iota)(x) \otimes \mu\big), T\rangle, \\ \langle R \cdot \lambda, T\rangle &= \langle J'(S), T \circ R\rangle = \langle S, \mu \otimes \iota(T(x))\rangle = \langle S(\mu), \iota(T(x))\rangle \\ &= \langle \phi_1\big(\kappa_E(x) \otimes (\iota' \circ S)(\mu)\big), T\rangle.\end{aligned}$$

Thus we have

$$\begin{aligned}
\langle S_1(\mu), \iota(x)\rangle &= \langle S_1, J(\mu\otimes x)\rangle = \langle J'(S_1), R\rangle = \langle \Phi\cdot\lambda, R\rangle = \langle \Phi, \lambda\cdot R\rangle \\
&= \langle \Phi, \phi_1((S'\circ\kappa_{l^\infty(I)}\circ\iota)(x)\otimes\mu)\rangle = \langle (\theta_1(\Phi)\circ S'\circ\kappa_{l^\infty(I)}\circ\iota)(x), \mu\rangle \\
&= \langle (\kappa'_{l^\infty(I)}\circ S''\circ\theta_1(\Phi)'\circ\kappa_{E'})(\mu), \iota(x)\rangle, \\
\langle S_2(\mu), \iota(x)\rangle &= \langle \lambda\cdot\Phi, R\rangle = \langle \Phi, R\cdot\lambda\rangle = \langle (\theta_1(\Phi)\circ\kappa_E)(x), (\iota'\circ S)(\mu)\rangle \\
&= \langle (\kappa'_E\circ\theta_1(\Phi)'\circ\kappa_{E'}\circ\iota'\circ S)(\mu), x\rangle,
\end{aligned}$$

as required. □

Proposition 2.9.4. *Let E be a Banach space. Then we have*

$$\theta_1(\mathfrak{Z}_t^{(1)}(\mathcal{K}(E)''))\subseteq\mathcal{B}(E')^a \quad , \quad \theta_1(\mathfrak{Z}_t^{(2)}(\mathcal{K}(E)''))\subseteq\kappa_E\circ\mathcal{B}(E'',E).$$

Furthermore, we have

$$\begin{aligned}
\psi_2(T)&\in\mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'') \qquad (T\in\mathcal{A}(E')^a), \\
\psi_1(T)&\in\mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'') \qquad (T\in\kappa_E\circ\mathcal{A}(E'',E)).
\end{aligned}$$

As such, when E is not reflexive, the topological centres of $\mathcal{K}(E)''$ are distinct, neither contains the other, and both lie strictly between $\kappa_{\mathcal{K}(E)}(\mathcal{K}(E))$ and $\mathcal{K}(E)''$.

Proof. The calculations for $\theta_1(\mathfrak{Z}_t^{(i)}(\mathcal{K}(E)'')$, for $i=1,2$, follow exactly as for $\mathcal{A}(E)$, as in Proposition 2.7.8.

Let $T=\Lambda\otimes\mu\in\mathcal{F}(E')$, $\lambda\in\mathcal{K}(E)'$ and $\Phi\in\mathcal{K}(E)''$. Let $S,S_1\in\mathcal{I}(E',l^\infty(I)')$ be such that $J'(S)=\lambda$ and $J'(S_1)=\Phi\cdot\lambda$. As S is weakly-compact, we have that $\kappa_{l^\infty(I)'}\circ\kappa'_{l^\infty(I)}\circ S''=S''$. Then we have, by the preceding lemma,

$$\begin{aligned}
\langle \psi_2(T')\Box\Phi, \lambda\rangle &= \langle \psi_2(T'), \Phi\cdot\lambda\rangle = \mathrm{Tr}(T\circ\iota'\circ S_1) \\
&= \mathrm{Tr}(T\circ\iota'\circ\kappa'_{l^\infty(I)}\circ S''\circ\theta_1(\Phi)'\circ\kappa_{E'}) \\
&= \langle \Lambda, (\iota'\circ\kappa'_{l^\infty(I)}\circ S''\circ\theta_1(\Phi)'\circ\kappa_{E'})(\mu)\rangle \\
&= \langle (S''\circ\theta_1(\Phi)'\circ\kappa_{E'})(\mu), \iota''(\Lambda)\rangle = \langle (\theta_1(\Phi)\circ S'\circ\iota'')(\Lambda), \mu\rangle \\
&= \langle \Phi, \phi_1((S'\circ\iota'')(\Lambda)\otimes\mu)\rangle = \langle \Phi, \lambda\cdot\psi_2(T')\rangle = \langle \psi_2(T')\Diamond\Phi, \lambda\rangle.
\end{aligned}$$

Thus $\psi_2(T')\in\mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'')$.

Similarly, for $T=M\otimes\kappa_E(x)\in\mathcal{F}(E'')$, let Φ,λ and S be as above. Then let $S_2\in\mathcal{I}(E',l^\infty(I)')$ be such that $J'(S_2)=\lambda\cdot\Phi$. As before, we have that $\kappa''_E\circ\kappa_E=\kappa_{E''}\circ\kappa_E$ and $S''(M)=\kappa_{l^\infty(I)'}(\nu)$ for some $\nu\in l^\infty(I)'$, so that $\iota'''(S''(M))=\kappa_{l^\infty(I)'}(\iota'(\nu))$. Then we have

$$
\begin{aligned}
\langle \Phi \Diamond \psi_1(T), \lambda \rangle &= \langle \psi_1(T), \lambda \cdot \Phi \rangle = \mathrm{Tr}(S_2' \circ \iota'' \circ T) = \mathrm{Tr}(S' \circ \iota'' \circ \mathcal{Q}(\theta_1(\Phi)) \circ T) \\
&= \langle M, (S' \circ \iota'' \circ \mathcal{Q}(\theta_1(\Phi)) \circ \kappa_E)(x) \rangle \\
&= \langle (\iota''' \circ S'')(M), (\kappa_{E'}' \circ \theta_1(\Phi)'' \circ \kappa_E'' \circ \kappa_E)(x) \rangle \\
&= \langle (\kappa_{E'}' \circ \theta_1(\Phi)'' \circ \kappa_{E''} \circ \kappa_E)(x), \iota'(\nu) \rangle = \langle (\theta_1(\Phi) \circ \kappa_E)(x), \iota'(\nu) \rangle \\
&= \langle (\theta_1(\Phi) \circ \kappa_E)(x), (\kappa_E' \circ \iota''' \circ S'')(M) \rangle = \langle \Phi, \phi_1(\kappa_E(x) \otimes (\kappa_E' \circ \iota''' \circ S'')(M)) \rangle \\
&= \langle \Phi, \psi_1(T) \cdot J'(S) \rangle = \langle \Phi \Box \psi_1(T), \lambda \rangle,
\end{aligned}
$$

so that $\psi_1(T) \in \mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'')$. $\Box$

Proposition 2.9.5. *Let E be a Banach space and $\theta_1 : \mathcal{K}(E)'' \to \mathcal{B}(E'')$ be as before. Let*

$$I_1 = \ker \theta_1 \subseteq \mathcal{K}(E)'' \quad , \quad I_2 = \ker(\mathcal{Q} \circ \theta_1) \subseteq \mathcal{K}(E)''.$$

Then I_1 is a closed ideal for either Arens product, and I_2 is a closed ideal in $(\mathcal{K}(E)'', \Diamond)$. Furthermore, we have

$$\mathcal{K}(E)'' \Box I_1 = I_1 \Diamond \mathcal{K}(E)'' = I_2 \Diamond \mathcal{K}(E)'' = \{0\}.$$

Proof. The first part follows exactly as in Proposition 2.7.12. Fix $\lambda \in \mathcal{K}(E)'$ and let $S \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S) = \lambda$. For $\Phi \in I_1$, let $S_1 \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S_1) = \Phi \cdot \lambda$, so that

$$\iota' \circ S_1 = \iota' \circ \kappa_{l^\infty(I)}' \circ S'' \circ \theta_1(\Phi)' \circ \kappa_{E'} = 0.$$

Hence we have, for $T \in \mathcal{K}(E)$,

$$\langle \Phi \cdot \lambda, T \rangle = \langle J'(S_1), T \rangle = \langle S_1, \iota \circ T \rangle = \mathrm{Tr}(S_1 \circ T' \circ \iota') = \mathrm{Tr}(\iota' \circ S_1 \circ T') = 0,$$

so that $\Phi \cdot \lambda = 0$, and hence

$$\langle \Psi \Box \Phi, \lambda \rangle = \langle \Psi, \Phi \cdot \lambda \rangle = 0 \qquad (\Psi \in \mathcal{K}(E)'').$$

As λ was arbitrary, we have $\mathcal{K}(E)'' \Box I_1 = \{0\}$. Similarly, let $S_2 \in \mathcal{I}(E', l^\infty(I)')$ be such that $J'(S_2) = \lambda \cdot \Phi$, so that

$$\iota' \circ S_2 = \eta(\theta_1(\Phi)) \circ \iota' \circ S = 0.$$

Hence we have

$$\langle \lambda \cdot \Phi, T \rangle = \langle J'(S_2), T \rangle = \langle S_2, J(T) \rangle = \mathrm{Tr}(T' \circ \iota' \circ S_2) = 0 \qquad (T \in \mathcal{K}(E)),$$

so that $\lambda \cdot \Phi = 0$. Thus $I_1 \Diamond \mathcal{K}(E)'' = \{0\}$.

Similarly, let $\Phi \in I_2$, so that $\eta(\theta_1(\Phi)) = 0$, and hence $\iota' \circ S_2 = 0$ when $J'(S_2) = \lambda \cdot \Phi$. Following the previous paragraph, we see that $I_2 \Diamond \mathcal{K}(E)'' = \{0\}$. $\square$

As before, we now turn our attention to when we can use nuclear and not integral operators. This takes us to a result shown in [Feder, Saphar, 1975].

Theorem 2.9.6. *Let E and F be Banach spaces such that E'' or F' has the Radon-Nikodým property. Define $V : E''\widehat{\otimes}F' \to \mathcal{K}(E,F)'$ by*

$$\langle V(\Phi \otimes \mu), T\rangle = \langle \Phi, T'(\mu)\rangle \qquad (\Phi \otimes \mu \in E''\widehat{\otimes}F', T \in \mathcal{K}(E,F)).$$

Then V is a quotient operator, and furthermore, for $\lambda \in \mathcal{K}(E,F)'$, there exists $u \in E''\widehat{\otimes}F'$ with $V(u) = \lambda$ and $\|u\| = \|\lambda\|$. Also, given $J_\pi : E''\widehat{\otimes}F' \to \mathcal{N}(E',F') \subseteq I(E',F')$, we have $\ker V \subseteq \ker J_\pi$.

Proof. This is [Feder, Saphar, 1975, Theorem 1]. We will sketch the easier case, which is when E'' has the Radon-Nikodým property. Form I and $\iota : F \to l^\infty(I)$ in a similar way to above, and define $J : \mathcal{K}(E,F) \to \mathcal{A}(E, l^\infty(I)$ by $J(T) = \iota \circ T$ for $T \in \mathcal{K}(E,F)$. Then

$$(l^\infty(I)\check{\otimes}E')' = \mathcal{I}(l^\infty(I), E'') = \mathcal{N}(l^\infty(I), E'') = l^\infty(I)'\widehat{\otimes}E'',$$

as E'' has the Radon-Nikodým property and $l^\infty(I)$ is a dual space with the approximation property. By applying the swap map to both sides, we see that

$$\mathcal{K}(E, l^\infty(I))' = (E'\check{\otimes}l^\infty(I))' = E''\widehat{\otimes}l^\infty(I)'.$$

Thus $J' : E''\widehat{\otimes}l^\infty(I)' \to \mathcal{K}(E,F)'$. Hence we have the following diagram.

$$\begin{tikzcd}
E''\widehat{\otimes}l^\infty(I)' \arrow[r, "J'"] \arrow[d, "\mathrm{Id}_{E''} \otimes \iota'"'] & \mathcal{K}(E,F)' \\
E''\widehat{\otimes}F' \arrow[ur, "V"']
\end{tikzcd}$$

We can verify that this diagram commutes, so as J is an isometry, J' is a quotient operator. As $\mathrm{Id}_{E''} \otimes \iota'$ is norm-decreasing, V must also be a quotient operator. We can then easily verify the other claims, and the case when F' has the Radon-Nikodým property follows in a similar manner. $\square$

In particular, when E' or E'' has the Radon-Nikodým property, we have a quotient operator $V : E''\widehat{\otimes}E' \to \mathcal{K}(E)'$, and this respects the usual identification of $\mathcal{A}(E)' =$

$\mathcal{I}(E') = \mathcal{N}(E')$. It is clear that V agrees with the map ϕ_1, and so $\theta_1 : \mathcal{K}(E)'' \to \mathcal{B}(E'')$ is an isometry onto its range which contains $\mathcal{A}(E'')$ by Proposition 2.9.2. Indeed, we have

$$\theta_1(\mathcal{K}(E)'') = (\ker \phi_1)^\circ = \{T \in \mathcal{B}(E'') : \langle T, \tau \rangle = 0 \ (\tau \in E''\widehat{\otimes} E', \phi_1(\tau) = 0)\}.$$

Theorem 2.9.7. *Let E be a Banach space such that E' or E'' has the Radon-Nikodým property. Then $\mathcal{K}(E)''$ is identified isometrically with $X = \theta_1(\mathcal{K}(E)'') \subseteq \mathcal{B}(E'')$ and we have*

$$\mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'') = X \cap \mathcal{B}(E')^a \quad , \quad \mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'') = X \cap (\kappa_E \circ \mathcal{B}(E'', E))$$
$$\mathfrak{Z}_t^{(1)}(\mathcal{K}(E)'') \cap \mathfrak{Z}_t^{(2)}(\mathcal{K}(E)'') = X \cap \mathcal{W}(E)^{aa}.$$

Proof. This follows exactly as in the $\mathcal{A}(E)$ case, Corollary 2.7.34. □

Notice that we cannot easily generalise Lemma 2.7.31 as we have no simple description of $\ker \phi_1$. Indeed, it is hard to say whether $\ker \phi_1$ is trivial or not.

Definition 2.9.8. Let E be a Banach space such that for each compact subset $K \subseteq E$ and each $\varepsilon > 0$, there exists $T \in \mathcal{K}(E)$ with $\|T(x) - x\| < \varepsilon$ for each $x \in K$. Then E has the *compact approximation property*. When we can control the norm of T, E has the *bounded compact approximation property* or the *metric compact approximation property*.

We might be tempted to suppose that ϕ_1 is injective when E' has the compact approximation property. This does not seem to be true in general, unlike the $\mathcal{A}(E)$ case.

The paper [Grønbæk, Willis, 1993] is a good source of information on the compact approximation property, when applied to algebraic questions about $\mathcal{K}(E)$. We will come back to this, but for now, we need a definition from [Grønbæk, Willis, 1993].

Definition 2.9.9. Let E be a Banach space. Then E' has the $\mathcal{K}(E)^a$*-approximation property* if, for each compact subset $K \subseteq E'$ and each $\varepsilon > 0$, there exists $T \in \mathcal{K}(E)$ such that $\|T'(\mu) - \mu\| \leq \varepsilon$ for each $\mu \in K$. Similarly, we have the idea of the *bounded* $\mathcal{K}(E)^a$*-approximation property*.

Thus the $\mathcal{K}(E)^a$-approximation property is stronger than E' having the compact approximation property, and [Grønbæk, Willis, 1993, Example 4.3] shows that, in general, these properties do not coincide. In [Grønbæk, Willis, 1993, Section 3], a sufficient condition on E is given for these properties to be the same, but given the lack of examples of Banach spaces without the (compact) approximation property, it is left open if this condition on E is common or not.

Then [Grønbæk, Willis, 1993, Corollaries 2.6, 2.7] states that E' has the bounded $\mathcal{K}(E)^a$-approximation property if and only if $\mathcal{K}(E)$ has a bounded right approximate identity, or equivalently, a bounded approximate identity.

Proposition 2.9.10. *Let E be a Banach space such that E' or E'' has the Radon-Nikodým property, so that $\mathcal{K}(E)''$ is identified with a subalgebra of $\mathcal{B}(E'')$. Then ϕ_1 is injective implies that E' has the metric $\mathcal{K}(E)^a$-approximation property. Conversely, when E' has the $\mathcal{K}(E)^a$-approximation property, we have $\mathcal{B}(E)^{aa} \subseteq \mathcal{K}(E)''$ and that E' has the metric $\mathcal{K}(E)^a$-approximation property.*

Proof. Given the above, we see that E' has the bounded $\mathcal{K}(E)^a$-approximation property if and only if $\mathcal{K}(E)''$ has a mixed identity. As $\mathcal{K}(E)' = \phi_1(E''\widehat{\otimes}E')$, we see that ϕ_1 is injective if and only if ϕ_1 is an isometry $E''\widehat{\otimes}E' \to \mathcal{K}(E)'$, which is if and only if $\theta_1 : \mathcal{K}(E)'' \to \mathcal{B}(E'')$ is surjective. We can easily see that $\Xi \in \mathcal{K}(E)''$ is a mixed identity if and only if $\theta_1(\Xi) = \mathrm{Id}_{E''}$, in which case, as θ_1 is an isometry, we have that E' has the metric $\mathcal{K}(E)^a$-approximation property. We see immediately that when ϕ_1 is injective, E' has the metric $\mathcal{K}(E)^a$-approximation property.

Conversely, suppose that $\tau \in E''\widehat{\otimes}E'$ is such that $\phi_1(\tau) = 0$. We can find a representative $\tau = \sum_{n=1}^\infty \Lambda_n \otimes \mu_n$ with $\sum_{n=1}^\infty \|\Lambda_n\| < \infty$ and $\|\mu_n\| \to 0$ as $n \to \infty$. Let $S \in \mathcal{B}(E)$. Then $(S'(\mu_n))_{n=1}^\infty$ is a compact subset of E', so as E' has the $K(E)^a$-approximation property, for each $\varepsilon > 0$, there exists $R \in \mathcal{K}(E)$ with $\|S'(\mu_n) - R'(S'(\mu_n))\| < \varepsilon$ for each n. As $S \circ R \in \mathcal{K}(E)$, we hence have

$$|\langle S'', \tau\rangle| = |\langle S'', \tau\rangle - \langle \phi_1(\tau), S \circ R\rangle| = \Big|\sum_{n=1}^\infty \langle \Lambda_n, S'(\mu_n) - R'(S'(\mu_n))\rangle\Big| \le \varepsilon \sum_{n=1}^\infty \|\Lambda_n\|.$$

As $\varepsilon > 0$ was arbitrary, we see that $\langle S'', \tau\rangle = 0$, and as $\tau \in \ker \phi_1$ was arbitrary, we see that $S'' \in \theta_1(\mathcal{K}(E)'')$, as required. Then $\mathrm{Id}_{E''} = \mathrm{Id}_E'' \in \theta_1(\mathcal{K}(E)'')$, so again, E' has the metric $\mathcal{K}(E)^a$-approximation property. □

The reason this is weaker than the corresponding result for $\mathcal{A}(E)$ is that we can easily show that $\mathcal{A}(E'') \subseteq \mathcal{A}(E)''$, but we do not know that $\mathcal{K}(E'') \subseteq \mathcal{K}(E)''$. Of course, when E is reflexive, this is not a problem.

Theorem 2.9.11. *Let E be a reflexive Banach space. Then $\mathcal{K}(E)$ is Arens regular, and $\mathcal{K}(E)''$ is identified, by θ_1, with an ideal in $\mathcal{B}(E)$. Furthermore, $\mathcal{K}(E)'' = \mathcal{B}(E)$ if and only if E has the compact approximation property.*

Proof. As E is reflexive, E' has the Radon-Nikodým property, and so $\phi_1 : E\widehat{\otimes}E' \rightarrow \mathcal{K}(E)'$ is a quotient operator, and $\theta_1 : \mathcal{K}(E)'' \rightarrow \mathcal{B}(E)$ is an isometry onto its range. We immediately see that $\mathcal{K}(E)$ is Arens regular (this is also shown in [Dales, 2000, Theorem 2.6.23], and, in a more limited case, in [Palmer, 1985, Theorem 3]). The proof is complete by applying the above proposition. $\square$

Corollary 2.9.12. *Let E be a reflexive Banach space with the compact approximation property. Then E has the metric compact approximation property.* $\square$

Example 2.9.13. In [Willis, 1992], Willis constructs a reflexive Banach space W which has the metric compact approximation property, but which does not have the approximation property. Thus we see that $\mathcal{K}(W)'' = \mathcal{B}(W)$, while $\mathcal{A}(W)''$ is, isometrically, a proper ideal in $\mathcal{B}(W)$. This example answers, in the affirmative, the question asked before Theorem 3 in [Palmer, 1985]. $\square$

There do exist Banach spaces without the compact approximation property, for example, those constructed in [Szankowski, 1978]. In general, however, we do not have a good supply of Banach spaces without the compact approximation property, a fact which explains the slightly hesitant approach taken in this section.

2.9.1 Radicals of biduals of ideals of compact operators

Studying the radicals of $\mathcal{K}(E)''$ is simple, given the work we have already done.

Theorem 2.9.14. *Let E be a Banach space. Then we have*

$$\operatorname{rad}(\mathcal{K}(E)'', \Box) = I_1 = \ker\theta_1 \quad , \quad \operatorname{rad}(\mathcal{K}(E)'', \Diamond) = I_2 = \ker(\mathcal{Q}\circ\theta_1).$$

Proof. Examining the proof of Theorem 2.8.2, we see that we only use properties of I_1, I_2, ψ_1 and ψ_2 which have been established for $\mathcal{K}(E)''$ in Proposition 2.9.5 and Proposition 2.9.2. Thus we simply use the same argument. $\square$

Chapter 3

Ultraproducts and Arens regularity

In this chapter we shall lay out the theory of ultraproducts of Banach spaces, and then extend some known results to dealing with modules. This will involve studying and extending the Principle of Local Reflexivity. We will then derive a representation of the dual of $\mathcal{B}(E)$ as a quotient of the projective tensor product of two ultrapowers, at least when $E = l^p$ for $1 < p < \infty$ and some related examples. In an alternative direction, we shall also give a representation of the dual of $\mathcal{B}(E)$ which is more complicated to analyse in detail, but which will allow us to show that $\mathcal{B}(E)$ is Arens regular for a wide class of Banach spaces.

3.1 Ultraproducts of Banach spaces

The idea of ultraproduct constructions is fundamental to model theory, and was applied to analysis by A. Robinson in his work on non-standard analysis, [Robinson, 1966]. However, ultraproducts of Banach spaces are different to non-standard analysis, and are accessible from a purely functional analysis viewpoint; the study of non-standard hulls of Banach spaces is related, but takes an approach closer to model theory. We will follow the survey article [Heinrich, 1980], and also the book [Haydon et al., 1991]. The key idea of ultraproducts is that they allow us to move from the local to the global.

Recall the notions of filter and ultrafilter from Section 1.3. Let $\mathcal{U}$ be a non-principal ultrafilter on a set I, and let E be a Banach space. We form the Banach space

$$l^\infty(E, I) = \Big\{ (x_i)_{i\in I} \subseteq E : \|(x_i)\| := \sup_{i\in I} \|x_i\| < \infty \Big\}.$$

For $(x_i) \in l^\infty(E, I)$, the family $(\|x_i\|)$ is a bounded subset of the reals (and so lies in a compact topological space), so that we can take its limit along $\mathcal{U}$. Thus define

$$\mathcal{N}_{\mathcal{U}} = \Big\{(x_i)_{i\in I} \in l^\infty(E,I) : \lim_{i\in\mathcal{U}} \|x_i\| = 0\Big\}.$$

We can show that $\mathcal{N}_{\mathcal{U}}$ is a closed subspace of $l^\infty(E,I)$. Thus we can form the quotient space, called the *ultrapower of E with respect to $\mathcal{U}$*,

$$(E)_{\mathcal{U}} := l^\infty(E,I)/\mathcal{N}_{\mathcal{U}}.$$

In general, this space will depend on $\mathcal{U}$, though most properties of $(E)_{\mathcal{U}}$ turn out to be independent of $\mathcal{U}$, as long as $\mathcal{U}$ is sufficiently "large" in some sense.

If we have a family of Banach spaces $(E_i)_{i\in I}$, then we can form $l^\infty((E_i),I)$ and $\mathcal{N}_{\mathcal{U}}$ in an analogous manner, leading to $(E_i)_{\mathcal{U}}$, the *ultraproduct of (E_i) with respect to $\mathcal{U}$*. We will not in general consider ultraproducts, but it should be clear that in many cases the results will carry over from ultrapowers to the more general case.

We can verify that, if $(x_i)_{i\in I}$ represents an equivalence class in $(E)_{\mathcal{U}}$, then

$$\|(x_i)_{i\in I} + \mathcal{N}_{\mathcal{U}}\| = \lim_{i\in I} \|x_i\|.$$

We will abuse notation and write (x_i) for the equivalence class it represents. Of course, whenever we define maps or operations on ultraproducts, these maps or operations will be well-defined, in the sense that they will not depend on the choice of representative.

Lemma 3.1.1. *Let E be a Banach space, $\mathcal{U}$ an ultrafilter on a set I, and $x \in (E)_{\mathcal{U}}$. Then we can find a representative $(x_i) \in l^\infty(E,I)$ of x such that $\|x_i\| = \|x\|$ for each $i \in I$.*

Proof. If $x = 0$, then we are done. Suppose otherwise, and pick a representative $x = (y_i)$. Then pick $z \in E$ with $z \neq 0$, and let

$$x_i = \begin{cases} y_i\|x\|\|y_i\|^{-1} & : y_i \neq 0, \\ z\|x\|\|z\|^{-1} & : y_i = 0. \end{cases}$$

Then $(x_i) \in (E)_{\mathcal{U}}$ and $\|x_i\| = \|x\|$ for each $i \in I$. For each $\varepsilon > 0$ with $\varepsilon < \|x\|$, we have, by the definition of the norm on $(E)_{\mathcal{U}}$, that $U_\varepsilon = \{i \in I : |\|y_i\| - \|x\|| < \varepsilon\} \in \mathcal{U}$. For each $i \in I$, either $y_i = 0$, and so $\|x_i - y_i\| = \|x_i\| = \|x\| > \varepsilon$, or otherwise $\|x_i - y_i\| = \|y_i\|x\|\|y_i\|^{-1} - y_i\| = \|y_i\||\|x\|\|y_i\|^{-1} - 1| = |\|x\| - \|y_i\||$. Thus

$$\{i \in I : \|x_i - y_i\| < \varepsilon\} = \{i \in I : |\|x\| - \|y_i\|| < \varepsilon\} = U_\varepsilon \in \mathcal{U},$$

so that $\lim_{i\in\mathcal{U}} \|x_i - y_i\| \leq \varepsilon$. As $\varepsilon > 0$ was arbitrary, we have $x = (y_i) = (x_i)$, as required. $\square$

We can check that the natural map $E \to (E)_{\mathcal{U}}$, given by mapping $x \in E$ to the constant family (x), is an isometry onto its range. We can check that it is a surjection if and only if E is finite-dimensional (in fact, $(E)_{\mathcal{U}}$ is either finite-dimensional or non-separable).

Definition 3.1.2. An ultrafilter $\mathcal{U}$ is *countably incomplete* when there exists a sequence $(U_n)_{n=1}^{\infty}$ in $\mathcal{U}$ such that $U_1 \supseteq U_2 \supseteq U_3 \cdots$ and such that $\bigcap_n U_n = \emptyset$.

This is a technical condition which we need to impose on ultrafilters. In fact, all the ultrafilters which we shall use will be countably incomplete by the way they are constructed. Notice that every non-principal ultrafilter on a countable set is countably incomplete. We often need to restrict to countably incomplete ultrafilters because of the following lemma, whose application allows us to "embed" the notion of convergence along a sequence into an ultrapower.

Lemma 3.1.3. *Let $\mathcal{U}$ be a countably incomplete ultrafilter on a set I. Then we can find a family of strictly positive reals $(\varepsilon_i)_{i\in I}$ such that $\lim_{i\in I} \varepsilon_i = 0$.*

Proof. Pick a nested sequence (U_n) in $\mathcal{U}$ with $\bigcap_n U_n = \emptyset$. Then, for example, define (ε_i) by

$$\varepsilon_i = \begin{cases} n^{-1} & : i \in U_n \setminus U_{n+1}, \\ 1 & : \text{otherwise}. \end{cases}$$

For each $\delta > 0$, pick $n_\delta \in \mathbb{N}$ with $n_\delta > \delta^{-1}$. Then we have

$$\begin{aligned} \{i \in I : |\varepsilon_i| < \delta\} &\supseteq \{i \in I : |\varepsilon_i| < n_\delta^{-1}\} \\ &= \{i \in I : i \in U_n \setminus U_{n+1} \ (n > n_\delta)\} = U_{n_\delta+1} \in \mathcal{U}, \end{aligned}$$

where the last equality is because $\bigcap_n U_n = \emptyset$. Thus $\{i \in I : |\varepsilon_i| < \delta\} \in \mathcal{U}$, so by definition, $\lim_{i\in\mathcal{U}} \varepsilon_i = 0$. ☐

Let $(T_i)_{i\in I}$ be a bounded family of operators in $\mathcal{B}(E, F)$, for Banach spaces E and F. Then we can define a map, denoted by (T_i), $(E)_{\mathcal{U}} \to (F)_{\mathcal{U}}$ by

$$(T_i) : (x_i) \mapsto (T_i(x_i)).$$

This is well-defined, and we claim that $\|(T_i)\| = \lim_{i\in\mathcal{U}} \|T_i\|$. This follows, as for arbitrary (x_i), we have

$$\|(T_i(x_i))\| = \lim_{i\in\mathcal{U}} \|T_i(x_i)\| \leq \lim_{i\in\mathcal{U}} \|T_i\|\|x_i\| = \left(\lim_{i\in\mathcal{U}} \|T_i\|\right)\left(\lim_{i\in\mathcal{U}} \|x_i\|\right).$$

Conversely, for $\varepsilon > 0$, for each $i \in I$, pick $x_i \in E$ with $\|x_i\| = 1$ and $\|T_i(x_i)\| > \|T_i\| - \varepsilon$. Then

$$\|(T_i(x_i))\| \geq \lim_{i \in \mathcal{U}} \|T_i\| - \varepsilon,$$

so we are done as $\varepsilon > 0$ was arbitrary. Using an argument similar to the above lemma, we can show that the norm is actually attained if $\mathcal{U}$ is countably incomplete. We have shown that the natural map

$$(\mathcal{B}(E, F))_{\mathcal{U}} \to \mathcal{B}((E)_{\mathcal{U}}, (F)_{\mathcal{U}})$$

is an isometry. It is not, in general, surjective; for example, see before Proposition 3.1.12. We thus have a canonical map

$$\mathcal{B}(E, F) \to (\mathcal{B}(E, F))_{\mathcal{U}} \to \mathcal{B}((E)_{\mathcal{U}}, (F)_{\mathcal{U}}),$$

which is an isometry. In general, we simply write T for the image of $T \in \mathcal{B}(E, F)$ in $\mathcal{B}((E)_{\mathcal{U}}, (F)_{\mathcal{U}})$, instead of the more correct (T).

Throughout the rest of this section, unless otherwise stated, $\mathcal{U}$ will be an ultrafilter on an index set I.

Proposition 3.1.4. *The classes of Banach algebras and C*-algebras are stable under ultraproduct constructions.*

Proof. We sketch what is [Heinrich, 1980, Proposition 3.1], in the case of ultrapowers. Let $\mathcal{A}$ be a Banach algebra, and define multiplication on $(\mathcal{A})_{\mathcal{U}}$ by

$$(a_i).(b_i) = (a_i b_i) \qquad ((a_i), (b_i) \in (\mathcal{A})_{\mathcal{U}}).$$

This is well-defined, and turns $(\mathcal{A})_{\mathcal{U}}$ into a Banach algebra. The natural map $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$ is a homomorphism.

If $\mathcal{A}$ is also a C*-algebra, then we define an involution on $(\mathcal{A})_{\mathcal{U}}$ by simply setting $(a_i)^* = (a_i^*)$. Then $(\mathcal{A})_{\mathcal{U}}$ becomes a C*-algebra. □

As the classes of $C(K)$ spaces and unital, commutative C*-algebras form the same class, we see that $C(K)$ spaces are stable under taking ultraproducts.

Theorem 3.1.5. *Let $1 \leq p < \infty$ and let (S_i, Σ_i, ν_i) be a family of measure spaces. For an ultrafilter $\mathcal{U}$, $(L^p(\nu_i))_{\mathcal{U}}$ is (order) isometric to $L^p(S, \Sigma, \nu)$ for some measure space (S, Σ, ν).*

Proof. This is [Heinrich, 1980, Theorem 3.3]. The proof uses the structure theory for abstract L^p spaces. □

Definition 3.1.6. Let E and F be Banach spaces, and $\varepsilon > 0$. Then $T \in \mathcal{B}(E, F)$ is a $(1+\varepsilon)$-*isomorphism* if T is surjective, and we have $(1-\varepsilon)\|x\| \leq \|T(x)\| \leq (1+\varepsilon)\|x\|$ for each $x \in E$. If such a T exists, we say that E is $(1+\varepsilon)$-isomorphic to F.

Similarly, we have the notion of a $(1+\varepsilon)$-isomorphism onto its range.

For Banach spaces E and F, E is *finitely representable* in F if, for each $M \in \mathrm{FIN}(E)$ and each $\varepsilon > 0$, M is $(1+\varepsilon)$-isomorphic to a subspace of F.

Proposition 3.1.7. *Let $(E_i)_{i\in I}$ be a family of Banach spaces, let M be a finite-dimensional subspace of $(E_i)_{\mathcal{U}}$ and let $\varepsilon > 0$. Then there exists $I_0 \in \mathcal{U}$ such that, for each $i \in I_0$, there exists a subspace $M_i \subseteq E_i$ which is $(1+\varepsilon)$-isomorphic to M.*

Proof. This is [Heinrich, 1980, Proposition 6.1]. Let M have a basis $\{x^{(1)}, \ldots, x^{(n)}\}$ and, for $1 \leq k \leq n$, let $x^{(k)} = (x_i^{(k)}) \in (E_i)_{\mathcal{U}}$. For each i, let $M_i = \mathrm{lin}\{x_i^{(k)} : 1 \leq k \leq n\} \in \mathrm{FIN}(E)$, and define $T_i : M \to M_i$ by $T_i(x^{(k)}) = x_i^{(k)}$. Then we can show that, for some $I_0 \in \mathcal{U}$ and each $i \in I_0$, we have that T_i is the required $(1+\varepsilon)$-isomorphism. $\square$

Proposition 3.1.8. *Let F be a Banach space and $\mathcal{C}$ be a family of Banach spaces. Suppose that for each $\varepsilon > 0$ and each $M \in \mathrm{FIN}(F)$, for some $E \in \mathcal{C}$, M is $(1+\varepsilon)$-isomorphic to a subspace of E. Then, for some ultrafilter $\mathcal{U}$ on I, there is a family $(E_i)_{i\in I}$ in $\mathcal{C}$ such that F is isometric to a subspace of $(E_i)_{\mathcal{U}}$.*

Proof. This is [Heinrich, 1980, Proposition 6.2]. $\square$

Theorem 3.1.9. *Let E and F be Banach spaces. Then F is finitely representable in E if and only if F is isometrically isomorphic to a subspace of $(E)_{\mathcal{U}}$ for some $\mathcal{U}$.*

Furthermore, if F is separable, then $\mathcal{U}$ can be any countably incomplete ultrafilter.

Proof. The first part is immediate. For the second, we use an argument similar to the above lemma, where we use the countable incompleteness to allow us to work with sequences, which is enough, as F is separable. See [Heinrich, 1980, Theorem 6.3] for more details. $\square$

The following comes from [James, 1972].

Definition 3.1.10. Let E be a Banach space such that when F is finitely-representable in E, F is reflexive. Then E is *super-reflexive*.

We immediately see that E is super-reflexive if and only if every closed subspace of $(E)_{\mathcal{U}}$ is reflexive, which holds if and only if $(E)_{\mathcal{U}}$ is reflexive for each $\mathcal{U}$. In fact, we can restrict to just one countably incomplete $\mathcal{U}$, by the fact that a Banach space is reflexive if and only if each separable subspace is reflexive (a fact which follows from the Eberlein-Smulian Theorem).

Proposition 3.1.11. *Let $E \in \mathrm{FIN}$ and $\mathcal{U}$ be an ultrafilter. Then the canonical map $E \to (E)_{\mathcal{U}}$ is an isometric isomorphism.*

Proof. As bounded subsets of E are relatively compact, for $(x_i) \in (E)_{\mathcal{U}}$, we know that (x_i) converges along the ultrafilter, so we can define

$$\sigma : (E)_{\mathcal{U}} \to E; (x_i) \mapsto \lim_{i \in \mathcal{U}} x_i.$$

For $(x_i) \in (E)_{\mathcal{U}}$, let $x = \sigma((x_i)) \in E$, so that, by definition, we have $\lim_{i\in\mathcal{U}} \|x - x_i\| = 0$, so that (x_i) and x define the same equivalence class in $(E)_{\mathcal{U}}$. $\square$

We might hope that the dual of $(E)_{\mathcal{U}}$ is $(E')_{\mathcal{U}}$, but this is not in general true. We can of course define a map

$$(E')_{\mathcal{U}} \to (E)'_{\mathcal{U}} \quad , \quad \langle (\mu_i), (x_i) \rangle = \lim_{i \in \mathcal{U}} \langle \mu_i, x_i \rangle \quad ((x_i) \in (E)_{\mathcal{U}}, (\mu_i) \in (E)_{\mathcal{U}}).$$

This map is well-defined, and we can check that it is an isometry. As $E' = \mathcal{B}(E, \mathbb{C})$ and $(\mathbb{C})_{\mathcal{U}} = \mathbb{C}$ by the preceding proposition, we have actually used the map $(\mathcal{B}(E, F))_{\mathcal{U}} \to \mathcal{B}((E)_{\mathcal{U}}, (F)_{\mathcal{U}})$ from above, with $F = \mathbb{C}$.

Proposition 3.1.12. *If $(E)_{\mathcal{U}}$ is reflexive, then $(E)'_{\mathcal{U}} = (E')_{\mathcal{U}}$. Furthermore, if $\mathcal{U}$ is countably incomplete, then $(E)'_{\mathcal{U}} = (E')_{\mathcal{U}}$ implies that $(E)_{\mathcal{U}}$ is reflexive.*

Proof. Suppose that $(E)_{\mathcal{U}}$ is reflexive. Assume that $(E')_{\mathcal{U}}$ forms a proper closed subspace of $(E)'_{\mathcal{U}}$, so that we can find a non-zero $\Phi \in (E)''_{\mathcal{U}}$ with $\langle \Phi, (\mu_i) \rangle = 0$ for each $(\mu_i) \in (E')_{\mathcal{U}}$. As $\Phi = \kappa_{(E)_{\mathcal{U}}}(x)$ for some $x = (x_i) \in (E)_{\mathcal{U}}$, we have $\lim_{i\in\mathcal{U}} \langle \mu_i, x_i \rangle = 0$ for each $(\mu_i) \in (E')_{\mathcal{U}}$, which implies that $x = 0$, a contradiction.

Conversely, if $(E)'_{\mathcal{U}} = (E')_{\mathcal{U}}$ and $\mathcal{U}$ is countably incomplete, then we shall show that each member of $(E)'_{\mathcal{U}}$ attains its norm on $(E)_{\mathcal{U}}$, which means that $(E)_{\mathcal{U}}$ is reflexive, by James's Theorem (see [James(2), 1972], or [Megginson, 1998, Theorem 1.13.11]). Pick $(\mu_i)_{i\in I} \in (E')_{\mathcal{U}} = (E)'_{\mathcal{U}}$ and pick (ε_i) with $\lim_{i\in\mathcal{U}} \varepsilon_i = 0$. For each $i \in I$, choose $x_i \in E$ with $\|x_i\| = 1$ and $|\langle \mu_i, x_i \rangle| > \|\mu_i\| - \varepsilon_i$. Then $x = (x_i)_{i\in I} \in (E)_{\mathcal{U}}$, $\|x\| = 1$ and $|\langle (\mu_i), x \rangle| = \lim_{i\in\mathcal{U}} |\langle \mu_i, x_i \rangle| = \|(\mu_i)\|$. $\square$

We can now show the power of ultrapower techniques. The following results are complicated to prove directly from the definitions (see [James, 1972] and [Enflo et al.,1975] for example).

Theorem 3.1.13. *Let E be a Banach space. Then E is super-reflexive if and only if E' is super-reflexive. Let F be a closed subspace of E. Then if two of the three spaces E, F and E/F are super-reflexive, they all are (the "three-space" problem).*

Proof. Let $\mathcal{U}$ be a countably incomplete ultrafilter. Then the following are equivalent: (1) E is super-reflexive; (2) $(E)_{\mathcal{U}}$ is reflexive; (3) $(E)'_{\mathcal{U}} = (E')_{\mathcal{U}}$. These together imply that $(E')_{\mathcal{U}}$ is reflexive, which implies that E' is super-reflexive. If E' is super-reflexive then E' is reflexive, so that by the above, $E = E''$ is super-reflexive.

The three space problem is (easily seen to be) true for reflexive spaces. So if we can show that $(E/F)_{\mathcal{U}}$ is naturally isomorphic to $(E)_{\mathcal{U}}/(F)_{\mathcal{U}}$, we are done. However, this is trivial, for map $(x_i + F) \in (E/F)_{\mathcal{U}}$ to $(x_i) + (F)_{\mathcal{U}}$ in $(E)_{\mathcal{U}}/(F)_{\mathcal{U}}$. This is well-defined, as if $(x_i + F) = (y_i + F)$ in $(E/F)_{\mathcal{U}}$, we have

$$\begin{aligned} 0 &= \lim_{i\in\mathcal{U}} \|x_i - y_i + F\| = \lim_{i\in\mathcal{U}} \inf\{\|x_i - y_i + z_i\| : z_i \in F\} \\ &= \inf\left\{ \lim_{i\in\mathcal{U}} \|x_i - y_i + z_i\| : (z_i) \in (F)_{\mathcal{U}} \right\} \\ &= \inf\{\|(x_i) - (y_i) + (z_i)\| : (z_i) \in (F)_{\mathcal{U}}\} = \|(x_i) - (y_i) + (F)_{\mathcal{U}}\|. \end{aligned}$$

The equality between lines one and two is trivial, if not obvious. This also shows that our map is an isometry, so we are done. □

From the Principle of Local Reflexivity, Theorem 1.4.9, E'' is finitely-representable in E, for each Banach space E, so that E'' is isometric to a subspace of $(E)_{\mathcal{U}}$, for some $\mathcal{U}$. Actually, Theorem 1.4.9 gives a lot more than this, and we can correspondingly strengthen the ultrapower result. As bounded subsets of E'' are compact in the weak*-topology, we can define a map $\sigma : (E)_{\mathcal{U}} \to E''$ by

$$\sigma((x_i)) = \operatorname*{weak^*\text{-}lim}_{i\in\mathcal{U}} x_i.$$

Thus, for $\mu \in E'$, we have

$$\langle \sigma((x_i)), \mu \rangle = \lim_{i\in\mathcal{U}} \langle \mu, x_i \rangle.$$

Theorem 3.1.14. *Let E be a Banach space. Then for some $\mathcal{U}$, there is an isometry $K : E'' \to (E)_{\mathcal{U}}$ such that $\sigma \circ K = \mathrm{Id}_{E''}$ and K restricted to $\kappa_E(E)$ is just the canonical embedding of E into $(E)_{\mathcal{U}}$.*

Proof. See [Heinrich, 1980, Proposition 6.7]. Also compare with Theorem 3.5.5. □

Similar ideas hold for duals of ultrapowers.

Theorem 3.1.15. *Let E be a Banach space, let $M \in \mathrm{FIN}((E)'_{\mathcal{U}})$, let $N \in \mathrm{FIN}((E)_{\mathcal{U}})$ and let $\varepsilon > 0$. Then there exists $T : M \to (E')_{\mathcal{U}}$ which is a $(1+\varepsilon)$-isomorphism onto its range, such that $T(\mu) = \mu$ for $\mu \in M \cap (E')_{\mathcal{U}}$, and $\langle T(\mu), x\rangle = \langle \mu, x\rangle$ for $\mu \in M$ and $x \in N$.*

Proof. This is [Heinrich, 1980, Theorem 7.3]. □

Corollary 3.1.16. *Let E be a Banach space, and let $\mathcal{U}$ be countably incomplete. Then the conclusions of Theorem 3.1.15 hold when M and N are merely separable closed subspaces. Indeed, we may also set $\varepsilon = 0$.*

Proof. This is [Heinrich, 1980, Corollary 7.5]. □

We can then apply these results to prove an analogy to Theorem 3.1.14. For a Banach space E and ultrafilters $\mathcal{U}$ and $\mathcal{V}$, define a map $J : ((E')_{\mathcal{U}})_{\mathcal{V}} \to (E)'_{\mathcal{U}}$ by

$$\langle J((\mu_j)), (x_i)\rangle = \lim_{j\in\mathcal{V}} \langle (\mu_i^{(j)}), (x_i)\rangle = \lim_{j\in\mathcal{V}} \lim_{i\in\mathcal{U}} \langle \mu_i^{(j)}, x_i\rangle \quad ((x_i) \in (E)_{\mathcal{U}}),$$

where $(\mu_j) \in ((E')_{\mathcal{U}})_{\mathcal{V}}$, and for each $j \in J$, $\mu_j = (\mu_i^{(j)}) \in (E')_{\mathcal{U}}$.

Corollary 3.1.17. *Let E be a Banach space and $\mathcal{U}$ be an ultrafilter. Then there exists an ultrafilter $\mathcal{V}$ and an isometry $K : (E)'_{\mathcal{U}} \to ((E')_{\mathcal{U}})_{\mathcal{V}}$ so that the restriction of K to $(E')_{\mathcal{U}}$ is the canonical map $(E')_{\mathcal{U}} \to ((E')_{\mathcal{U}})_{\mathcal{V}}$, and $J \circ K$ is the identity on $(E)'_{\mathcal{U}}$.* □

For ultrafilters $\mathcal{U}$ and $\mathcal{V}$ on index sets I and J respectively, and a Banach space E, then we can form the iterated ultrapower $((E)_{\mathcal{U}})_{\mathcal{V}}$. Define $\mathcal{U} \times \mathcal{V}$ to be the collection of subsets $K \subseteq I \times J$ such that

$$\{j \in J : \{i \in I : (i,j) \in K\} \in \mathcal{U}\} \in \mathcal{V}.$$

Then we can show that $\mathcal{U} \times \mathcal{V}$ is an ultrafilter on $I \times J$.

Lemma 3.1.18. *Let $I, J, \mathcal{U}, \mathcal{V}$ and E be as above. Let $(x_{ij})_{i\in I, j\in J}$ be a family in a Hausdorff topological space X. Then*

$$\lim_{j\in\mathcal{V}} \left(\lim_{i\in\mathcal{U}} x_{ij} \right) = \lim_{(i,j)\in\mathcal{U}\times\mathcal{V}} x_{ij},$$

where one limit exists if and only if the other one does. Consequently, $((E)_{\mathcal{U}})_{\mathcal{V}}$ is canonically isometrically isomorphic to $(E)_{\mathcal{U}\times\mathcal{V}}$.

Proof. Suppose the left-hand limit exists and equals $x \in X$. Let $\mathcal{O}$ be an open neighbourhood of x, and let $J_0 \in \mathcal{V}$ be the set of $j \in J$ such that $y_j = \lim_{i\in\mathcal{U}} x_{ij}$ exists. Then we have

$$J_1 = \{j \in J_0 : y_j \in \mathcal{O}\} \in \mathcal{V}.$$

For $j \in J_1$, $y_j \in \mathcal{O}$, so we have

$$U_j = \{i \in I : x_{ij} \in \mathcal{O}\} \in \mathcal{U}.$$

Let $K = \{(i,j) \in I \times J : j \in J_1, i \in U_j\}$. Then

$$\begin{aligned}\{j \in J : \{i \in I : (i,j) \in K\} \in \mathcal{U}\} &= \{j \in J : \{i \in I : j \in J_1, i \in U_j\} \in \mathcal{U}\} \\ &= \{j \in J : j \in J_1, U_j \in \mathcal{U}\} = J_1 \in \mathcal{V},\end{aligned}$$

so that $K \in \mathcal{U} \times \mathcal{V}$. As $(i,j) \in K$ if and only if $y_j \in \mathcal{O}$ and $x_{ij} \in \mathcal{O}$, and $\mathcal{O}$ was arbitrary, we see that

$$x = \lim_{(i,j)\in\mathcal{U}\times\mathcal{V}} x_{ij}.$$

The converse is similar.

For $x = (x_j) \in ((E)_{\mathcal{U}})_{\mathcal{V}}$, for each j let $x_j = (x_{ij}) \in (E)_{\mathcal{U}}$, and let $T(x) = (x_{ij}) \in (E)_{\mathcal{U}\times\mathcal{V}}$. Then

$$\|T(x)\| = \lim_{(i,j)\in\mathcal{U}\times\mathcal{V}} \|x_{ij}\| = \lim_{j\in\mathcal{V}} \lim_{i\in\mathcal{U}} \|x_{ij}\| = \lim_{j\in\mathcal{V}} \|x_j\| = \|x\|,$$

so that $T : ((E)_{\mathcal{U}})_{\mathcal{V}} \to (E)_{\mathcal{U}\times\mathcal{V}}$ is well-defined, and an isometry. We can similarly check that T is a surjection. □

We see immediately that E is super-reflexive if and only if each ultrapower of E is super-reflexive.

3.2 Ultrapowers and tensor products

We will see later that it would be nice to have a relationship between $(E\widehat{\otimes}F)_{\mathcal{U}}$ and $(E)_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}}$. For $M \in$ FIN, by Proposition 3.1.11, we have $(M)_{\mathcal{U}} = M$. From [Heinrich, 1980, Lemma 7.4], we have

$$(M\check{\otimes}E)_{\mathcal{U}} = M\check{\otimes}(E)_{\mathcal{U}} \quad , \quad (M\widehat{\otimes}E)_{\mathcal{U}} = M\widehat{\otimes}(E)_{\mathcal{U}}$$

for every Banach space E and $M \in$ FIN, with equality of norms. Note that we can re-write these equalities in terms of spaces of operators as $(\mathcal{A}(M',E))_{\mathcal{U}} = \mathcal{A}(M',(E)_{\mathcal{U}})$ and $(\mathcal{N}(M',E))_{\mathcal{U}} = \mathcal{N}(M',(E)_{\mathcal{U}})$.

For infinite-dimensional Banach spaces, these equalities are no longer necessarily true. However, we can make some useful statements, and also explain why we shall have to work harder to prove related results in later sections.

Define a map $\psi_0 : (E)_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}} \to (E\widehat{\otimes}F)_{\mathcal{U}}$. We do this by using the tensorial property of $\widehat{\otimes}$. Firstly we define $\psi_0 : (E)_{\mathcal{U}} \times (F)_{\mathcal{U}} \to (E\widehat{\otimes}F)_{\mathcal{U}}$ by

$$\psi_0(x,y) = (x_i \otimes y_i) \qquad (x = (x_i) \in (E)_{\mathcal{U}}, y = (y_i) \in (F)_{\mathcal{U}}).$$

Then we have

$$\|(x_i \otimes y_i)\| = \lim_{i\in\mathcal{U}} \|x_i \otimes y_i\| = \lim_{i\in\mathcal{U}} \|x_i\|\|y_i\| = \Big(\lim_{i\in\mathcal{U}} \|x_i\|\Big)\Big(\lim_{i\in\mathcal{U}} \|y_i\|\Big) = \|x\|\|y\|,$$

so that ψ_0 is well-defined, and is a norm-decreasing bilinear map. Thus ψ_0 extends to a norm-decreasing map $\psi_0 : (E)_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}} \to (E\widehat{\otimes}F)_{\mathcal{U}}$. For $u \in (E)_{\mathcal{U}} \otimes (F)_{\mathcal{U}}$, choose a representative $u = \sum_{k=1}^n x_k \otimes y_k$. Let, for each k, $x_k = (x_i^{(k)}) \in (E)_{\mathcal{U}}$ and $y_k = (y_i^{(k)}) \in (E)_{\mathcal{U}}$. Then we see that

$$\psi_0(u) = \left(\sum_{k=1}^n x_i^{(k)} \otimes y_i^{(k)}\right)_{i\in I} \in (E\widehat{\otimes}F)_{\mathcal{U}}.$$

The following idea is based upon a private communication to the author by C.J. Read (in particular, Example 3.2.3 below). For Banach spaces E and F, let

$$\Lambda_n = \left\{\sum_{i=1}^n e_i \otimes f_i \in E\widehat{\otimes}F\right\} \qquad (n \in \mathbb{N}).$$

Then $\operatorname{lin}\{\Lambda_n : n \in \mathbb{N}\} = \cup_{n=1}^\infty \Lambda_n = E \otimes F$. Then define

$$d_n(\tau) = \inf\{\pi(\tau - \sigma, E\widehat{\otimes}F) : \sigma \in \Lambda_n\} \qquad (\tau \in E\widehat{\otimes}F).$$

Proposition 3.2.1. *Let E and F be Banach spaces, and let $\tau = (\tau_i) \in (E\widehat{\otimes}F)_{\mathcal{U}}$ be in the image of ψ_0. Then*

$$\lim_{n\to\infty} \lim_{i\in\mathcal{U}} d_n(\tau_i) = 0.$$

Proof. Let $\tau = \psi_0(\sigma)$ for some $\sigma \in (E)_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}}$, where we have

$$\sigma = \sum_{k=1}^\infty (x_i^{(k)}) \otimes (y_i^{(k)}),$$

with $\|x_i^{(k)}\| = \|y_i^{(k)}\| = \alpha_k$ for each $k \in \mathbb{N}$ and $i \in I$, and such that $\sum_{k=1}^\infty \alpha_k^2 < \infty$. Thus we have

$$\tau_i = \sum_{k=1}^\infty x_i^{(k)} \otimes y_i^{(k)} \qquad (i \in I).$$

In particular, for each $i \in I$, we have

$$d_n(\tau_i) \leq \pi\Big(\tau_i - \sum_{k=1}^n x_i^{(k)} \otimes y_i^{(k)}, E\widehat{\otimes}F\Big) \leq \sum_{k=n+1}^\infty \|x_i^{(k)}\|\|y_i^{(k)}\| = \sum_{k=n+1}^\infty \alpha_k^2.$$

Hence we have

$$\lim_{n\to\infty}\lim_{i\in\mathcal{U}} d_n(\tau_i) \leq \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\alpha_k^2 = 0,$$

as required. $\square$

Corollary 3.2.2. *Let E and F be Banach spaces and $\mathcal{U}$ be a countably incomplete ultrafilter on an index set I. Suppose that $\psi_0 : (E)_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}} \to (E\widehat{\otimes}F)_{\mathcal{U}}$ has dense range. Then we have*

$$\lim_{n\to\infty}\sup\left\{d_n(\tau) : \tau\in E\widehat{\otimes}F, \pi(\tau)\leq 1\right\} = 0.$$

Proof. Define

$$\delta_n = \sup\left\{d_n(\tau) : \tau\in E\widehat{\otimes}F, \pi(\tau)\leq 1\right\} \qquad (n\in\mathbb{N}).$$

Then (δ_n) is a decreasing sequence of positive reals. Suppose towards a contradiction that (δ_n) does not tend to zero. Then we can find $\delta > 0$ such that for each $n\in\mathbb{N}$, we can find $\tau_n\in E\widehat{\otimes}F$ with $\pi(\tau_n)\leq 1$ and $d_n(\tau_n)\geq\delta$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and let $\tau = (\tau_n)\in(E\widehat{\otimes}F)_{\mathcal{U}}$. Let $\sigma\in(E)_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}}$, and let $\psi_0(\sigma) = (\sigma_i)\in(E\widehat{\otimes}F)_{\mathcal{U}}$. For $n\in\mathbb{N}$, notice that

$$d_n(\tau_i)\leq\pi(\tau_i-\sigma_i)+d_n(\sigma_i) \qquad (i\in\mathbb{N}).$$

By the above proposition, we hence have

$$\begin{aligned}\lim_{n\to\infty}\lim_{i\in\mathcal{U}} d_n(\tau_i) &\leq \lim_{n\to\infty}\lim_{i\in\mathcal{U}}\Big(\pi(\tau_i-\sigma_i)+d_n(\sigma_i)\Big)\\ &= \|\tau-\psi_0(\sigma)\| + \lim_{n\to\infty}\lim_{i\in\mathcal{U}} d_n(\sigma_i) = \|\tau-\psi_0(\sigma)\|.\end{aligned}$$

Now, for $i\geq n$, we have $d_n(\tau_i)\geq d_i(\tau_i)\geq\delta$. Thus we have

$$\|\tau-\psi_0(\sigma)\|\geq\lim_{n\to\infty}\lim_{i\in\mathcal{U}} d_n(\tau_i)\geq\lim_{n\to\infty}\delta=\delta>0.$$

Hence ψ_0 does not have dense range, giving us the required contradiction.

We can easily adapt the above argument to allow $\mathcal{U}$ to be an arbitrary countably incomplete ultrafilter. $\square$

Example 3.2.3. We consider l^2. Let $(e_n)_{n=1}^{\infty}$ be the standard orthonormal basis for l^2. Define

$$\tau_n = n^{-1}\sum_{i=1}^{n} e_i\otimes e_i\in l^2\widehat{\otimes}l^2 \qquad (n\in\mathbb{N}),$$

so that, by [Ryan, 2002, Example 2.10], $\pi(\tau_n, l^2\widehat{\otimes}l^2) = 1$.

Suppose that $\sigma = \sum_{j=1}^{N} x_j\otimes y_j\in l^2\otimes l^2$. As $\mathcal{A}(l^2)' = \mathcal{N}(l^2) = l^2\widehat{\otimes}l^2$, we have

$$\pi(\sigma-\tau_n, l^2\widehat{\otimes}l^2) = \sup\{|\langle\sigma-\tau_n, T\rangle| : T\in\mathcal{F}(l^2), \|T\|\leq 1\} \qquad (n\in\mathbb{N}).$$

Suppose that $n \geq N$, and define $P_n \in \mathcal{F}(l^2)$ by $P_n(e_j) = e_j$ for $j \leq n$, and $P_n(e_j) = 0$ for $j > n$. Then $\|P_n\| = 1$, and consequently we have

$$\pi(\sigma - \tau_n, l^2 \widehat{\otimes} l^2) \geq \sup\{|\langle \sigma - \tau_n, P_n T P_n \rangle| : T \in \mathcal{F}(l^2), \|T\| = 1\}.$$

We can verify that $\langle \tau_n, P_n T P_n \rangle = \langle \tau_n, T \rangle$ and $\langle \sigma, P_n T P_n \rangle = \langle P_n \cdot \sigma \cdot P_n, T \rangle$ where

$$P_n \cdot \sigma \cdot P_n = \sum_{j=1}^{N} P_n(x_j) \otimes P_n(y_j).$$

Hence, if we are trying to make $\pi(\sigma - \tau_n, l^2 \widehat{\otimes} l^2)$ small, we may assume that $\sigma = P_n \cdot \sigma \cdot P_n$. That is, we may assume x_j and y_j are in $E_n = \operatorname{lin}\{e_i : 1 \leq i \leq n\}$ for each j. By taking linear combinations and decreasing N if necessary, we can suppose that $(x_j)_{j=1}^N$ is an orthonormal set in E_n. Then choose $(x_j)_{j=N+1}^n$ such that $(x_j)_{j=1}^n$ is an orthonormal basis for E_n. Thus we can write $\tau_n = n^{-1} \sum_{j=1}^n x_j \otimes x_j$, so that

$$\tau_n - \sigma = \sum_{j=1}^{N} x_j \otimes (n^{-1} x_j - y_j) + n^{-1} \sum_{j=N+1}^{n} x_j \otimes x_j. \tag{3.1}$$

Finally, define $Q \in \mathcal{F}(E_n)$ by $Q(x_j) = x_j$ for $j > N$, and $Q(x_j) = 0$ for $j \leq N$. Isometrically extend Q to l^2, so that $\|Q\| = 1$, and by equation 3.1, we have

$$\begin{aligned}\pi(\tau_n - \sigma, l^2 \widehat{\otimes} l^2) \geq |\langle \tau_n - \sigma, Q \rangle| &= \left| \sum_{j=1}^{N} \langle Q(x_j), n^{-1} x_j - y_j \rangle + n^{-1} \sum_{j=N+1}^{n} \langle Q(x_j), x_j \rangle \right| \\ &= \frac{n - N}{n}.\end{aligned}$$

We conclude that $d_N(\tau_n) \geq n^{-1}(n - N)$ for $n \geq N$, and so $\delta_N = 1$ for each $N \in \mathbb{N}$. Thus $\psi_0 : (l^2)_{\mathcal{U}} \widehat{\otimes} (l^2)_{\mathcal{U}} \to (l^2 \widehat{\otimes} l^2)_{\mathcal{U}}$ does not have dense range. $\square$

We note that the above example relies rather heavily on the fact that any $M \in \operatorname{FIN}(l^2)$ admits a projection $P : E \to M$ with $\|P\| = 1$. Thus perhaps there is some hope of ψ_0 being surjective for Banach spaces which admit far fewer operators than l^2.

For the following, we refer the reader to [Heinrich, 1980, Section 9], where Heinrich gives a description of when $(E)_{\mathcal{U}}$ has the approximation property. In particular, $(E)_{\mathcal{U}}$ has the approximation property for some non-principal $\mathcal{U}$ if and only if $(E)_{\mathcal{V}}$ has the approximation property for all $\mathcal{V}$. Notice that, by Theorem 3.1.5, $(L^P(\nu))_{\mathcal{U}}$ has the approximation property for any measure ν and $1 \leq p \leq \infty$.

Proposition 3.2.4. *Let E and F be Banach spaces such that F is super-reflexive. Let $\mathcal{U}$ be an ultrafilter such that $(F)_{\mathcal{U}}$ has the approximation property. Then $\psi_0 : (E)_{\mathcal{U}} \widehat{\otimes} (F)_{\mathcal{U}} \to (E \widehat{\otimes} F)_{\mathcal{U}}$ is an isometry onto its range.*

Proof. As $(F)_{\mathcal{U}}$ is reflexive and $(F)_{\mathcal{U}}$ has the approximation property, we have

$$\mathcal{A}((E)_{\mathcal{U}}, (F')_{\mathcal{U}})' = ((E)'_{\mathcal{U}}\check{\otimes}(F')_{\mathcal{U}})' = \mathcal{I}((E)'_{\mathcal{U}}, (F)_{\mathcal{U}}) = (E)''_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}}.$$

As the map $\kappa_{(E)_{\mathcal{U}}} \otimes \mathrm{Id} : (E)_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}} \to (E)''_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}}$ is an isometry onto its range by Proposition 2.1.7, we see that

$$\pi(\sigma) = \sup\{|\langle \sigma, S\rangle| : S \in \mathcal{F}((E)_{\mathcal{U}}, (F')_{\mathcal{U}})_{[1]}\} \qquad (\sigma \in (E)_{\mathcal{U}}\widehat{\otimes}(F)_{\mathcal{U}}).$$

Fix $\sigma \in (E)_{\mathcal{U}} \otimes (F)_{\mathcal{U}}$. Let $\sigma = \sum_{k=1}^{n} y^{(k)} \otimes z^{(k)}$ and let $N = \mathrm{lin}\{y^{(k)} : 1 \leq k \leq n\} \in \mathrm{FIN}((F)_{\mathcal{U}})$. For each k, let $y^{(k)} = (y_i^{(k)})$ and $z^{(k)} = (z_i^{(k)})$ where $\|y^{(k)}\| = \|y_i^{(k)}\|$ and $\|z^{(k)}\| = \|z_i^{(k)}\|$ for each i. Thus

$$\psi_0(\sigma) = (\sigma_i) = \Big(\sum_{k=1}^{n} y_i^{(k)} \otimes z_i^{(k)}\Big)_{i\in I}.$$

Choose $\varepsilon > 0$ and let $S \in \mathcal{F}((E)_{\mathcal{U}}, (F')_{\mathcal{U}})$ be such that $\|S\| \leq 1$ and $|\langle \sigma, S\rangle| > \pi(\sigma) - \varepsilon$. Let $M = S((E)_{\mathcal{U}}) \in \mathrm{FIN}((F')_{\mathcal{U}})$ have a basis $\{x^{(1)}, \dots, x^{(m)}\}$ where $x^{(k)} = (x_i^{(k)}) \in (E')_{\mathcal{U}}$ for each k. As in the proof of Proposition 3.1.7, let $M_i = \mathrm{lin}\{x_i^{(k)} : 1 \leq k \leq m\} \in \mathrm{FIN}(F')$ and $T_i : M \to M_i$ be defined by $T_i(x^{(k)}) = x_i^{(k)}$. Then, for some $I_0 \in \mathcal{U}$, T_i is a $(1+\varepsilon)$-isomorphism for each $i \in I_0$.

We can write $S = \sum_{k=1}^{m} \mu^{(k)} \otimes x^{(k)}$ for some $(\mu^{(k)})_{k=1}^{m} \subseteq (E)'_{\mathcal{U}}$. Let $P = \mathrm{lin}\{\mu^{(k)} : 1 \leq k \leq m\}$. By Theorem 3.1.15, let $T : P \to (E')_{\mathcal{U}}$ be such that $\|T\| \leq 1+\varepsilon$ and

$$\langle T(\mu^{(k)}), z\rangle = \langle \mu^{(k)}, z\rangle \qquad (1 \leq k \leq m, z \in N).$$

For each k, let $T(\mu^{(k)}) = (\mu_i^{(k)}) \in (E')_{\mathcal{U}}$. Then let $Q = T(P)$, let $Q_i = \mathrm{lin}\{\mu_i^{(k)} : 1 \leq k \leq m\} \in \mathrm{FIN}(E')$ and let $R_i : Q \to Q_i$ be given by $R_i(T(\mu^{(k)})) = \mu_i^{(k)}$. Let $I_1 \in \mathcal{U}$ be such that R_i is a $(1+\varepsilon)$-isomorphism for each $i \in I_1$.

For each $i \in I_0 \cap I_1$, let

$$S_i = \sum_{k=1}^{m} R_iT(\mu^{(k)}) \otimes T_i(x^{(k)}) = (R_iT \otimes T_i)S \in F' \otimes E' = \mathcal{F}(F, E'),$$

so that $\|S_i\| \leq \|R_i\|\|T\|\|T_i\|\|S\| \leq (1+\varepsilon)^3$. Then we have

$$\begin{aligned}\langle \sigma, S\rangle &= \sum_{j=1}^{n} \langle S(y^{(j)}), z^{(j)}\rangle = \sum_{j=1}^{n}\sum_{k=1}^{m} \langle \mu^{(k)}, y^{(j)}\rangle\langle x^{(k)}, z^{(j)}\rangle \\ &= \sum_{j=1}^{n}\sum_{k=1}^{m} \langle T(\mu^{(k)}), y^{(j)}\rangle\langle x^{(k)}, z^{(j)}\rangle = \sum_{j=1}^{n}\sum_{k=1}^{m} \lim_{i\in\mathcal{U}} \langle \mu_i^{(k)}, y_i^{(j)}\rangle\langle x_i^{(k)}, z_i^{(j)}\rangle \\ &= \lim_{i\in\mathcal{U}} \sum_{j=1}^{n}\sum_{k=1}^{m} \langle R_iT(\mu^{(k)}), y_i^{(j)}\rangle\langle T_i(x^{(k)}), z_i^{(j)}\rangle = \lim_{i\in\mathcal{U}} \langle S_i, \sigma_i\rangle.\end{aligned}$$

As ψ_0 is norm-decreasing, we conclude that

$$\pi(\sigma) - \varepsilon < \lim_{i\in\mathcal{U}} |\langle S_i, \sigma_i\rangle| \leq \lim_{i\in\mathcal{U}} \|S_i\| \pi(\sigma_i) \leq (1+\varepsilon)^3 \|\psi_0(\sigma)\| \leq (1+\varepsilon)^3 \pi(\sigma).$$

As $\varepsilon > 0$ was arbitrary, we see that $\pi(\sigma) = \|\psi_0(\sigma)\|$, and so by continuity, we see that ψ_0 is an isometry onto its range. □

If we examine the above proof, then we could weaken the conditions on F to be that $(F)''_{\mathcal{U}}$ has the approximation property and the Radon-Nikodým property (and then use Theorem 3.1.15 again to move form $(F)''_{\mathcal{U}}$ to $(F)_{\mathcal{U}}$).

It seems that ψ_0 is rarely, if ever, surjective. For a Banach space E, by Theorem 3.1.14, we see that $\mathcal{B}(E')'$ is a one-complemented subspace of $(E'\widehat{\otimes}E)_{\mathcal{U}}$ for some $\mathcal{U}$. We can thus form the chain of maps

$$(E')_{\mathcal{U}}\widehat{\otimes}(E)_{\mathcal{U}} \xrightarrow{\psi_0} (E'\widehat{\otimes}E)_{\mathcal{U}} \twoheadrightarrow \mathcal{B}(E')',$$

and can ask the question of whether the composition map is surjective, or has dense range. We will see later that for $E = l^p$ for $1 < p < \infty$ (and some other similar cases), this is indeed true (see Theorem 3.4.18). In the other direction, we have the following. We identify $\mathcal{K}(E)$ with the subspace $\mathcal{K}(E)^a \subseteq \mathcal{B}(E')$ so that $\mathcal{K}(E)'$ is a quotient of $\mathcal{B}(E')'$.

Proposition 3.2.5. *Let E be a Banach space and $\mathcal{U}$ be an ultrafilter such that the map*

$$\psi : (E')_{\mathcal{U}}\widehat{\otimes}(E)_{\mathcal{U}} \to (E'\widehat{\otimes}E)_{\mathcal{U}} \to \mathcal{B}(E')' \to \mathcal{K}(E)'$$

has dense range. Then the map $V : E''\widehat{\otimes}E' \to \mathcal{K}(E)'$ has dense range (compare to Theorem 2.9.6).

Proof. Let $T \in \mathcal{K}(E)$, and let $u = \sum_{n=1}^\infty (\mu_i^{(n)}) \otimes (x_i^{(n)}) \in (E')_{\mathcal{U}}\widehat{\otimes}(E)_{\mathcal{U}}$, where we have $\|x_i^{(n)}\| = \|x_n\|$ and $\|\mu_i^{(n)}\| = \|\mu_n\|$ for each $i \in I$ and $n \in \mathbb{N}$. As $T(E_{[1]})$ is a relatively compact subset of E, we see that the limits

$$y_n = \lim_{i\in\mathcal{U}} T(x_i^{(n)}) \in E \qquad (n \in \mathbb{N})$$

exist. Similarly, as $E'_{[1]}$ and $E''_{[1]}$ are weak*-compact, we can let $\lambda_n = \text{weak}^*\text{-}\lim_{i\in\mathcal{U}} \mu_i^{(n)}$, and $\Phi_n = \text{weak}^*\text{-}\lim_{i\in\mathcal{U}} \kappa_E(x_i^{(n)}) \in E''$. Then we have

$$\begin{aligned}\langle \psi(u), T\rangle &= \sum_{n=1}^\infty \lim_{i\in\mathcal{U}} \langle \mu_i^{(n)}, T(x_i^{(n)})\rangle = \sum_{n=1}^\infty \lim_{i\in\mathcal{U}} \langle \mu_i^{(n)}, y_n\rangle = \sum_{n=1}^\infty \langle \lambda_n, y_n\rangle \\ &= \sum_{n=1}^\infty \lim_{i\in\mathcal{U}} \langle \lambda_n, T(x_i^{(n)})\rangle = \sum_{n=1}^\infty \lim_{i\in\mathcal{U}} \langle T'(\lambda_n), x_i^{(n)}\rangle = \sum_{n=1}^\infty \langle \Phi_n, T'(\lambda_n)\rangle \\ &= \langle V\Big(\sum_{n=1}^\infty \Phi_n \otimes \lambda_n\Big), T\rangle.\end{aligned}$$

Thus we are done. □

The key point in this proof is that the limit $y_n = \lim_{i\in\mathcal{U}} T(x_i^{(n)})$ is in norm, and not in a weak topology. Suppose that E has the approximation property, so that $\mathcal{A}(E) = \mathcal{K}(E)$, and suppose that ψ has dense range. Then $\mathcal{A}(E)' = \mathcal{I}(E')$, so we conclude that $V(E''\widehat{\otimes}E') = \mathcal{N}(E')$ is dense in $\mathcal{I}(E')$. Consequently, there is a wide class of Banach spaces (for example, those such that E' has the bounded approximation property and $\mathcal{I}(E') \neq \mathcal{N}(E')$) for which ψ (and so certainly also ψ_0) has no hope of having dense range.

We have a way of getting around this problem, which is to replace the Banach space E by one which encodes some of the "summation" structure which we lose when moving from $(E'\widehat{\otimes}E)_{\mathcal{U}}$ to $(E')_{\mathcal{U}}\widehat{\otimes}(E)_{\mathcal{U}}$.

3.3 Arens regularity of $\mathcal{B}(E)$

It has been brought to my attention, by Volker Runde, that some of the following argument is essentially the same as that used in [Cowling, Fendler, 1984]. I was unaware of this paper at the time of writing the article [Daws, 2004], but wish to state that, in particular, the construction used in Theorem 3.3.1 below is not new. Indeed, one could use the main result from [Cowling, Fendler, 1984] in place of Theorem 3.3.1, if one wished. The application to Arens products remains new, however.

Let E be a Banach space. We cannot ensure that the map $(E')_{\mathcal{U}}\widehat{\otimes}(E)_{\mathcal{U}} \to \mathcal{B}(E)'$ is surjective. However, fix $p \in (1,\infty)$ and turn $l^p(E)$ into a Banach left $\mathcal{B}(E)$-module by defining the module action co-ordinate wise. That is,

$$T\cdot(x_n) = (T(x_n)) \qquad ((x_n)\in l^p(E), T\in\mathcal{B}(E)).$$

Then, for each ultrafilter $\mathcal{U}$, $(l^p(E))_{\mathcal{U}}$ becomes a Banach left $\mathcal{B}(E)$-module in a similar way, by the canonical map $\mathcal{B}(l^p(E)) \to \mathcal{B}((l^p(E))_{\mathcal{U}})$. Thus we can form the map

$$\phi_1 : (l^p(E))''_{\mathcal{U}}\widehat{\otimes}(l^p(E))'_{\mathcal{U}} \to \mathcal{B}(E)',$$

and thus get the first Arens representation, $\theta_1 : \mathcal{B}(E)'' \to \mathcal{B}((l^p(E))''_{\mathcal{U}})$.

Theorem 3.3.1. *Let E be a Banach space, and let $p\in(1,\infty)$. For some ultrafilter $\mathcal{U}$, the map ϕ_1, as defined above, is a quotient operator.*

Proof. The map $\mathcal{B}(E) \to \mathcal{B}(E)^{aa} \subseteq \mathcal{B}(E'')$ is an isometry, so we can view $\mathcal{B}(E)'$ as a quotient of $\mathcal{B}(E'')'$. As $(E''\widehat{\otimes}E')' = \mathcal{B}(E'')$, for some ultrafilter $\mathcal{U}$, by Theorem 3.1.14, we can find an isometry $K : \mathcal{B}(E'')' \to (E''\widehat{\otimes}E')_{\mathcal{U}}$ such that $\sigma \circ K = \mathrm{Id}_{\mathcal{B}(E'')'}$, where $\sigma : (E''\widehat{\otimes}E')_{\mathcal{U}} \to \mathcal{B}(E'')'$ is the usual map.

Pick $\lambda \in \mathcal{B}(E)'$ with $\lambda \neq 0$. Then there exists $\Lambda \in \mathcal{B}(E'')'$ with $\|\Lambda\| = \|\lambda\|$ and such that $\langle \Lambda, T''\rangle = \langle \lambda, T\rangle$ for each $T \in \mathcal{B}(E)$. Let $(u_i) = K(\Lambda)$, where we may suppose that $\pi(u_i, E''\widehat{\otimes}E') = \|\Lambda\|$ for each i. As we may suppose that $\mathcal{U}$ is countably incomplete, pick (ε_i) in $\mathbb{R}^{>0}$ so that $\lim_{i\in\mathcal{U}} \varepsilon_i = 0$. Then we can find representatives $u_i = \sum_{n=1}^\infty \Phi_n^i \otimes \mu_n^i \in E''\widehat{\otimes}E'$ so that, for each i,

$$\pi(u_i) \leq \sum_{n=1}^\infty \|\Phi_n^i\|\|\mu_n^i\| \leq \pi(u_i) + \varepsilon_i.$$

Let q be the conjugate index to p, so that $q^{-1} = 1 - p^{-1}$. For each i and n, let

$$\Psi_n^i = \|\Phi_n^i\|^{-1+1/p}\|\mu_n^i\|^{1/p}\Phi_n^i \quad , \quad \lambda_n^i = \|\Phi_n^i\|^{1/q}\|\mu_n^i\|^{-1+1/q}\mu_n^i.$$

Then we have, for each i,

$$\left(\sum_{n=1}^\infty \|\Psi_n^i\|^p\right)^{1/p} = \left(\sum_{n=1}^\infty \|\Phi_n^i\|\|\mu_n^i\|\right)^{1/p} \leq (\|\Lambda\| + \varepsilon_i)^{1/p},$$

and similarly

$$\left(\sum_{n=1}^\infty \|\lambda_n^i\|^q\right)^{1/q} \leq (\|\Lambda\| + \varepsilon_i)^{1/q}.$$

Thus, for each i, let $\Psi_i = (\Psi_n^i) \in l^p(E'')$ and $\lambda_i = (\lambda_n^i) \in l^q(E')$. Then let $\lambda = (\lambda_i) \in (l^q(E'))_{\mathcal{U}} \subseteq (l^p(E))'_{\mathcal{U}}$, and let $\Psi = (\Psi_i) \in (l^p(E''))_{\mathcal{U}}$. As $(l^q(E'))_{\mathcal{U}}$ is a closed subspace of $(l^p(E))'_{\mathcal{U}}$, and $(l^p(E''))_{\mathcal{U}}$ is a closed subspace of $(l^q(E'))'_{\mathcal{U}}$, we can let Ψ_0 be a Hahn-Banach extension of Ψ, so that $\Psi_0 \in (l^p(E))''_{\mathcal{U}}$.

Now let $u = \Psi_0 \otimes \lambda$, so that

$$\|u\| = \|\Psi_0\|\|\lambda\| = \|\Psi\|\|\lambda\| = \left(\lim_{i\in\mathcal{U}} \|\Psi_i\|\right)\left(\lim_{i\in\mathcal{U}} \|\lambda_i\right) = \|\Lambda\|^{1/p}\|\Lambda\|^{1/q} = \|\lambda\| = \|\Lambda\|.$$

For $T \in \mathcal{B}(E)$ and $x = (x_i) = (x_n^i) \in (l^p(E))_{\mathcal{U}}$, we have

$$\langle \lambda \cdot T, x\rangle = \langle \lambda, T \cdot x\rangle = \langle (\lambda_n^i), (T(x_n^i))\rangle = \lim_{i\in\mathcal{U}} \sum_{n=1}^\infty \langle \lambda_n^i, T(x_n^i)\rangle = \lim_{i\in\mathcal{U}} \sum_{n=1}^\infty \langle T'(\lambda_n^i), x_n^i\rangle,$$

so that $\lambda \cdot T \in (l^q(E'))_{\mathcal{U}}$. Thus we have

$$\begin{aligned}\langle \phi_1(u), T\rangle = \langle \Psi_0, \lambda \cdot T\rangle = \langle \Psi, \lambda \cdot T\rangle &= \lim_{i\in\mathcal{U}} \sum_{n=1}^\infty \langle \Psi_n^i, T'(\lambda_n^i)\rangle \\ &= \lim_{i\in\mathcal{U}} \sum_{n=1}^\infty \langle \Phi_n^i, T'(\mu_n^i)\rangle = \lim_{i\in\mathcal{U}} \langle u_i, T\rangle = \langle \sigma(u_i), T\rangle = \langle \lambda, T\rangle.\end{aligned}$$

This completes the proof. □

We can show similarly that $\phi_2 : (l^p(E))'_{\mathcal{U}}\widehat{\otimes}(l^p(E))_{\mathcal{U}} \to \mathcal{B}(E)'$ is a quotient operator. By Theorem 3.1.14, $(l^p(E))''_{\mathcal{U}}$ is a complemented subspace of $(l^p(E))_{\mathcal{U}\times\mathcal{V}}$ for some ultrafilter $\mathcal{V}$. That is, we can find an isometry $K : (l^p(E))''_{\mathcal{U}} \to (l^p(E))_{\mathcal{U}\times\mathcal{V}}$ such that $\sigma \circ K$ is the identity on $(l^p(E))''_{\mathcal{U}}$, where $\sigma : (l^p(E))_{\mathcal{U}\times\mathcal{V}} \to (l^p(E))''$ is the usual map taking the limit in the weak*-topology. We can show that σ is always a $\mathcal{B}(E)$-module homomorphism, and we shall show in Proposition 3.5.6 that K can be chosen to be a $\mathcal{B}(E)$-module homomorphism. Assuming this, we can define

$$\theta_0 : \mathcal{B}(E)'' \to \mathcal{B}((l^p(E))_{\mathcal{U}\times\mathcal{V}}); \quad \theta_0(\Phi) = K \circ \theta_1(\Phi) \circ \sigma \qquad (\Phi \in \mathcal{B}(E)'').$$

Then, for $\Phi, \Psi \in \mathcal{B}(E)''$, we have

$$\theta_0(\Phi\Box\Psi) = K \circ \theta_1(\Phi) \circ \theta_1(\Psi) \circ \sigma = K \circ \theta_1(\Phi) \circ \sigma \circ K \circ \theta_1(\Psi) \circ \sigma = \theta_0(\Phi) \circ \theta_0(\Psi).$$

However, for $T \in \mathcal{B}(E)$ and $x \in (l^p(E))_{\mathcal{U}\times\mathcal{V}}$, we have

$$\theta_0(\kappa_{\mathcal{B}(E)}(T))(x) = K \circ T'' \circ \sigma(x) = K\big(T \cdot \sigma(x)\big) = T \cdot K(\sigma(x)).$$

This is not, in general, equal to $T \cdot x = T(x)$, where we allow T to act on $(l^p(E))_{\mathcal{U}\times\mathcal{V}}$ in the canonical way. The problem here is that $(l^p(E))_{\mathcal{U}\times\mathcal{V}}$ does not carry a weak*-like topology, unlike $(l^p(E))''_{\mathcal{U}}$.

When $l^p(E)$ is super-reflexive, $(l^p(E))_{\mathcal{U}}$ is also super-reflexive, and so $\theta_1 : \mathcal{B}(E)'' \to \mathcal{B}((l^p(E))_{\mathcal{U}})$ is an isometry onto its range, and a homomorphism for either Arens product. In particular, $\mathcal{B}(E)$ is Arens regular, and $\mathcal{B}(E)''$ can be identified with a subalgebra of $\mathcal{B}(F)$ for some super-reflexive Banach space F.

Indeed, it turns out that $l^p(E)$ is super-reflexive if and only if E is super-reflexive. However, we need to take a slightly convoluted path to this result.

Definition 3.3.2. Let E be a Banach space. The *modulus of convexity*, $\delta_E(\varepsilon)$, for $0 < \varepsilon \leq 2$, is defined as

$$\delta_E(\varepsilon) = \inf\{1 - \|x + y\|/2 : x, y \in E, \|x\| = \|y\| = 1, \|x - y\| = \varepsilon\}.$$

Then E is *uniformly convex* if $\delta_E(\varepsilon) > 0$ for each $\varepsilon \in (0, 2]$.

We can easily show (see [Lindenstrauss, Tzafriri, 1979, Section 1.e]) that

$$\delta_E(\varepsilon) = \inf\{1 - \|x + y\|/2 : x, y \in E_{[1]}, \|x - y\| \geq \varepsilon\}.$$

For example, one can show that l^p is uniformly convex for $1 < p < \infty$. Notice that uniform convexity is only an isometric invariant, not an isomorphic invariant.

Theorem 3.3.3. *Let $(E, \|\cdot\|)$ be a Banach space. Then the following are equivalent:*

1. *there exists a norm $\|\cdot\|_0$, equivalent to $\|\cdot\|$, so that $(E, \|\cdot\|_0)$ is uniformly convex;*
2. *$(E, \|\cdot\|)$ is super-reflexive.*

Proof. This is detailed in [Habala et al., 1996, Chapter 11]. In particular, this result is [Habala et al., 1996, Theorem 345] and is due to James and Enflo. We can use some ultrapower results to show that (1)⇒(2), but the reverse implication is harder. □

Lemma 3.3.4. *Let E be a Banach space with modulus of convexity δ_E. Then, for each ultrafilter $\mathcal{U}$, we have $\delta_{(E)_\mathcal{U}} = \delta_E$.*

Proof. Let $\mathcal{U}$ be an ultrafilter on an index set I, let $\varepsilon \in (0,2]$, and let $x, y \in (E)_\mathcal{U}$ be such that $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| > \varepsilon$. Then, using a similar argument to that in Lemma 3.1.1, we may suppose that $x = (x_i)$ and $y = (y_i)$ with $\|x_i\| \leq 1$, $\|y_i\| \leq 1$ and $\|x_i - y_i\| > \varepsilon$ for each $i \in I$. For each i, we thus have $1 - \|x_i - y_i\|/2 \geq \delta_E(\varepsilon)$, and so, taking limits, we have $1 - \|x-y\|/2 \geq \delta_E(\varepsilon)$. Thus we have $\delta_{(E)_\mathcal{U}}(\varepsilon) \geq \delta_E(\varepsilon)$. As E is isometrically a subspace of $(E)_\mathcal{U}$, we must have equality, as required. □

Proposition 3.3.5. *Let E be a uniformly convex Banach space. Then E is reflexive.*

Proof. Assume towards a contradiction that E is not reflexive. We shall see later, in Theorem 4.1.1, that for $\theta \in (0,1)$ we can find sequences of unit vectors (x_n) and (μ_n) in E and E', respectively, such that

$$\langle \mu_m, x_n \rangle = \begin{cases} \theta & : m \leq n, \\ 0 & : m > n. \end{cases}$$

Then let $x = x_1$ and $y = x_2$, so that $\|x\| = \|y\| = 1$. Then $\langle \mu_2, x-y \rangle = -\theta$ so that $\|x-y\| \geq \theta$; similarly, $\langle \mu_1, x+y \rangle = 2\theta$ so that $\|x+y\| \geq 2\theta$. Thus, by choosing θ arbitrarily close to 1, we see that, for example, $\delta_E(1/2) = 0$, and thus that E is not uniformly convex, giving the required contradiction. □

Corollary 3.3.6. *Let E be a Banach space which admits an equivalent, uniformly convex norm. Then E is super-reflexive.*

Proof. Let $\mathcal{U}$ be an ultrafilter, so that we are required to prove that $(E)_\mathcal{U}$ is reflexive. We may suppose, by re-norming, that E is uniformly convex, so that $(E)_\mathcal{U}$ is uniformly convex, by the above lemma. Thus we are done by the above proposition. □

Proposition 3.3.7. *Let E be a Banach space and $p \in (1,\infty)$. Then E is super-reflexive if and only if $l^p(E)$ is super-reflexive.*

Proof. By the previous theorem, it is enough to show that when E is uniformly convex, so is $l^p(E)$. However, this is precisely the result in [Day, 1941]. □

When, for example, $E = l^p$, we have that $l^p(E) = l^p(l^p)$ is isometrically isomorphic to l^p. However, we need to be careful, as for this to be useful in the above work, we need that $l^p(l^p)$ and l^p are isomorphic as left $\mathcal{B}(l^p)$-modules, which is not true.

Theorem 3.3.8. *Let E be a super-reflexive Banach space. Then $\mathcal{B}(E)$ is Arens regular. Furthermore, $\mathcal{B}(E)''$ can be identified with a subalgebra of $\mathcal{B}(F)$ for a super-reflexive Banach space F. Consequently, every even dual of $\mathcal{B}(E)$ is also Arens regular.*

Proof. This is shown, using much the same method, by the author in [Daws, 2004]. As E is super-reflexive, $l^2(E)$ is super-reflexive, and so $l^2(E)' = l^2(E')$. By Theorem 3.3.1, there exists an ultrafilter $\mathcal{U}$ such that

$$\phi_1 : (l^2(E))_{\mathcal{U}} \widehat{\otimes} (l^2(E'))_{\mathcal{U}} \to \mathcal{B}(E)'$$

is a quotient operator. Thus $\theta_1 : \mathcal{B}(E)'' \to \mathcal{B}((l^2(E))_{\mathcal{U}})$ is an isometry onto its range. As $(l^2(E))_{\mathcal{U}}$ is reflexive, θ_1 is a homomorphism for either Arens product, so that $\mathcal{B}(E)$ is Arens regular.

Repeating the argument, we see that $\mathcal{B}((l^2(E))_{\mathcal{U}})$ is also Arens regular, as $(l^2(E))_{\mathcal{U}}$ is actually super-reflexive. Thus $\mathcal{B}(E)''$, as a subalgebra of $\mathcal{B}((l^2(E))_{\mathcal{U}})$, is also Arens regular, and $\mathcal{B}(E)^{[4]}$ is a subalgebra of $\mathcal{B}(F)$ for some super-reflexive Banach space F. By induction, we see that every even dual space of $\mathcal{B}(E)$ is Arens regular.

This answers in the affirmative the conjecture made in [Young, 1976] (once we realise that super-reflexive spaces, and not uniformly convex spaces, is the correct category to study). □

The above construction compares with the C*-algebra case. Let $\mathcal{A}$ be a C*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(H)$ be a representation on a Hilbert space H such that, for each state $\mu \in \mathcal{A}'$, we have

$$\langle \mu, a \rangle = [\pi(a)(x), x] \qquad (a \in \mathcal{A}),$$

for some $x \in H$. We can certainly find such a representation by Theorem 1.8.5.

Now define $\phi_1 : H \widehat{\otimes} H' \to \mathcal{A}'$ as before, so that

$$\langle \phi_1(x \otimes \mu), a \rangle = \langle \mu, \pi(a)(x) \rangle \qquad (x \otimes \mu \in H \widehat{\otimes} H', a \in \mathcal{A}),$$

where we have used π to turn H into a Banach left $\mathcal{A}$-module. Notice that as $(H\widehat{\otimes}H')' = \mathcal{B}(H)$, we actually have $\phi_1 = \pi' \circ \kappa_{H\widehat{\otimes}H'}$. Then, as H is reflexive, $\theta_1 : \mathcal{A}'' \to \mathcal{B}(H)$ is a homomorphism for either Arens product, $\theta_1 \circ \kappa_{\mathcal{A}} = \pi$, and $\mathcal{A}$ is Arens regular if ϕ_1 has dense range.

For $\mu \in H'$, recall that the Riesz Representation theorem says that we can find $y \in H$ so that $\langle \mu, x\rangle = [x, y]$ for each $x \in H$. Thus we see that $\phi_1(H\widehat{\otimes}H')$ certainly contains the linear span of the states. By the next proposition, ϕ_1 is thus a surjection, and we are done.

Proposition 3.3.9. *Let $\mathcal{A}$ be a unital C*-algebra and $\mu \in \mathcal{A}'$. Then μ is a linear combination of at most four states.*

Proof. See [Dales, 2000, Corollary 3.2.17], for example. ☐

We hence see that our above construction for a super-reflexive Banach space E mirrors the C*-algebra case by constructing a super-reflexive Banach space F and a representation $\pi : \mathcal{B}(E) \to \mathcal{B}(F)$ such that $\phi_1 = \pi' \circ \kappa_{F\widehat{\otimes}F'}$ is a surjection.

Definition 3.3.10. Let E be a reflexive Banach space, so that $(E\widehat{\otimes}E')' = \mathcal{B}(E)$. The *weak operator topology* is the weak*-topology on $\mathcal{B}(E)$ induced by this duality. This is easily seen to be the topology induced by saying that the net (T_α) converges to T if and only if

$$\langle \mu, T(x)\rangle = \lim_\alpha \langle \mu, T_\alpha(x)\rangle \qquad (x \in E, \mu \in E').$$

Let $\mathcal{A} \subseteq \mathcal{B}(E)$ be a subalgebra. The *weak operator closure* of $\mathcal{A}$ is the closure of $\mathcal{A}$ in $\mathcal{B}(E)$ with respect to this topology.

Let E be a Banach space, and let $\mathcal{A} \subseteq \mathcal{B}(E)$. Define the *commutant* of $\mathcal{A}$ to be

$$\mathcal{A}^c = \{T \in \mathcal{B}(E) : T \circ S = S \circ T \ (S \in \mathcal{A})\}.$$

Proposition 3.3.11. *Let $\mathcal{A}$ be a C*-algebra, and let $\pi : \mathcal{A} \to \mathcal{B}(H)$ be an injective representation, so that π is automatically an isometry. For ϕ_1 and θ_1 defined as above, $\theta_1(\mathcal{A}'')$ is equal to the weak operator closure of $\pi(\mathcal{A})$ in $\mathcal{B}(H)$.*

Furthermore, suppose that for each $x \in H$, there exists $a \in \mathcal{A}$ with $\pi(a)(x) \neq 0$. Then $\theta_1(\mathcal{A}'') = \pi(\mathcal{A})^{cc}$, the double-commutant of $\pi(\mathcal{A})$.

Proof. For $\Phi \in \mathcal{A}''$, let $\Phi = \text{weak}^*\text{-}\lim_{i\in\mathcal{U}} a_i$ for some bounded family $(a_i)_{i\in I} \subseteq \mathcal{A}$ and some ultrafilter $\mathcal{U}$. For $\mu \in H'$ and $x \in H$, we have

$$\langle \mu, \theta_1(\Phi)(x)\rangle = \langle \Phi, \phi_1(x \otimes \mu)\rangle = \lim_{i\in\mathcal{U}} \langle \phi_1(x \otimes \mu), a_i\rangle = \lim_{i\in\mathcal{U}} \langle \mu, \pi(a_i)(x)\rangle.$$

Consequently, we see that $\theta_1(\mathcal{A}'')$ is contained in the weak operator closure of $\pi(\mathcal{A})$.

Conversely, let $(T_i)_{i\in I}$ be a bounded family in $\pi(\mathcal{A})$ and $T \in \mathcal{B}(H)$ be such that $\langle \mu, T(x)\rangle = \lim_{i\in\mathcal{U}} \langle \mu, T_i(x)\rangle$ for each $\mu \in H'$ and $x \in H$. As π is an isometry, we can let $T_i = \pi(a_i)$ for each i, for some bounded family $(a_i)_{i\in I} \subseteq \mathcal{A}$. Let $\Phi = \text{weak}^*\text{-}\lim_{i\in\mathcal{U}} a_i \in \mathcal{A}''$. Then we have

$$\begin{aligned}\langle \mu, T(x)\rangle &= \lim_{i\in\mathcal{U}} \langle \mu, T_i(x)\rangle = \lim_{i\in\mathcal{U}} \langle \mu, \pi(a_i)(x)\rangle = \lim_{i\in\mathcal{U}} \langle \phi_1(\mu\otimes x), a_i\rangle \\ &= \langle \Phi, \phi_1(\mu\otimes x)\rangle = \langle \mu, \theta_1(\Phi)(x)\rangle,\end{aligned}$$

so that $T = \theta_1(\Phi)$, as required.

The remark about the double-commutant is a standard result about self-adjoint subalgebras of $\mathcal{B}(H)$. For example, see [Arveson, 1976, Theorem 1.2.1]. $\square$

Notice that the trivial representation $\pi : \mathcal{B}(H) \to \mathcal{B}(H)$, for some Hilbert space H, shows that we can have π injective but θ_1 not injective.

Example 3.3.12. Let H be a Hilbert space and consider the C*-algebra $\mathcal{A} = \mathcal{K}(H) = \mathcal{A}(H)$. As is by now standard, we have $\mathcal{A}' = H\widehat{\otimes}H'$ and $\mathcal{A}'' = \mathcal{B}(H)$. As the trivial representation $\pi : \mathcal{A} \to \mathcal{B}(H)$ is injective, we see that the this specific calculation agrees with the above proposition. We also clearly have $\mathcal{A}(H)^c = \mathbb{C}\mathrm{Id}_H$ so that $\mathcal{A}(H)^{cc} = \mathcal{B}(H)$ as required.

Note that this also holds for a reflexive Banach space E with the approximation property. However, when E is a reflexive Banach space without the approximation property, then we still have $\mathcal{A}(E)^c = \mathbb{C}\mathrm{Id}_E$, so that $\mathcal{A}(E)^{cc} = \mathcal{B}(E)$, but now $\mathcal{A}(E)''$ is a proper ideal in $\mathcal{B}(E)$. $\square$

Notice that we could have shown the C*-algebra case by proving the result for $\mathcal{B}(H)$ and then applying the Gelfand-Naimark theorem. A natural question to now raise is if every Arens regular Banach algebra arises as a subalgebra of $\mathcal{B}(E)$ for some super-reflexive Banach space E. As shown in [Kaijser, 1981], this is true if we allow just reflexive Banach spaces E. Indeed, more than this is true.

Theorem 3.3.13. *Let $\mathcal{A}$ be an Arens regular Banach algebra. There exists a reflexive Banach space E and a homomorphism $\pi : \mathcal{A} \to \mathcal{B}(E)$ such that π is an isomorphism onto its range and, when ϕ_1 and θ_1 are defined using π, we have that $\theta_1 : \mathcal{A}'' \to \mathcal{B}(E)$ is also an isometry onto its range.*

Proof. This is [Kaijser, 1981, Theorem 4.10], which we shall now sketch. We may suppose that $\mathcal{A}$ is unital, for if not, use $\mathcal{A}^\sharp$ which must also be Arens regular. For $\mu \in \mathcal{A}'$, we know that the map $a \mapsto a \cdot \mu, \mathcal{A} \to \mathcal{A}'$ is weakly compact, as $\mathcal{A}$ is Arens regular. By Theorem 2.2.9, we can factor this map through a reflexive Banach space E_μ. In fact, this Banach space is naturally a Banach left $\mathcal{A}$-module and the maps in the factorisation are left $\mathcal{A}$-module homomorphisms (this follows by an examination of the proof of Theorem 2.2.9). Hence we find a Banach left $\mathcal{A}$-module E_μ and left $\mathcal{A}$-module homomorphisms $R_\mu : \mathcal{A} \to E_\mu$ and $S_\mu : E_\mu \to \mathcal{A}'$ such that $a \cdot \mu = S_\mu(R_\mu(a))$ for each $a \in \mathcal{A}$, and $\|S_\mu\| \leq 1$ and $\|R_\mu\| \leq \|\mu\|$.

Let $p \in (1, \infty)$ and let $E = l^p(\bigoplus_{\mu \in \mathcal{A}'_{[1]}} E_\mu)$, so that E is a reflexive Banach space. Define $\pi : \mathcal{A} \to \mathcal{B}(E)$ by $\pi(a)(x) = (a \cdot x_\mu)$ for $x = (x_\mu) \in E$, so that π is norm-decreasing. For $a \in \mathcal{A}$, let $\mu \in \mathcal{A}'_{[1]}$ be such that $\langle \mu, a \rangle = \|a\|$. Then we have

$$\begin{aligned}\|\pi(a)\| &\geq \|\pi(a)(R_\mu(e_\mathcal{A}))\| = \|a \cdot R_\mu(e_\mathcal{A})\| \geq \|S_\mu(a \cdot R_\mu(e_\mathcal{A}))\| \\ &\geq \|S_\mu(R_\mu(a))\| = \|a \cdot \mu\| \geq |\langle a \cdot \mu, e_\mathcal{A} \rangle| = |\langle \mu, a \rangle| = \|a\|.\end{aligned}$$

Thus π is an isometry. For $\lambda \in \mathcal{A}'_{[1]}$, let $x^{(\lambda)}$ be the family in E defined by $x^{(\lambda)} = (x^{(\lambda)}_\mu)$, where

$$x^{(\lambda)}_\mu = \begin{cases} R_\lambda(e_\mathcal{A}) & : \lambda = \mu, \\ 0 & : \text{otherwise}. \end{cases} \qquad (\mu \in \mathcal{A}'_{[1]})$$

Similarly define $\psi^{(\lambda)} \in E'$ by $\psi^{(\lambda)}_\lambda = S'_\lambda(\kappa_\mathcal{A}(e_\mathcal{A}))$ and $\psi^{(\lambda)}_\mu = 0$ for $\mu \neq \lambda$. For $a \in \mathcal{A}$, we hence have

$$\begin{aligned}\langle \psi^{(\lambda)}, \pi(a)(x^{(\lambda)}) \rangle &= \langle S'_\lambda(\kappa_\mathcal{A}(e_\mathcal{A})), a \cdot R_\lambda(e_\mathcal{A}) \rangle = \langle S_\lambda(R_\lambda(a)), e_\mathcal{A} \rangle \\ &= \langle a \cdot \lambda, e_\mathcal{A} \rangle = \langle \lambda, a \rangle.\end{aligned}$$

We hence see that ϕ_1 is a quotient operator, as $\|\psi^{(\lambda)}\| \leq 1$ and $\|x^{(\lambda)}\| \leq \|R_\lambda\| \leq \|\lambda\|$. Similarly, θ_1 is an isometry onto its range. $\square$

Young also proves some similar results in [Young, 1976]. For example, [Young, 1976, Theorem 1] shows that a Banach algebra $\mathcal{A}$ arises isometrically as a subalgebra of $\mathcal{B}(E)$ for some reflexive Banach space E if and only if the set

$$WAP(\mathcal{A}) := \{\mu \in \mathcal{A}'_{[1]} : (a \mapsto a \cdot \mu) \in \mathcal{W}(\mathcal{A}, \mathcal{A}')\}$$

is norming for $\mathcal{A}$.

There exist reflexive, but not super-reflexive, Banach spaces E_0 for which $\mathcal{B}(E_0)$ is not Arens regular (see [Young, 1976, Corollary 1] and also Proposition 4.1.2). In particular, $\mathcal{A}(E_0)$ is Arens regular, and the following shows that $\mathcal{A}(E_0)$ cannot be a subalgebra of $\mathcal{B}(E)$ for any super-reflexive Banach space E.

The following appears to be a new result, although it does seem too simple to have been unknown until now.

Proposition 3.3.14. *Let E and F be Banach spaces, and let $\pi : \mathcal{A}(E) \to \mathcal{B}(F)$ be an injective, continuous homomorphism. Then E is isomorphic to a subspace of F.*

Proof. Let $x_0 \in E$ and $\mu_0 \in E'$ be such that $\langle \mu_0, x_0 \rangle = 1$. Let $P_0 = \pi(\mu_0 \otimes x_0)$ so that $P_0^2 = \pi((\mu_0 \otimes x_0)^2) = P_0$, and, as π is injective, $P_0 \neq 0$. Let $y \in F$ and $\lambda \in F'$ be such that $\langle \lambda, P_0(y) \rangle = 1$. Then define $T : E \to F$ and $S : E' \to F'$ by

$$T(x) = \pi(\mu_0 \otimes x)(y) \quad , \quad S(\mu) = \pi(\mu \otimes x_0)'(\lambda) \qquad (x \in E, \mu \in E').$$

As π is bounded, T and S are bounded, and are clearly linear. Then, for $x \in E$ and $\mu \in E'$, we have

$$\begin{aligned}\langle (S' \circ \kappa_F \circ T)(x), \mu \rangle &= \langle S(\mu), T(x) \rangle = \langle \lambda, \pi(\mu \otimes x_0)\pi(\mu_0 \otimes x)(y) \rangle \\ &= \langle \mu, x \rangle \langle \lambda, \pi(\mu_0 \otimes x_0)(y) \rangle = \langle \mu, x \rangle \langle \lambda, P_0(y) \rangle = \langle \mu, x \rangle.\end{aligned}$$

Thus we have $S' \circ \kappa_F \circ T = \kappa_E$, so that T is an isomorphism onto its range. □

3.4 The dual of $\mathcal{B}(l^p)$

In this section, we shall show, for $p \in (1, \infty)$ and a suitable ultrafilter $\mathcal{U}$, that the map

$$(E')_{\mathcal{U}} \widehat{\otimes} (E)_{\mathcal{U}} \xrightarrow{\psi_0} (E' \widehat{\otimes} E)_{\mathcal{U}} \longrightarrow \mathcal{B}(E)'$$

is surjective (actually, is a quotient map) for $E = l^p$ (and some related Banach spaces). This is an extension of work first shown by the author in [Daws, Read, 2004].

By Proposition 3.2.5, it is reasonable to insist that E be reflexive (so that $\mathcal{A}(E)' = \mathcal{I}(E') = \mathcal{N}(E')$) and that E has the approximation property (implies metric approximation property for reflexive spaces) so that $\mathcal{K}(E) = \mathcal{A}(E)$. Actually, we shall insist on E having a far stronger structure than this. If one were to examine the following results closely, then the early ones could be formulated, and proved, in an analogous way for spaces just with the metric approximation property. However, the later results seem to require that a far stronger condition be imposed upon E.

3.4.1 Schauder bases

We shall study spaces E which have some notion of a co-ordinate system. The following section sketches the theory of Schauder bases in Banach spaces. For proofs, see [Megginson, 1998] or [Lindenstrauss, Tzafriri, 1977], for example.

Definition 3.4.1. A sequence (e_n) in a Banach space E is a *Schauder basis* (or, from now on, simply a *basis*) when, for each $x \in E$, there is a unique sequence of scalars (a_n) such that $x = \sum_{n=1}^{\infty} a_n e_n$, with convergence in norm.

A sequence (x_n) which is a basis for its closed linear span is called a *basic sequence*.

The order of summation is important here. Notice that the proof of Proposition 1.2.6 shows that every (algebraic) basis of a finite-dimensional Banach space is a Schauder basis. A Schauder basis can be thought of as a way of giving an infinite-dimensional Banach space a co-ordinate system. We can clearly suppose that $\|e_n\| = 1$ for each n (so that (e_n) is a *normalised* basis). Henceforth, all our bases will be normalised.

Theorem 3.4.2. *Let E be a Banach space and (e_n) be a sequence in E. Then (e_n) is a basis for E if and only if:*

1. *each e_n is non-zero;*
2. *the linear span of (e_n) is dense in E; and*
3. *for some constant $K > 0$, for each sequence of scalars (a_n) and each $n, m \in \mathbb{N}$, we have*
$$\Big\| \sum_{i=1}^{n} a_i e_i \Big\| \leq K \Big\| \sum_{i=1}^{n+m} a_i e_i \Big\|.$$

Proof. We shall sketch this (which is, for example, [Lindenstrauss, Tzafriri, 1977, Proposition 1.a.3]). Suppose that (e_n) is a basis for E. For $x \in E$, define
$$\|x\|_0 = \sup_{n \in \mathbb{N}} \Big\| \sum_{i=1}^{n} a_i e_i \Big\| \qquad \Big(x = \sum_{i=1}^{\infty} a_i e_i \Big).$$
Note that this is well-defined, for, by definition, we have $x = \lim_{n\to\infty} \sum_{i=1}^{n} a_i e_i$, so that the sequence $(\sum_{i=1}^{n} a_i e_i)_{n=1}^{\infty}$ is bounded. Furthermore, we also see that $\|x\|_0 \geq \|x\|$ for each $x \in E$. We can check that $\|\cdot\|_0$ is norm, and claim (without proof here) that $(E, \|\cdot\|_0)$ is a Banach space. Thus we can let $\iota : (E, \|\cdot\|_0) \to (E, \|\cdot\|)$ be the formal identity, and have that ι is a bounded linear map. It is clearly a bijection, so that by the open mapping theorem, ι is bounded below, that is, for some $K > 0$ we have $\|x\|_0 \leq K\|x\|$ for each $x \in E$. Then we have condition (3), as letting $x = \sum_{i=1}^{n+m} a_i e_i$, we have
$$\Big\| \sum_{i=1}^{n} a_i e_i \Big\| \leq \|x\|_0 \leq K\|x\|,$$
as required. Conditions (1) and (2) are clear.

Conversely, let F be the linear span of (e_n), and define $\|\cdot\|_0$ on F, as above. By condition (3), we have that $\|x\|_0 \leq K\|x\|$ for each $x \in F$, so that $\|\cdot\|_0$ and $\|\cdot\|$ are equivalent norms on F. We can then extend $\|\cdot\|_0$ to the completion of F, which by (2) is hence isomorphic to E. For $n \in \mathbb{N}$, define $P_n : F \to F$ by

$$P_n\Big(\sum_{i=1}^m a_i e_i\Big) = \sum_{i=1}^n a_i e_i \qquad \Big(m \geq n, \sum_{i=1}^m a_i e_i \in F\Big),$$

so that P_n is norm-decreasing, with respect to $\|\cdot\|_0$. Thus P_n extends to a bounded linear map on E, with $\|P_n\| \leq K$. Then, for $n \in \mathbb{N}$, define $e_n^* \in E'$ by setting, for $x \in E$, $\langle e_n^*, x\rangle e_n = P_n(x) - P_{n-1}(x)$, so that $\|e_n^*\| \leq 2K$. For $x \in E$, let (x_n) be a sequence in F which tends to x, so that, for each $m \in \mathbb{N}$, we have

$$\begin{aligned}\Big\|\sum_{i=1}^n \langle e_n^*, x\rangle e_n - x\Big\|_0 &= \|P_n(x) - x\|_0 \\ &\leq \|P_n(x) - P_n(x_m)\|_0 + \|P_n(x_m) - x_m\|_0 + \|x_m - x\|_0 \\ &\leq 2\|x - x_m\|_0 + \|P_n(x_m) - x_m\|_0.\end{aligned}$$

Clearly, as $x_m \in \mathrm{lin}(e_k)$, we have that $P_n(x_m) = x_m$ for sufficiently large n, so we see that

$$x = \sum_{i=1}^\infty \langle e_i^*, x\rangle e_i,$$

with convergence with respect to $\|\cdot\|_0$, or equivalently $\|\cdot\|$. This shows that each $x \in E$ has a, necessarily unique by condition (1), expansion of the form $x = \sum_{n=1}^\infty a_n e_n$, as required. $\square$

We call the bounded family (P_n) the *natural projections* associated with the basis (e_n), and (e_n^*) the *co-ordinate functionals*. The above proof shows that we can re-norm a Banach space with a basis (e_n) to be *monotone*, that is, so that $\|P_n\| = 1$ for each n.

Suppose that E has a basis (e_n). Then, for $x \in E$, we define the *support* of x, with respect to (e_n), to be

$$\mathrm{supp}(x) = \{n \in \mathbb{N} : \langle e_n^*, x\rangle \neq 0\}.$$

Proposition 3.4.3. *Let E be a Banach space with a monotone basis (e_n). Then E has the metric approximation property.*

Proof. We simply note that, for $x \in E$, $x = \lim_{n\to\infty} P_n(x)$, where $\|P_n\| = 1$ for each n. $\square$

In particular, there are certainly (separable) Banach spaces without a basis.

We thus see that the spaces $l^p(\mathbb{N})$, for $1 \leq p < \infty$, each have a natural monotone basis formed by letting e_n be the sequence which is 0 apart from a 1 in the nth co-ordinate. That is, $e_n = (\delta_{nm})_{m=1}^\infty$, where δ_{nm} is the Kronecker delta. The same also holds for c_0.

Definition 3.4.4. Let E be a Banach space with a basis (e_n). Suppose that for each $x = \sum_{n=1}^\infty a_n e_n \in E$, the sum $\sum_{n=1}^\infty a_n e_n$ converges unconditionally (that is, the order of summation is not important). Then we say that (e_n) is an *unconditional basis*.

Theorem 3.4.5. *Let E be a Banach space with a basis (e_n). Then the following are equivalent:*

1. *(e_n) is an unconditional basis;*
2. *for each $x = \sum_{n=1}^\infty a_n e_n \in E$, and each $\varepsilon > 0$, there exists a finite set $A \subseteq \mathbb{N}$ so that, if $B \subseteq \mathbb{N}$ is finite and $A \subseteq B$, then $\|x - \sum_{n\in B} a_n e_n\| < \varepsilon$;*
3. *there exists $K > 0$ so that, for each $A \subseteq \mathbb{N}$, we can define $P_A : E \to E$ by*
$$P_A\Big(\sum_{n=1}^\infty a_n e_n\Big) = \sum_{n\in A} a_n e_n,$$
and have $\|P_A\| \leq K$;
4. *for $x = \sum_{n=1}^\infty a_n e_n$, we can define*
$$\|x\|_m = \sup\Big\{\Big\|\sum_{n=1}^\infty b_n a_n e_n\Big\| : |b_n| \leq 1 \ (n \in \mathbb{N})\Big\},$$
and $\|\cdot\|_m$ is equivalent to $\|\cdot\|$ on E;

Proof. These follow from either standard results about unconditional summation, or from arguments similar to that used above. See, for example, [Megginson, 1998, Section 4.2] for further details. □

The infimum of possible values for K arising in (3) is called the *unconditional basis constant* of (e_n). By using (4) above, we can re-norm a Banach space E with an unconditional basis (e_n) so that
$$\Big\|\sum_{n=1}^\infty b_n a_n e_n\Big\| \leq \|(b_n)\|_\infty \Big\|\sum_{n=1}^\infty a_n e_n\Big\| \qquad \Big(\sum_{n=1}^\infty a_n e_n \in E\Big),$$
for any bounded sequence $(b_n) \in l^\infty$. In particular, we can then take $K = 1$ in condition (3), that is, we may suppose that (e_n) is 1-*unconditional*. Notice that the standard unit vector bases of l^p, for $1 \leq p < \infty$, and c_0 already satisfy this condition.

When E has a basis (e_n), it is easy to show that (e_n^*) is a basic sequence in E', as the natural projections $(\hat{P}_n)$ associated with (e_n^*) are simply the (restrictions of the) adjoints of P_n, that is, for each n, $\hat{P}_n$ is P_n' restricted to the closed linear span of (e_n^*).

Definition 3.4.6. Let E be a Banach space with a basis (e_n) such that (e_n^*) forms a basis for E'. Then (e_n) is a *shrinking basis*.

Suppose that, whenever a sequence of scalars (a_n) satisfies $\sup_n \|\sum_{i=1}^n a_i e_i\| < \infty$, we have that $\sum_{n=1}^\infty a_n e_n$ converges. Then the basis (e_n) is *boundedly complete*.

Theorem 3.4.7. *Let E be a Banach space with a basis (e_n). Then (e_n) is shrinking if and only if the basic sequence (e_n^*) is boundedly complete. Also, (e_n) is boundedly complete if and only if the basic sequence (e_n^*) is shrinking. Furthermore, E is reflexive if and only if (e_n) is both boundedly complete and shrinking.*

Proof. See, for example, [Megginson, 1998, Theorem 4.4.11, Theorem 4.4.14, Theorem 4.4.15]. □

For example, we see that the standard unit vector basis of c_0 is not boundedly complete, and that the standard unit vector basis of l^1 is not shrinking (in this latter case, (e_i^*) spans $\kappa_{c_0}(c_0) \subseteq l^\infty$).

3.4.2 Dual of $\mathcal{B}(E)$

For the moment, we shall work with Banach spaces E which are reflexive and which have a monotone basis (e_n). Then $\mathcal{K}(E) = \mathcal{A}(E)$ and $\mathcal{A}(E)' = \mathcal{I}(E') = \mathcal{N}(E') = E''\widehat{\otimes}E' = E\widehat{\otimes}E'$. Thus, as before, $\mathcal{A}(E)'' = \mathcal{B}(E)$, and the map $\kappa_{\mathcal{A}(E)}$ is just the inclusion map $\mathcal{A}(E) \to \mathcal{B}(E)$. Then $\kappa'_{\mathcal{A}(E)} : \mathcal{B}(E)' \to E\widehat{\otimes}E'$ is a projection, in that $\kappa'_{\mathcal{A}(E)} \circ \kappa_{E\widehat{\otimes}E'}$ is the identity. We can form the quotient space $\mathcal{B}(E)/\mathcal{A}(E)$, which has the quotient norm

$$\|T + \mathcal{A}(E)\| = \inf\{\|T + S\| : S \in \mathcal{A}(E)\} \qquad (T \in \mathcal{B}(E)).$$

By Theorem 1.4.10, $(\mathcal{B}(E)/\mathcal{A}(E))' = \mathcal{A}(E)^\circ \subseteq \mathcal{B}(E)'$. We thus have $\mathcal{A}(E)^\circ = \ker \kappa'$, so that

$$\mathcal{B}(E)' = \mathcal{A}(E)' \oplus \ker \kappa' = E\widehat{\otimes}E' \oplus \mathcal{A}(E)^\circ.$$

We know all about $E\widehat{\otimes}E'$, so the interesting space to study is $\ker \kappa' = \mathcal{A}(E)^\circ$.

To avoid repetition, we will assume unless otherwise stated that E is a reflexive Banach space with a normalised, 1-unconditional basis (e_n), and that (P_n) are the natural projections associated with (e_n).

Lemma 3.4.8. *Let E be as above, and let $T \in \mathcal{A}(E)$. Then $TP_n \to T$ and $P_nT \to T$ as $n \to \infty$, in the operator norm.*

Proof. Let $\varepsilon > 0$ and let $S \in \mathcal{F}(E)$ be such that $\|T - S\| < \varepsilon$. Let $S = \sum_{i=1}^N \mu_i \otimes x_i \in E' \otimes E$. For each i, we have that $\lim_{n\to\infty} P_n'(\mu_i) = \mu_i$, as (e_n) is a shrinking basis. Thus, for n sufficiently large, we have $\|\mu_i - P_n'(\mu_i)\| < \varepsilon\|x_i\|^{-1}$ for each i. Then we have

$$\begin{aligned}\|TP_n - T\| &\leq \|TP_n - SP_n\| + \|SP_n - S\| + \|S - T\| < \varepsilon\|P_n\| + \|SP_n - S\| + \varepsilon \\ &\leq 2\varepsilon + \sum_{i=1}^N \|P_n'(\mu_i) - \mu_i\|\|x_i\| < (N+2)\varepsilon.\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we are done. The claim that $P_nT \to T$ follows similarly (and does not require that the basis be shrinking). □

Proposition 3.4.9. *Let E be as above, and let $Q_n = \mathrm{Id}_E - P_n$. For $T \in \mathcal{B}(E)$, we have*

$$\|T + \mathcal{A}(E)\| = \lim_{n\to\infty} \|TQ_n\| = \lim_{n\to\infty} \|Q_nTQ_n\|.$$

We may also replace $\lim_{n\to\infty}$ by $\inf_n$.

Proof. Notice that $Q_n = P_{\{m\in\mathbb{N}:m>n\}}$, so that, as (e_n) is a 1-unconditional basis, $\|Q_n\| = 1$. For $n \in \mathbb{N}$, we have that $Q_{n+1}Q_n = Q_{n+1} = Q_nQ_{n+1}$, so that

$$\|TQ_{n+1}\| = \|TQ_nQ_{n+1}\| \leq \|TQ_n\|\|Q_{n+1}\| = \|TQ_n\|.$$

Thus $(\|TQ_n\|)_{n=1}^\infty$ is a decreasing sequence, and so we can interchange taking limits and taking infima. We can show the same for $(\|Q_nTQ_n\|)_{n=1}^\infty$.

For $n \in \mathbb{N}$, as $TQ_n = T - TP_n$ and $TP_n \in \mathcal{A}(E)$, we have $\|T + \mathcal{A}(E)\| \leq \|TQ_n\|$. Assume that we have $S \in \mathcal{A}(E)$ with $\|T + S\| < \inf_n \|TQ_n\|$, so that as $S = \lim_n SP_n$, we have $\lim_n \|SQ_n\| = 0$, and so $\lim_n \|TQ_n\| = \lim_n \|(T+S)Q_n\| \leq \|T + S\| < \lim_n \|TQ_n\|$. This contradiction shows that

$$\|T + \mathcal{A}(E)\| = \lim_n \|TQ_n\|.$$

For $n \in \mathbb{N}$, we have $Q_nTQ_n = T - TP_n - P_nT + P_nTP_n$, and so $\|T + \mathcal{A}(l^p)\| \leq \|Q_nTQ_n\|$. Hence

$$\|T + \mathcal{A}(E)\| \leq \lim_n \|Q_nTQ_n\| \leq \lim_n \|TQ_n\| = \|T + \mathcal{A}(E)\|$$

so we must have equality throughout, completing the proof. □

The following is a variant of Helley's Lemma (see Theorem 3.5.2).

Proposition 3.4.10. *Let F be a Banach space, $\Phi \in F''$ and $M \in \mathrm{FIN}(F')$. Then for $\varepsilon > 0$ we can find $x \in F$ so that $\langle \mu, x \rangle = \langle \Phi, \mu \rangle$ for each $\mu \in M$, and*

$$\|x\| \leq \varepsilon + \max\{|\langle \Phi, \mu \rangle| : \mu \in M_{[1]}\}.$$

Proof. This follows easily from [Guerre-Delabriére, 1992, Lemma I.6.2]. Alternatively, we can use the Hahn-Banach Theorem to find $\Psi \in F''$ with $\langle \Psi, \mu \rangle = \langle \Phi, \mu \rangle$ for each $\mu \in M$, and

$$\|\Psi\| = \max\{|\langle \Phi, \mu \rangle| : \mu \in M_{[1]}\}.$$

The result then follows from Theorem 3.5.2. $\square$

For $x \in E$, notice that we have $P_n(x) = x$ if and only if $\mathrm{supp}(x) \subseteq \{1, \ldots, n\}$, and that $Q_n(x) = x$ if and only if $\mathrm{supp}(x) \subseteq \{n+1, n+2, \ldots\}$.

Lemma 3.4.11. *Let E be as before. Let $M \subset \mathcal{B}(E)$ be a finite-dimensional subspace, and let $\varepsilon > 0$. Then we have:*

1. *for each $x \in E$, there exists $N_0 \in \mathbb{N}$ such that $\|Q_n T(x)\| < \varepsilon \|T\|$ for each $T \in M$ and $n \geq N_0$;*
2. *for each $m \in \mathbb{N}$, there exists $N_1 \in \mathbb{N}$ such that $\|P_m T Q_n\| < \varepsilon \|T\|$ for each $T \in M$ and $n \geq N_1$;*
3. *there exists $N_2 \in \mathbb{N}$ such that $\|T Q_n\| < \varepsilon \|T\|$ for each $T \in M \cap \mathcal{A}(E)$ and $n \geq N_2$.*

Proof. Firstly, assume towards a contradiction that we can find $n(1) < n(2) < \cdots$ such that for each $k \in \mathbb{N}$, there exists $T_k \in M$ with $\|T_k\| = 1$ and $\|Q_{n(k)}(T_k(x))\| \geq \varepsilon \|T_k\| = \varepsilon$. Then, as M has compact unit ball, we can find $T \in M$ and a sequence $k(1) < k(2) < \cdots$ such that $T_{k(j)} \to T$ as $j \to \infty$. Then we have, with reference to Lemma 3.4.8,

$$0 = \lim_j \|Q_{n(k(j))}(T(x))\| = \lim_j \|Q_{n(k(j))}(T_{k(j)}(x))\| \geq \varepsilon,$$

which is the required contradiction.

For the second part, by the compactness of the unit ball of M, let $(T_i)_{i=1}^N \subseteq M$ be such that $\|T_i\| = 1$ for each i, and such that

$$\min_{1 \leq i \leq N} \|T - T_i\| \leq \varepsilon/2 \qquad (T \in M_{[1]}).$$

Fix $m \in \mathbb{N}$. Then we claim that we can find $N_1 \in \mathbb{N}$ such that $\|P_m T_i Q_n\| < \delta \|T_i\|$ for $n \geq N_1$ and $1 \leq i \leq N$.

It is enough to show this for each separate i, as we have only finitely many to consider. It is enough to show that $\lim_n \|P_m T_i Q_n\| = 0$, so assume towards a contradiction that, for some $\theta > 0$, there is an increasing sequence $(n_k)_{k=1}^\infty$ such that $\|P_m T_i Q_{n_k}\| \geq 2\theta$ for each k. Then we can find $(x_j)_{j=1}^\infty$ with $\|x_j\| = 1$ and $Q_{n_j}(x_j) = x_j$, so that $\|P_m T_i(x_j)\| \geq \theta$ for each j. However, we then have

$$\begin{aligned}\lim_{j\to\infty} \|P_m T_i(x_j)\| &= \lim_{j\to\infty} \Big\| \sum_{k=1}^m \langle e_k^*, T_i(x_j)\rangle e_k \Big\| \leq \lim_{j\to\infty} \sum_{k=1}^m |\langle T_i'(e_k^*), x_j\rangle| \\ &= \sum_{k=1}^m \lim_{j\to\infty} |\langle T_i'(e_k^*), Q_{n_j}(x_j)\rangle| \leq \sum_{k=1}^m \lim_{j\to\infty} \|Q_{n_j}' T_i'(e_k^*)\| = 0,\end{aligned}$$

as (e_n) is a shrinking basis, giving the required contradiction.

Then, for $n \geq N_1$ and $T \in M$ with $\|T\| = 1$, for some i we have $\|T - T_i\| \leq \varepsilon/2$, and so

$$\|P_m T Q_n\| \leq \|P_m T_i Q_n\| + \varepsilon/2 < \varepsilon\|T_i\|/2 + \varepsilon/2 = \varepsilon,$$

as required.

Finally, for (3), suppose towards a contradiction that we can find $n(1) < n(2) < \cdots$ such that, for each k, we can find $T_k \in M \cap \mathcal{A}(E)$ with $\|T_k\| = 1$ and $\|T_k Q_{n(k)}\| \geq \varepsilon$. Again, we can find $T \in M \cap \mathcal{A}(E)$ and $k(1) < k(2) < \cdots$ such that $T_{k(j)} \to T$ as $j \to \infty$. Then we have

$$0 = \lim_{j\to\infty} \|T Q_{n(k(j))}\| = \lim_{j\to\infty} \|T_{k(j)} Q_{n(k(j))}\| \geq \varepsilon,$$

which is our required contradiction. □

Definition 3.4.12. A *block-basis* in E, with respect to a basis (e_n) of E, is a sequence of norm-one vectors $(x_n)_{n=1}^\infty$ in E such that $\operatorname{supp}(x_n)$ is finite for each n, and such that $\max\operatorname{supp}(x_n) < \min\operatorname{supp}(x_{n+1})$ for each n.

Some definitions of a block-basic do not require that $\|x_n\| = 1$ for each n. We can show that when (x_n) is a block-basis, it is a basic sequence with basis constant no greater than that of the original basis (e_n), so that, in our case, every block-basis is a monotone basic sequence (see [Megginson, 1998, Proposition 4.3.16]).

Definition 3.4.13. Let (A_n) and (B_n) be sequences of subsets of $\mathbb{N}$, such that, for each $n \in \mathbb{N}$, we have $\max A_n < \min A_{n+1}$ and $\max B_n < \min B_{n+1}$. We say that (A_n) is *union subordinate* to (B_n) if, for each $n \in \mathbb{N}$, B_n intersects at most one of the (A_m).

Proposition 3.4.14. *Let $\lambda \in \mathcal{A}(E)^\circ$ with $\|\lambda\| = 1$, and let $M \subset \mathcal{B}(E)$ be a finite-dimensional subspace with $M \cap \mathcal{A}(E) = \{0\}$. Let (ε_n) be a sequence of positive reals,* *let (s_n) be an increasing sequence of natural numbers, and let $n_1 \in \mathbb{N}$. Then we can find a block-basis (x_n) in E, and $(A_n)_{n=1}^\infty$ a sequence of pairwise-disjoint subsets of $\mathbb{N}$, such that:*

1. $|\langle \lambda, T\rangle| \leq (1+\varepsilon_1)\sup_n \|T(x_n)\|$ *for each* $T \in M$*;*
2. $\|P_{\mathbb{N}\setminus A_n}(T(x_n))\| < \varepsilon_n \|T\|$ *and* $\|P_{A_n}(T(x_m))\| < \varepsilon_m\|T\|$ *for each* $n, m \in \mathbb{N}$ *with* $n \neq m$*, and each* $T \in M$*;*
3. $\operatorname{supp}(x_n) \subseteq \{n_1+1, n_1+2, \dots\}$ *for each* $n \in \mathbb{N}$*;*
4. *setting* $B_m = \{i \in \mathbb{N} : s_m \leq i < s_{m+1}\}$*, we have that* $(\operatorname{supp}(x_n))$ *is union subordinate to* (B_m)*, and that* (A_n) *is union subordinate to* (B_m)*.*

Proof. As M has a compact unit ball, let $(T_n)_{n=1}^\infty$ be a dense sequence in $M_{[1]}$. For T_1, we can find x_1 in E with finite support contained in $\{n_1+1, n_2+1, \dots\}$, $\|x_1\| = 1$, and $(1+\varepsilon_1)\|T_1(x_1)\| > |\langle \lambda, T_1\rangle|$. We can do this because, using the fact that $\lambda \in \mathcal{A}(E)^\circ$, we have $|\langle \lambda, T_1\rangle| = |\langle \lambda, T_1 Q_{n_1}\rangle| \leq \|T_1 Q_{n_1}\|$. Then, by Lemma 3.4.11(1), we can find $r_1 \in \mathbb{N}$ so that $\|Q_{r_1}T(x_1)\| < \frac{1}{2}\varepsilon_1\|T\|$ for each $T \in M$. Let $A_1 = \{1, 2, \dots, r_1\}$.

Assume inductively that we have found $(x_i)_{i=1}^k \subseteq E$ of norm one and with finite support, and with $\max\operatorname{supp}(x_i) < \min\operatorname{supp}(x_{i+1})$ for each $1 \leq i < k$, and that we have pairwise-disjoint, finite subsets of $\mathbb{N}$, $(A_i)_{i=1}^k$, such that:

1. for $1 \leq i \leq k$, $|\langle \lambda, T_i\rangle| \leq (1+\varepsilon_1)\|T_i(x_i)\|$;
2. for $1 \leq i \leq k$ and $T \in M$, setting $r_i = \max A_i$, we have $\|Q_{r_i}T(x_i)\| < \frac{1}{2}\varepsilon_i\|T\|$;
3. for $1 \leq i \leq k$ and $T \in M$, $\|P_{(\min A_i)-1}T(x_i)\| < \frac{1}{2}\varepsilon_i\|T\|$;
4. we have that $(\operatorname{supp}(x_i))_{i=1}^k$ and $(A_i)_{i=1}^k$ are each union subordinate to $(B_i)_{i=1}^\infty$.

We shall show how to choose x_{k+1} and A_{k+1}. Choose $r \in \mathbb{N}$ such that $r > \max(A_j)$ for $1 \leq j \leq k$, and with

$$r > \max\{s_{i+1} : B_i \cap A_j \neq \emptyset \text{ for some } 1 \leq j \leq k\}.$$

By Lemma 3.4.11(2) we can find $m \in \mathbb{N}$ such that $\|P_r T Q_m\| < \frac{1}{2}\varepsilon_{k+1}\|T\|$ for each $T \in M$. We may suppose that $m > \max\operatorname{supp}(x_k)$ and that

$$m > \max\{s_{i+1} : B_i \cap \operatorname{supp}(x_j) \neq \emptyset \text{ for some } 1 \leq j \leq k\}.$$

Then we have

$$|\langle \lambda, T_{k+1}\rangle| = |\langle \lambda, T_{k+1}Q_m\rangle| \leq \|T_{k+1}Q_m\|,$$

so that we can find a unit vector $x_{k+1} \in E$ with finite support such that $\min \operatorname{supp}(x_{k+1}) > m$ and $|\langle \lambda, T_{k+1} \rangle| \leq (1+\varepsilon_1)\|T_{k+1}(x_{k+1})\|$. By Lemma 3.4.11(1), we can find $r_{k+1} \in \mathbb{N}$ such that

$$\|Q_{r_{k+1}} T(x_{k+1})\| < \tfrac{1}{2}\varepsilon_{k+1}\|T\| \qquad (T \in M).$$

Let $A_{k+1} = \{r+1, r+2, \dots, r_{k+1}\}$ so that $(A_j)_{j=1}^{k+1}$ is a family of pairwise-disjoint, finite subsets of $\mathbb{N}$, which is union subordinate to (B_i). As $\min \operatorname{supp}(x_{k+1}) > m$, by the choice of m, we see that $(\operatorname{supp}(x_i))_{i=1}^{k+1}$ is union subordinate to (B_i). We thus have conditions (1), (2) and (4). For (3), we have $\|P_{(\min A_{k+1})-1} T(x_{k+1})\| = \|P_r T Q_m(x_{k+1})\| \leq \|P_r T Q_m\| < \frac{1}{2}\varepsilon_{k+1}\|T\|$, as required, for $T \in M$.

So by induction we can find (x_n) and (A_n) with the above properties. We certainly have conditions (3) and (4) from the statement of the proposition. We now verify (2). For $n \in \mathbb{N}$, we have

$$\begin{aligned}\|P_{\mathbb{N}\setminus A_n} T(x_n)\| &\leq \|Q_{\max A_n} T(x_n)\| + \|P_{(\min A_n)-1} T(x_n)\| \\ &< \tfrac{1}{2}\varepsilon_n\|T\| + \tfrac{1}{2}\varepsilon_n\|T\| = \varepsilon_n\|T\|,\end{aligned}$$

as required. For $n, m \in \mathbb{N}$ with $n < m$, we have $\max A_n \leq (\min A_m) - 1$, and so

$$\|P_{A_n} T(x_m)\| \leq \|P_{(\min A_m)-1} T(x_m)\| < \varepsilon_m\|T\|.$$

For $n, m \in \mathbb{N}$ with $n > m$, we have $\min A_n > \max A_m$, and so

$$\|P_{A_n} T(x_m)\| \leq \|Q_{\max A_m} T(x_m)\| < \varepsilon_m\|T\|.$$

Finally, for $T \in M$, and for each $\delta > 0$, there exists an $n \in \mathbb{N}$ so that $\|T - T_n\| < \delta$, and thus

$$\begin{aligned}|\langle \lambda, T \rangle| &< |\langle \lambda, T_n \rangle| + \delta \leq (1+\varepsilon_1)\|T_n(x_n)\| + \delta \\ &\leq (1+\varepsilon_1)\|T(x_n)\| + \delta(2+\varepsilon_1).\end{aligned}$$

As this holds for each $\delta > 0$, we see that $|\langle \lambda, T \rangle| \leq (1+\varepsilon_1)\sup_n \|T(x_n)\|$. $\square$

Definition 3.4.15. Let E be a reflexive Banach space with a 1-unconditional basis (e_n). We say that (e_n) is of *block p-type*, for $1 < p < \infty$, if we can find an increasing sequence (s_n) of integers, such that for each sequence of scalars (a_n), we have

$$\Big\| \sum_{n=1}^{\infty} a_n e_n \Big\| = \left(\sum_{m=1}^{\infty} \Big\| \sum_{n=s_m}^{s_{m+1}-1} a_n e_n \Big\|^p \right)^{1/p}.$$

Thus, for $1 < p < \infty$, the standard unit vector basis of l^p is of block p-type. For example, the Figiel spaces $l^p\Big(\bigoplus_{n=1}^\infty l_{m_n}^{p_n}\Big)$ are also of block p-type, where $(p_n)_{n=1}^\infty \subseteq (1,\infty)$ and $(m_n)_{n=1}^\infty$ is a sequence of integers (see [Figiel, 1972] or [Laustsen, Loy, 2003] for more details).

We introduce this definition as it will allow us to "sum-up" a sequence of vectors in E into a single vector in E, provided these vectors have a suitable pairwise-disjoint support. This is the motivation behind this whole construction: the intuitive idea is to approximate $\lambda \in \mathcal{A}(E)^\circ$ by an elementary tensor, and then to use Proposition 3.4.9 to "move-along" the support until we can ignore our elementary tensor (that is, apply Q_n to everything for a suitably large n), and then re-approximate, hoping for convergence. Of course, we then have to sum our elementary tensors, or we have gained nothing.

Lemma 3.4.16. *Let E be a reflexive Banach space with a 1-unconditional basis (e_n) which is of block p-type, for some $p \in (1,\infty)$, with respect to (s_n). Then (e_n^*) is of block q-type, where $p^{-1}+q^{-1}=1$, with respect to (s_n).*

Proof. For each $m \in \mathbb{N}$, let $M_n = \operatorname{lin}\{e_i : s_m \leq i < s_{m+1}\} \in \mathrm{FIN}(E)$. We can verify that E is then naturally isometrically isomorphic to $l^p(\bigoplus_n M_n)$, and so E' is naturally isometrically isomorphic to $l^q(\bigoplus_n M_n')$. We can then easily check that M_n' is isometrically isomorphic to $\operatorname{lin}\{e_i^* : s_m \leq i < s_{m+1}\} \in \mathrm{FIN}(E')$, so that we are done. □

We can now prove our key result, which tells us that for a Banach space E of block p-type, any member of $\mathcal{A}(E)^\circ$ can be approximated, on a finite-dimensional subspace of $\mathcal{B}(E)$, by an elementary tensor in $E\widehat{\otimes}E'$.

Theorem 3.4.17. *Let E be of block p-type with respect to a 1-unconditional basis (e_n), for $1<p<\infty$. Let $\lambda \in \mathcal{A}(E)^\circ$, let $M \in \mathrm{FIN}(\mathcal{B}(E))$, let $\varepsilon > 0$, and let $p^{-1}+q^{-1}=1$. Then we can find $x \in E$ and $\mu \in E'$ with $\|x\| < \|\lambda\|^{1/p}(1+\varepsilon)^{1/p}$, $\|\mu\| < \|\lambda\|^{1/q}(1+\varepsilon)^{1/q}$, and such that*

$$|\langle \lambda, T\rangle - \langle \mu, T(x)\rangle| < \varepsilon\|\lambda\|\|T\| \qquad (T \in M).$$

Proof. By Lemma 3.4.11(3), we can find n_1 such that $\|TQ_{n_1}\| < \frac{1}{2}\varepsilon\|T\|$ for each $T \in M \cap \mathcal{A}(E)$. Let $M_0 \subseteq M$ be a subspace of M such that $M_0 \cap \mathcal{A}(E) = \{0\}$ and $M = M_0 \oplus (M \cap \mathcal{A}(E))$. Let (ε_n) be a sequence of positive reals such that $\sum_{n=1}^\infty \varepsilon_n < \varepsilon/3$.

If the result is true in the case where $\|\lambda\| = 1$, then we can find x and μ with $\|x\| < (1+\varepsilon)^{1/p}$ and $\|\mu\| < (1+\varepsilon)^{1/q}$ and with $|\|\lambda\|^{-1}\langle\lambda, T\rangle - \langle\mu, T(x)\rangle| < \varepsilon\|T\|$ for each $T \in M$. Then let $\hat{x} = \|\lambda\|^{1/p}x$ and $\hat{\mu} = \|\lambda\|^{1/q}\mu$ so that $\|\hat{x}\| < \|\lambda\|^{1/p}(1+\varepsilon)^{1/p}$ and

$\|\hat{\mu}\| < \|\lambda\|^{1/q}(1+\varepsilon)^{1/q}$ and, for each $T \in M$, we have $|\langle \lambda, T\rangle - \langle \hat{\mu}, T(\hat{x})\rangle| < \varepsilon\|\lambda\|\|T\|$, as required. Thus we may suppose henceforth that $\|\lambda\| = 1$.

Let (e_n) be of block p-type with respect to (s_n). We can use Proposition 3.4.14, applied to M_0, (ε_n) and (s_n), to find sequences (x_n) and (A_n). Recall the definitions of $l^1(E')$ and $l^\infty(E) = l^1(E')'$. Let

$$X = \{(T(x_n))_{n=1}^\infty : T \in M_0\} \subset l^\infty(E),$$

so that X is a finite-dimensional subspace of $l^\infty(E)$. Define $\Phi \in X'$ by

$$\langle \Phi, (T(x_n))\rangle = \langle \lambda, T\rangle \qquad (T \in M_0).$$

As $|\langle \lambda, T\rangle| \leq (1+\varepsilon_1)\|(T(x_n))\|_\infty$, we have that $\|\Phi\| \leq 1+\varepsilon_1$. Then, by Proposition 3.4.10, as X is finite-dimensional, we can find $(\mu_n) \in l^1(E')$ such that $\sum_{n=1}^\infty \|\mu_n\| \leq 1+\varepsilon_1+\varepsilon_2 < 1+\varepsilon$ and $\langle \Phi, (T(x_n))\rangle = \sum_{n=1}^\infty \langle \mu_n, T(x_n)\rangle$ for each $T \in M_0$.

For each $n \in \mathbb{N}$, set $\hat{\mu}_n = P_{A_n}(\mu_n)$, and set

$$x = \sum_{n=1}^\infty x_n \|\hat{\mu}_n\|^{1/p} \quad \text{and} \quad \mu = \sum_{n=1}^\infty \hat{\mu}_n \|\hat{\mu}_n\|^{-1+1/q}.$$

For $m \in \mathbb{N}$, let $B_m = \{i \in \mathbb{N} : s_m \leq i < s_{m+1}\}$. As the (x_n) have pairwise-disjoint support, and $(\operatorname{supp}(x_n))$ is union subordinate to (B_m), we have, by the block p-type nature of the basis (e_n), that

$$\begin{aligned}\|x\| &= \Big(\sum_{n=1}^\infty \|x_n\|^p \|\hat{\mu}_n\|\Big)^{1/p} = \Big(\sum_{n=1}^\infty \|\hat{\mu}_n\|\Big)^{1/p} = \Big(\sum_{n=1}^\infty \|P_{A_n}(\mu_n)\|\Big)^{1/p} \\ &\leq \Big(\sum_{n=1}^\infty \|\mu_n\|\Big)^{1/p} < (1+\varepsilon)^{1/p}.\end{aligned}$$

Similarly, by the above lemma, and the fact that $(A_n) = (\operatorname{supp} \hat{\mu}_n)$ is union subordinate to (B_m), we have

$$\|\mu\| = \Big(\sum_{n=1}^\infty \|\hat{\mu}_n\|^q \|\hat{\mu}_n\|^{-q+1}\Big)^{1/q} = \Big(\sum_{n=1}^\infty \|\hat{\mu}_n\|\Big)^{1/q} < (1+\varepsilon)^{1/q}.$$

Then, for $T \in M_0$, we have

$$\langle \mu, T(x)\rangle = \sum_{n=1}^\infty \sum_{m=1}^\infty \langle P_{A_n}(\mu_n), T(x_m)\rangle.$$

By condition (2) in Proposition 3.4.14, for each $T \in M_0$, we have

$$\begin{aligned}\Big|\sum_{n\neq m} \langle P_{A_n}(\mu_n), T(x_m)\rangle\Big| &\leq \sum_{n=1}^\infty \Big|\langle \mu_n, \sum_{m\neq n} P_{A_n}(T(x_m))\rangle\Big| \\ &\leq \sum_{n=1}^\infty \|\mu_n\| \sum_{m=1}^\infty \varepsilon_m \|T\| \leq \|T\| \left(\sum_{m=1}^\infty \varepsilon_m\right)\left(\sum_{n=1}^\infty \|\mu_n\|\right) < \tfrac{1}{3}\varepsilon(1+\varepsilon_1+\varepsilon_2)\|T\|.\end{aligned}$$

Then, again by condition (2), for $T \in M_0$, we have

$$\begin{aligned}\left|\langle\lambda, T\rangle - \sum_{n=1}^{\infty}\langle\hat{\mu}_n, T(x_n)\rangle\right| &\le \sum_{n=1}^{\infty}\|\mu_n\|\|P_{A_n}(T(x_n)) - T(x_n)\| \\ &< \sum_{n=1}^{\infty}\varepsilon_n\|\mu_n\|\|T\| < \|T\|\left(\sup_n\|\mu_n\|\right)\left(\sum_{n=1}^{\infty}\varepsilon_n\right) < \tfrac{1}{3}\varepsilon(1+\varepsilon_1+\varepsilon_2)\|T\|.\end{aligned}$$

Consequently, for $T \in M_0$, we have

$$|\langle\lambda, T\rangle - \langle\mu, T(x)\rangle| < \tfrac{2}{3}\varepsilon(1+\varepsilon_1+\varepsilon_2)\|T\|,$$

and we may suppose that $\frac{2}{3}\varepsilon(1+\varepsilon_1+\varepsilon_2) < \varepsilon$. Finally, for $T \in M \cap \mathcal{A}(l^p)$, by the choice of n_1, we have

$$\begin{aligned}|\langle\mu, T(x)\rangle| &\le \sum_{n=1}^{\infty}|\langle P_{A_n}(\mu_n), T(x_n)\rangle| \le \sum_{n=1}^{\infty}\|\mu_n\|\|TQ_{n_1}\| \\ &< \tfrac{1}{2}\varepsilon(1+\varepsilon_1+\varepsilon_2)\|T\| < \varepsilon\|T\|,\end{aligned}$$

as required, since $\langle\lambda, T\rangle = 0$ and $\|\lambda\| = 1$. □

Now suppose that E is super-reflexive. Then, for an ultrafilter $\mathcal{U}$, $(E)_{\mathcal{U}}$ is reflexive, and we have $\phi_1 : (E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}} \to \mathcal{B}(E)'$, so that $\theta_1 : \mathcal{B}(E)'' \to \mathcal{B}((E)_{\mathcal{U}})$ is a homomorphism, for either Arens product. We can define a bilinear map $\rho : E \times E' \to (E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}}$ by $\rho(x, \mu) = (x) \otimes (\mu)$ for $x \in E$ and $\mu \in E'$. It is clear that ρ is norm-decreasing, so that ρ extends to a norm-decreasing map $\rho : E\widehat{\otimes}E' \to (E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}}$. Then, for $T \in \mathcal{B}(E)$, we have

$$\langle\phi_1(\rho(x \otimes \mu)), T\rangle = \langle(\mu), T(x)\rangle = \langle T, x \otimes \mu\rangle,$$

for $x \in E$ and $\mu \in E'$. Thus the following diagram commutes:

$$\begin{tikzcd} E\widehat{\otimes}E' \arrow[r, "\kappa_{E\widehat{\otimes}E'}"] \arrow[d, "\rho"'] & \mathcal{B}(E)' \\ (E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}} \arrow[ur, "\phi_1"] & \end{tikzcd}$$

We conclude that ρ is an isometry, as ϕ_1 is norm-decreasing.

Theorem 3.4.18. *Let E be a super-reflexive Banach space and be of block p-type with respect to a 1-unconditional basis (e_n), for $1 < p < \infty$. Then the map $\phi_1 : (E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}} \to \mathcal{B}(E)'$ is surjective for a suitable ultrafilter $\mathcal{U}$. In fact, for $\lambda \in \mathcal{B}(E)'$, we can find $\sigma \in (E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}}$ with $\phi_1(\sigma) = \lambda$ and $\|\sigma\| = \|\lambda\|$.*

Proof. Let I be the collection of finite-dimensional subspaces of $\mathcal{B}(E)$, partially ordered by inclusion. Let $\mathcal{U}$ be an ultrafilter on I which refines the order filter, so that, for $M \in I$, we have $\{N \in I : M \subseteq N\} \in \mathcal{U}$.

Pick $\lambda \in \mathcal{A}(E)^\circ$ and, for $M \in I$, let $x_M \in E$ and $\mu_M \in E'$ be given by Theorem 3.4.17, applied with $\varepsilon_M = (\dim M)^{-1}$. Then $\|x_M\| < (1+\varepsilon_M)^{1/p}\|\lambda\|^{1/p}$ and $\|\mu_M\| < (1+\varepsilon_M)^{1/q}\|\lambda\|^{1/q}$, so that, if we set $x = (x_M)$ and $\mu = (\mu_M)$, then $x \in (E)_\mathcal{U}$, $\mu \in (E')_\mathcal{U}$, and

$$\|x\|\|\mu\| = \lim_{M\in\mathcal{U}} \|x_M\|\|\mu_M\| \leq \lim_{M\in\mathcal{U}} (1+\varepsilon_M)\|\lambda\| = \|\lambda\|.$$

Then, for each $T \in \mathcal{B}(E)$, we have

$$|\langle \lambda, T\rangle - \langle \phi_1(x\otimes\mu), T\rangle| = |\langle \lambda, T\rangle - \lim_{M\in\mathcal{U}} \langle \mu_M, T(x_M)\rangle| < \lim_{M\in\mathcal{U}} \varepsilon_M \|\lambda\|\|T\| = 0,$$

so that $\phi_1(x\otimes\mu) = \lambda$, and hence $\|x\|\|\mu\| = \|\lambda\|$.

Now let $\lambda \in \mathcal{B}(E)'$. Then let $\tau = \kappa'_{\mathcal{A}(E)}(\lambda) \in E\widehat{\otimes}E'$ and $\lambda_0 = \lambda - \kappa_{E\widehat{\otimes}E'}(\tau) \in \mathcal{A}(E)^\circ$. Then let $\sigma \in (E)_\mathcal{U}\widehat{\otimes}(E')_\mathcal{U}$ be such that $\phi_1(\sigma) = \lambda_0$ and $\|\sigma\| = \|\lambda_0\|$. Thus $\phi_1(\sigma + \rho(\tau)) = \lambda$, and so

$$\|\lambda\| \leq \|\sigma + \rho(\tau)\| \leq \|\sigma\| + \|\rho(\tau)\| = \|\lambda_0\| + \|\tau\|.$$

For $\varepsilon > 0$, let $T \in \mathcal{B}(E)_{[1]}$ be such that $|\langle \lambda_0, T\rangle| \geq \|\lambda_0\| - \varepsilon$. Similarly, let $S \in \mathcal{F}(E)_{[1]}$ be such that $|\langle S, \tau\rangle| \geq \|\tau\| - \varepsilon$. We may suppose that $S = \sum_{i=1}^n \mu_i \otimes x_i \in E' \otimes E$, where each x_i and μ_i has finite support. Thus, there exists $N \in \mathbb{N}$ so that $P_N S P_N = S$. Recall that $Q_N = \mathrm{Id}_E - P_N$. By increasing N, we may suppose that $|\langle Q_N R Q_N, \tau\rangle| < \varepsilon\|R\|$ for each $R \in \mathcal{B}(E)$ (we do this by picking a representative of τ in $E\widehat{\otimes}E'$, and approximation). Then, by again increasing N, we may suppose that $\|x\|^p = \|Q_N(x) + P_N(x)\|^p = \|Q_N(x)\|^p + \|P_N(x)\|^p$ for each $x \in E$ (as E is of block p-type). As $P_N S P_N = S$, we thus have, for each $x \in E$,

$$\begin{aligned}
\|S(x) + Q_N T Q_N(x)\| &= \Big(\|P_N S P_N(x)\|^p + \|Q_N T Q_N(x)\|^p\Big)^{1/p} \\
&\leq \Big(\|S\|^p\|P_N(x)\|^p + \|Q_N T Q_N\|^p\|Q_N(x)\|^p\Big)^{1/p} \\
&\leq \|x\| \max\{\|S\|, \|T\|\} \leq \|x\|,
\end{aligned}$$

so that $\|S + Q_N T Q_N\| \leq 1$. Then we have,

$$\begin{aligned}
\|\lambda\| = \|\lambda_0 + \kappa_{E\widehat{\otimes}E'}(\tau)\| &\geq |\langle \lambda_0 + \kappa_{E\widehat{\otimes}E'}(\tau), S + Q_N T Q_N\rangle| \\
&= |\langle \lambda_0, T\rangle + \langle S + Q_N T Q_N, \tau\rangle| > \|\lambda_0\| + \|\tau\| - 3\varepsilon.
\end{aligned}$$

As $\varepsilon > 0$ was arbitrary, we conclude that

$$\|\lambda\| \leq \|\sigma + \rho(\tau)\| \leq \|\lambda_0\| + \|\tau\| = \|\lambda\|,$$

so that, in particular, $\|\lambda\| = \|\sigma + \rho(\tau)\|$. $\square$

Thus $\phi_1 : (E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}} \to \mathcal{B}(E)'$ is a quotient operator, and so $\theta_1 : \mathcal{B}(E)'' \to \mathcal{B}((E)_{\mathcal{U}})$ is an isometry onto its range.

There is another way to look at this result. The natural embedding $\mathcal{B}(E) \to \mathcal{B}((E)_{\mathcal{U}})$ induces a quotient map $\mathcal{B}((E)_{\mathcal{U}})' \to \mathcal{B}(E)'$, with the following diagram commuting:

$$\begin{tikzcd}
\mathcal{B}((E)_{\mathcal{U}})' \arrow[r, two heads] & \mathcal{B}(E)' \\
(E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}} \arrow[u, hook, "\kappa"] \arrow[ur, "\phi_1"'] &
\end{tikzcd}$$

where $\kappa = \kappa_{(E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}}}$. This means for each $\lambda \in \mathcal{B}(E)'$, if we view λ as a map from a subspace of $\mathcal{B}((E)_{\mathcal{U}})$ to $\mathbb{C}$, then there is a Hahn-Banach extension of λ to a member of $(E)_{\mathcal{U}}\widehat{\otimes}(E')_{\mathcal{U}}$.

3.5 Ultrapowers of modules

We will improve upon the Principle of Local Reflexivity and apply the results to ultrapowers of modules. These results are close to those in [Behrends, 1991], and we shall use the results of this paper below.

We start by proving an improved version of Helley's Lemma, giving a result first shown in [Barton, Yu, 1996] (where they follow much the same presentation as here).

Lemma 3.5.1. *Let E be a Banach space and let $N \subseteq E'$ be a closed subspace. If N is reflexive, then $({}^{\circ}N)^{\circ} = N$.*

Proof. As $({}^{\circ}N)^{\circ}$ is the weak*-closure of N, we see that $({}^{\circ}N)^{\circ} = N$ if and only if N is weak*-closed in E'. Suppose that N is reflexive, and that (μ_α) is a bounded net in N such that (μ_α) tends to $\mu \in E'$ in the weak*-topology. Let $\lambda = \text{weak-}\lim_\alpha \mu_\alpha$ so that $\lambda \in N$, as N is reflexive. Then, for $x \in E$, let $\hat{x} = \kappa_E(x) + N^\circ \in E''/N^\circ = N'$, so that we have

$$\begin{aligned}
\langle \mu, x \rangle = \lim_\alpha \langle \mu_\alpha, x \rangle = \lim_\alpha \langle \kappa_E(x) + N^\circ, \mu_\alpha \rangle = \lim_\alpha \langle \hat{x}, \mu_\alpha \rangle = \langle \hat{x}, \lambda \rangle \\
= \langle \kappa_E(x) + N^\circ, \lambda \rangle = \langle \lambda, x \rangle.
\end{aligned}$$

As $x \in E$ was arbitrary, we see that $\mu = \lambda \in N$, as required. $\square$

We can now give an improvement of Helley's Lemma.

Theorem 3.5.2. *Let E be a Banach space, $N \subseteq E'$ be a closed, reflexive subspace, $\Phi \in E''$ and $\varepsilon > 0$. Then there exists $x \in E$ with $\|x\| \leq (1+\varepsilon)\|\Phi\|$ and $\langle \Phi, \mu \rangle = \langle \mu, x \rangle$ for each $\mu \in N$.*

Proof. Let $X = E/{}^\circ N$ so that $X' = ({}^\circ N)^\circ = N$ as N is reflexive. Then $X'' = E''/N^\circ$, so let $\hat{\Phi} = \Phi + N^\circ \in X''$. As X is reflexive, $\hat{\Phi} = \hat{x} \in X = E/{}^\circ N$. Thus we can find $x \in E$ with $x + {}^\circ N = \hat{x}$ and $\|x\| \leq (1+\varepsilon)\|\hat{x}\| = (1+\varepsilon)\|\hat{\Phi}\| \leq (1+\varepsilon)\|\Phi\|$. Finally, for $\mu \in N$, we have

$$\langle \mu, x \rangle = \langle \mu, x + {}^\circ N \rangle = \langle \mu, \hat{x} \rangle = \langle \hat{\Phi}, \mu \rangle = \langle \Phi + N^\circ, \mu \rangle = \langle \Phi, \mu \rangle. \qquad \square$$

We will now sketch how to "bootstrap" this result to give the principle of local reflexivity. Note that the above is precisely the principle of local reflexivity when $M \subseteq E''$ is one-dimensional (that is, $M = \mathbb{C}\Phi$), so the idea is to work with a space of functions, thus reducing to a one-dimensional problem.

The following is from [Behrends, 1991].

Definition 3.5.3. Let E be a Banach space, $M \in \mathrm{FIN}(E'')$ and $N \in \mathrm{FIN}(E')$. A map $T : M \to E$ is an *ε-isomorphism along N* when T is a $(1+\varepsilon)$-isomorphism onto its range and $\langle \mu, T(\Phi) \rangle = \langle \Phi, \mu \rangle$ for $\Phi \in M$ and $\mu \in N$.

Let $(F_i)_{i=1}^n$ and $(G_j)_{j=1}^m$ be families of Banach spaces. Let $A_i : \mathcal{B}(M, E) \to F_i$ be an operator, for $1 \leq i \leq n$, and let $\psi_j : \mathcal{B}(M, E) \to G_j$ be an operator, for $1 \leq j \leq m$. For $1 \leq i \leq n$, let $f_i \in F_i$, and for $1 \leq j \leq m$, let $C_j \subseteq G_j$ be a convex set. Then M satisfies:

1. the *exact conditions* (A_i, f_i), for $1 \leq i \leq n$, and
2. the *approximate conditions* (ψ_j, C_j), for $1 \leq j \leq m$,

if for each $N \in \mathrm{FIN}(E')$ and $\varepsilon > 0$, there exists an ε-isomorphism $T : M \to E$ along N such that $A_i(T) = f_i$, for $1 \leq i \leq n$, and $\psi_j(T) \in (C_j)_\varepsilon = \{y + z : y \in C_j, z \in G_j, \|z\| \leq \varepsilon\}$, for $1 \leq j \leq m$.

When M is a finite-dimensional Banach space and E is a Banach space, we have that M' has the approximation property and the Radon-Nikodým property, so that $\mathcal{B}(M, E) = M' \check{\otimes} E$, $\mathcal{B}(M, E)' = M \widehat{\otimes} E'$ and $\mathcal{B}(M, E)'' = \mathcal{B}(M, E'')$.

So, for $M \in \mathrm{FIN}(E'')$, the inclusion map $M \to E''$ lies in $\mathcal{B}(M, E)''$; we denote this map by Id_M. Define $A_M : \mathcal{B}(M, E) \to \mathcal{B}(M \cap \kappa_E(E), E)$ to be the restriction operator, and let $\iota_M : M \cap \kappa_E(E) \to E$ be the map $\iota_M(\kappa_E(x)) = x$. Then the usual Principle of Local Reflexivity simply asserts that each $M \in \mathrm{FIN}(E'')$ satisfies the exact condition (A_M, ι_M).

Given (A_i) as in the above definition, we have that $A_i' : F_i' \to M\widehat{\otimes}E'$, and that $A_i'' : \mathcal{B}(M, E'') \to F_i''$.

Theorem 3.5.4. *Let E be a Banach space, $M \in \mathrm{FIN}(E'')$, and let (F_i), (A_i), (y_i), (G_j), (ψ_j) and (C_j) be as in the above definition. Then the following are equivalent:*

1. *M satisfies the exact conditions $(A_i, y_i)_{i=1}^n$ and the approximate conditions $(\psi_j, C_j)_{j=1}^m$;*
2. *Id_M is weak*-continuous on the weak*-closure of $A_1'(F_1')+\cdots+A_n'(F_n')$, $A_i''(\mathrm{Id}_M) = \kappa_{F_i}(y_i)$ for each i, and $\psi_j''(\mathrm{Id}_M)$ lies in the weak*-closure of $\kappa_{G_j}(C_j)$, for each j.*

Suppose that the map $T \mapsto (A_i(T))_{i=1}^n$ from $\mathcal{B}(M, E)$ to $A_1 \oplus \cdots \oplus A_n$ has a closed range. Then we may replace Id_M being weak-continuous on the weak*-closure of $\sum_{i=1}^n A_i'(F_i')$ by there existing some $T : M \to E$ which satisfies $A_i(T) = y_i$, for $1 \leq i \leq n$ (T need not satisfy any other condition).*

Proof. This is [Behrends, 1991, Theorem 2.3], and the remark thereafter. □

For example, the Principle of Local Reflexivity follows directly from this, as with $A_M : \mathcal{B}(M, E) \to \mathcal{B}(M \cap \kappa_E(E), E)$ as above, we see that A_M is surjective, so the second condition holds in a simplified form. Clearly there is some $T : M \to E$ with $A_M(T) = \iota_M$ (as M is finite-dimensional). A calculation shows that $A_M'' : \mathcal{B}(M, E'') \to \mathcal{B}(M \cap \kappa_E(E), E'')$ is also the restriction map, so that $A_M''(\mathrm{Id}_M) = \iota_M$, and so we immediate see that each M satisfies the exact condition (A_M, ι_M).

To prove the principle of local reflexivity directly, we apply Helley's lemma to Id_M and a suitable subspace of $M\widehat{\otimes}E'$ to find $T \in \mathcal{B}(M, E)$. Indeed, this subspace can be $N_0 = M\widehat{\otimes}N \subseteq M\widehat{\otimes}E'$, so that

$$\langle \mu, T(\Phi) \rangle = \langle \Phi \otimes \mu, T \rangle = \langle \mathrm{Id}_M, \Phi \otimes \mu \rangle = \langle \Phi, \mu \rangle \qquad (\Phi \in M, \mu \in N).$$

The other conditions follow by a suitable refinement of this argument (see, for example, [Ryan, 2002, Theorem 5.54]). With some more work, we can also allow N to be a reflexive subspace of E', given the strengthened version of Helley's lemma above.

We want to extend the principle of local reflexivity to (bi)modules of Banach algebras. Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach left (or right, or bi-module) $\mathcal{A}$-module, so that we can certainly apply the principle of local reflexivity to E. However, we also want to take account of the $\mathcal{A}$-module structure, that is, ensure that $T : M \to E$ is in some sense an $\mathcal{A}$-module homomorphism.

It will be helpful to recall that κ_E is a $\mathcal{A}$-module homomorphism. For the following, note that for $L \in \mathrm{FIN}(\mathcal{A})$ and $M \in \mathrm{FIN}(E'')$, we have

$$L \cdot M = \{a \cdot \Phi : a \in L, \Phi \in M\} \in \mathrm{FIN}(E'').$$

Theorem 3.5.5. *Let $\mathcal{A}$ be a Banach algebra and E be a Banach left $\mathcal{A}$-module. Let $M \in \mathrm{FIN}(E'')$, $L \in \mathrm{FIN}(\mathcal{A})$, $N \in \mathrm{FIN}(E')$, and $\varepsilon > 0$. Let $M_0 \in \mathrm{FIN}(E'')$ be such that $L \cdot M + M \subseteq M_0$. Then there exists $T : M_0 \to E$, a $(1 + \varepsilon)$-isomorphism onto its range, such that:*

1. *$\langle \Phi, \mu\rangle = \langle \mu, T(\Phi)\rangle$ for $\Phi \in M_0$ and $\mu \in N$;*
2. *$T(\kappa_E(x)) = x$ for $\kappa_E(x) \in M_0 \cap \kappa_E(E)$;*
3. *$\|a \cdot T(\Phi) - T(a \cdot \Phi)\| \leq \varepsilon\|a\|\|\Phi\|$ for $a \in L$ and $\Phi \in M$.*

A similar result holds for Banach right $\mathcal{A}$-modules and Banach $\mathcal{A}$-bimodules with condition (3) changed in the obvious way.

Proof. Let $\delta = \varepsilon/5$ or 1, whichever is smaller. Let $(a_i)_{i=1}^n$ be a set in L such that $\|a_i\| = 1$ for each i, and such that

$$\min_{1\leq i\leq n} \|a - a_i\| < \delta \qquad (a \in L, \|a\| = 1).$$

For $1 \leq i \leq n$, define $\psi_i : \mathcal{B}(M_0, E) \to \mathcal{B}(M, E)$ by

$$\psi_i(T)(\Phi) = T(a_i \cdot \Phi) - a_i \cdot T(\Phi) \qquad (T \in \mathcal{B}(M_0, E), \Phi \in M).$$

Then $\psi_i' : M\widehat{\otimes}E' \to M_0\widehat{\otimes}E'$, and for $\Phi \in M$, $\mu \in E'$ and $T \in \mathcal{B}(M_0, E)$, we have

$$\langle \psi_i'(\Phi \otimes \mu), T\rangle = \langle \mu, \psi_i(T)(\Phi)\rangle = \langle \mu, T(a_i \cdot \Phi) - a_i \cdot T(\Phi)\rangle.$$

Thus we have $\psi_i'(\Phi \otimes \mu) = a_i \cdot \Phi \otimes \mu - \Phi \otimes \mu \cdot a_i$. Then, for $\Phi \in M$ and $\mu \in E'$, we have

$$\langle \psi_i''(\mathrm{Id}_{M_0}), \Phi \otimes \mu\rangle = \langle \mathrm{Id}_{M_0}, a_i \cdot \Phi \otimes \mu - \Phi \otimes \mu \cdot a_i\rangle = \langle a_i \cdot \Phi, \mu\rangle - \langle \Phi, \mu \cdot a_i\rangle = 0.$$

Hence M satisfies the approximate conditions $(\psi_i, \{0\})$, for $1 \leq i \leq n$.

We can the apply the above theorem (with A_{M_0} as well) to find $T \in \mathcal{B}(M_0, E)$, a $(1+\delta)$-isomorphism onto its range, with conditions (1) and (2), and such that $\|\psi_i(T)\| < \delta$

for $1 \leq i \leq n$. Then, for $a \in L$ and $\Phi \in M$ with $\|a\| = \|M\| = 1$, we can find i with $\|a - a_i\| < \delta$. Then we have

$$\begin{aligned}
&\|a \cdot T(\Phi) - T(a \cdot \Phi)\| \\
&\qquad \leq \|(a - a_i) \cdot T(\Phi)\| + \|a_i \cdot T(\Phi) - T(a_i \cdot \Phi)\| + \|T(a_i \cdot \Phi - a.\Phi)\| \\
&\qquad < \delta(1 + \delta) + \|\psi_i(T)\| + (1 + \delta)\delta < 3\delta + 2\delta^2 < \varepsilon.
\end{aligned}$$

Thus we are done, as $\delta < \varepsilon$.

Similarly, we can easily adapt the above argument to give the result for right $\mathcal{A}$-modules and $\mathcal{A}$-bimodules. □

We can then apply this to an ultrapower construction. We state this for left modules; the other cases are entirely similar. Note that, if E is a Banach left $\mathcal{A}$-module, then so is $(E)_{\mathcal{U}}$ with co-ordinate-wise operations. We then have the map $\sigma : (E)_{\mathcal{U}} \to E'', (x_i) \mapsto$ weak*-$\lim_{i \in \mathcal{U}} x_i$. We claim that this is a left $\mathcal{A}$-module homomorphism. Indeed, for $a \in \mathcal{A}$, $x = (x_i) \in (E)_{\mathcal{U}}$ and $\mu \in E'$, we have

$$\langle \mu, a \cdot \sigma(x) \rangle = \langle \mu \cdot a, \sigma(x) \rangle = \lim_{i \in \mathcal{U}} \langle \mu \cdot a, x_i \rangle = \lim_{i \in \mathcal{U}} \langle \mu, a \cdot x_i \rangle = \langle \mu, \sigma(a \cdot x) \rangle.$$

Proposition 3.5.6. *Let $\mathcal{A}$ be a Banach algebra, and let E be a Banach left $\mathcal{A}$-module. Then there exist an ultrafilter $\mathcal{U}$ and an isometric (onto its range) left $\mathcal{A}$-module homomorphism $K : E'' \to (E)_{\mathcal{U}}$ such that $\sigma \circ K = \mathrm{Id}_{E''}$ and with K restricted to E being the canonical map $E \to (E)_{\mathcal{U}}$. Consequently, we can view E'' has a one-complemented submodule of $(E)_{\mathcal{U}}$.*

An analogous statement holds for right- and bi-modules.

Proof. This is a standard ultrapower argument, similar to the proof of Theorem 3.4.17. □

Example 3.5.7. Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is a Banach $\mathcal{A}$-bimodule, so there exists an ultrafilter $\mathcal{U}$ and an $\mathcal{A}$-bimodule homomorphism $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ with the above properties. Now, $(\mathcal{A})_{\mathcal{U}}$ has a natural algebra structure, and $\mathcal{A}''$ has the Arens products, but it seems unlikely that K is a homomorphism. However, we do have

$$K(\kappa_{\mathcal{A}}(a) \Box \Phi) = K(a \cdot \Phi) = a \cdot K(\Phi) \qquad (a \in \mathcal{A}, \Phi \in \mathcal{A}''),$$

where the module action of $\mathcal{A}$ on $(\mathcal{A})_{\mathcal{U}}$ is the same as the action induced by the canonical embedding $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$ followed by the algebra product on $(\mathcal{A})_{\mathcal{U}}$. Similarly, this holds for

$\Diamond$, and "on the right", so that, for $\Phi, \Psi \in \mathcal{A}''$, we have

$$K(\Phi \Box \Psi) = K(\Phi)K(\Psi) = K(\Phi \Diamond \Psi)$$

when at least one of Φ, Ψ is in $\kappa_{\mathcal{A}}(\mathcal{A})$. $\Box$

Notice that the above example gives a way to define another bilinear "product" on $\mathcal{A}''$ by defining

$$\Phi * \Psi = \sigma\big(K(\Phi)K(\Psi)\big).$$

This idea is studied in detail in [Iochum, Loupias, 1989] and [Godefroy, Iochum, 1988], although in this paper, the authors study $*$ for general $\mathcal{U}$ and map K, as opposed to our approach of trying to construct $\mathcal{U}$ and K which have special properties. In general, it turns out that one cannot say a great deal about $*$ (so that, for example, $*$ need not be associative in general). However, for unital C*-algebras $\mathcal{A}$, this product $*$ actually always agrees with the Arens products on $\mathcal{A}''$, a fact shown in [Godefroy, Iochum, 1988, Theorem II.1].

Theorem 3.5.8. *Let $\mathcal{A}$ be an Arens regular Banach algebra, $M \in \mathrm{FIN}(\mathcal{A}'')$, $N \in \mathrm{FIN}(\mathcal{A}')$ and $\varepsilon > 0$. Let $M_0 = M + M \Box M$ and $N_0 = N + M \cdot N$. Then there exists a $(1+\varepsilon)$-isomorphism onto its range $T : M_0 \to \mathcal{A}$ such that:*

1. *$\langle \Phi, \mu \rangle = \langle \mu, T(\Phi) \rangle$ for $\Phi \in M_0$ and $\mu \in N_0$;*
2. *$T(\kappa_{\mathcal{A}}(a)) = a$ for $\kappa_{\mathcal{A}}(a) \in M_0 \cap \kappa_{\mathcal{A}}(\mathcal{A})$;*
3. *$|\langle \mu, T(\Phi \Box \Psi) - T(\Phi)T(\Psi) \rangle| \leq \varepsilon \|\mu\| \|\Phi\| \|\Psi\|$ for $\mu \in N$ and $\Phi, \Psi \in M$.*

Proof. Let $\delta > 0$ be such that $\delta < \varepsilon$ and $\delta(1+\delta)(3+\delta) < \varepsilon$. Let $(\mu_i)_{i=1}^n \subseteq N$ be such that $\|\mu_i\| = 1$ for each i, and such that we have

$$\min_{1 \leq i \leq n} \|\mu_i - \mu\| < \delta \qquad (\mu \in N, \|\mu\| = 1).$$

For $1 \leq i \leq n$, define $\psi_i : \mathcal{B}(M_0, \mathcal{A}) \to \mathcal{B}(M_0, \mathcal{A}')$ by

$$\psi_i(T)(\Phi) = T(\Phi) \cdot \mu_i \qquad (T \in \mathcal{B}(M_0, \mathcal{A}), \Phi \in M_0),$$

and define $T_i \in \mathcal{B}(M_0, \mathcal{A}')$ by $T_i(\Phi) = \Phi \cdot \mu_i$ for $\Phi \in M_0$. Then we have $\psi_i' : M_0 \widehat{\otimes} \mathcal{A}'' \to M_0 \widehat{\otimes} \mathcal{A}'$, and, for $T \in \mathcal{B}(M_0, \mathcal{A})$, $\Phi \in M_0$ and $\Lambda \in \mathcal{A}''$, we have

$$\langle \phi_i'(\Phi \otimes \Lambda), T \rangle = \langle \Lambda, \phi_i(T)(\Phi) \rangle = \langle \Lambda, T(\Phi) \cdot \mu_i \rangle = \langle \mu_i \cdot \Lambda, T(\Phi) \rangle.$$

Thus we have $\phi_i'(\Phi \otimes \Lambda) = \Phi \otimes \mu_i \cdot \Lambda$, and so

$$\langle \phi_i''(\mathrm{Id}_{M_0}), \Phi \otimes \Lambda \rangle = \langle \Phi, \mu_i \cdot \Lambda \rangle = \langle \Lambda \Diamond \Phi, \mu_i \rangle = \langle \Lambda, \Phi \cdot \mu_i \rangle = \langle \kappa_{\mathcal{A}'}(T_i(\Phi)), \Lambda \rangle,$$

as $\mathcal{A}$ is Arens regular. Thus $\phi_i''(\mathrm{Id}_{M_0}) = \kappa_{\mathcal{B}(M_0,\mathcal{A}')}(T_i)$. Again, we can then find $T \in \mathcal{B}(M_0, \mathcal{A})$ satisfying (1) and (2), and such that $\|\psi_i(T) - T_i\| < \delta$ for $1 \leq i \leq n$.

For $\mu \in N$ and $\Phi, \Psi \in M$ with $\|\mu\| = \|\Phi\| = \|\Psi\| = 1$, let i be such that $\|\mu - \mu_i\| < \delta$. Then $\Phi\Box\Psi \in M_0$ and $\Psi \cdot \mu \in N_0$ so that we have

$$\langle \mu, T(\Phi\Box\Psi)\rangle = \langle \Phi\Box\Psi, \mu\rangle = \langle \Phi, \Psi \cdot \mu\rangle = \langle \Psi \cdot \mu, T(\Phi)\rangle.$$

As $\|\psi_i(T) - T_i\| < \delta$, we have $\|T(\Psi) \cdot \mu_i - \Psi \cdot \mu_i\| < \delta$, and so

$$\begin{aligned}
&\|T(\Psi) \cdot \mu - \Psi \cdot \mu\| \\
&\qquad \leq \|T(\Psi) \cdot \mu - T(\Psi) \cdot \mu_i\| + \|T(\Psi) \cdot \mu_i - \Psi \cdot \mu_i\| + \|\Psi \cdot \mu_i - \Psi \cdot \mu\| \\
&\qquad < \delta\|T(\Psi)\| + \delta + \delta\|\Psi\| \leq \delta(1+\delta) + 2\delta.
\end{aligned}$$

Putting these together, we then get

$$\begin{aligned}
|\langle \mu, T(\Phi\Box\Psi) - T(\Phi)T(\Psi)\rangle| &= |\langle \Psi \cdot \mu, T(\Phi)\rangle - \langle T(\Psi) \cdot \mu, T(\Phi)\rangle| \\
&< \|T(\Phi)\|\big(\delta(1+\delta) + 2\delta\big) \leq (1+\delta)\big(\delta(1+\delta) + 2\delta\big) \\
&= \delta(1+\delta)(3+\delta) < \varepsilon,
\end{aligned}$$

as required, completing the proof. □

Theorem 3.5.9. *Let $\mathcal{A}$ be an Arens regular Banach algebra. Then there exists an ultrafilter $\mathcal{U}$ and an isometry $K : \mathcal{A}'' \to (\mathcal{A})_{\mathcal{U}}$ such that $K \circ \kappa_{\mathcal{A}}$ is the canonical map $\mathcal{A} \to (\mathcal{A})_{\mathcal{U}}$, $\sigma \circ K = \mathrm{Id}_{\mathcal{A}''}$ and such that the map $*$, defined above, agrees with the Arens products on $\mathcal{A}''$.*

Proof. This is a standard ultrafilter argument, given the above theorem. □

Let $\mathcal{A}$ be Arens regular, and K be given as above. For $\Phi, \Psi \in \mathcal{A}''$, let $K(\Phi) = (a_i)$ and $K(\Psi) = (b_i)$, so that

$$\Phi\Box\Psi = \Phi\Diamond\Psi = \Phi * \Psi = \operatorname*{weak^*\text{-}lim}_{i\in\mathcal{U}} a_i b_i,$$

which gives a symmetric definition of the Arens products (compare to Section 1.7).

Chapter 4

Structure and semi-simplicity of $\mathcal{B}(E)''$

In this chapter we shall construct some examples of reflexive Banach spaces E for which $\mathcal{B}(E)$ is not Arens regular. We shall also study the question of whether $\mathcal{B}(E)''$ is semi-simple, presenting some joint work with Charles Read (see [Daws, Read, 2004]) which shows, in particular, that $\mathcal{B}(l^p)''$ is not semi-simple unless $p = 2$.

4.1 Arens regularity of $\mathcal{B}(E)$

We now know that when E is super-reflexive, $\mathcal{B}(E)$ is Arens regular. In [Young, 1976, Corollary 1], Young produced a reflexive Banach space E such that $\mathcal{B}(E)$ is not Arens regular. Young's approach is via group algebras (which are never Arens regular unless they are finite-dimensional). We shall present a shorter, direct construction, and shall also discuss why it seems to be difficult to prove (or find a counter-example) to the conjecture that E is super-reflexive if and only if $\mathcal{B}(E)$ is Arens regular.

Firstly we need a good characterisation of when E is a reflexive Banach space.

Theorem 4.1.1. *Let E be a Banach space. Then the following are equivalent:*

1. *E is not reflexive;*
2. *for each $\theta \in (0, 1)$ there exist sequences of unit vectors (x_n) and (μ_n) in E and E', respectively, such that*

$$\langle \mu_m, x_n \rangle = \begin{cases} \theta & : m \leq n, \\ 0 & : m > n. \end{cases}$$

Proof. This is [James, 1972, Lemma 1]. We can use some of the theory we have previously developed. When E is not reflexive, let $\theta \in (0, 1)$ and let $M \in E'''$ be such that

$\kappa'_E(M) = 0$ and $\theta < \|M\| < 1$. Let $\Phi \in E''$ be such that $\langle M, \Phi\rangle = \theta$ and $\|\Phi\| < 1$. Let $\mu_1 \in E'_{[1]}$ be such that $\langle \Phi, \mu_1\rangle = \theta$. By adding a suitable vector from $\ker(\Phi) \subseteq E'$, we may suppose that $\|\mu_1\| = 1$. Then apply Helley's Lemma (Theorem 3.5.2) to find $x_1 \in E_{[1]}$ with $\langle \mu_1, x_1\rangle = \langle \Phi, \mu_1\rangle = \theta$. Similarly, we may suppose that $\|x_1\| = 1$ (as $\|\Phi\| < 1$).

Suppose we have found $(x_i)_{i=1}^n$, $(\mu_i)_{i=1}^n$ and $(\Phi_i)_{i=1}^n$ with:

$$\langle \mu_i, x_j\rangle = \begin{cases} \theta & : i \leq j, \\ 0 & : i > j, \end{cases} \qquad (1 \leq i, j \leq n),$$

$$\langle \Phi, \mu_i\rangle = \theta, \|x_i\| = \|\mu_i\| = 1 \qquad (1 \leq i \leq n).$$

Apply Helley's Lemma to M to find $\mu_{n+1} \in E'$ with (we may again similarly suppose) $\|\mu_{n+1}\| = 1$, $\langle \Phi, \mu_{n+1}\rangle = \theta$, and, for $1 \leq i \leq n$, $\langle \mu_{n+1}, x_i\rangle = 0$. Similarly, apply Helley's Lemma to Φ to find $x_{n+1} \in E$ with $\|x\| = 1$ and, for $1 \leq i \leq n+1$, $\langle \mu_i, x\rangle = \theta$. Thus, by induction, we have shown that (1)⇒(2).

Conversely, suppose that we have sequences (x_n) and (μ_n) for some $\theta \in (0,1)$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$ and set

$$\Phi = \operatorname{weak}^*\text{-}\lim_{n\in\mathcal{U}} x_n \in E'' \quad , \quad M = \operatorname{weak}^*\text{-}\lim_{n\in\mathcal{U}} \mu_n \in E'''.$$

Then we have

$$\langle \Phi, \mu_m\rangle = \lim_{n\in\mathcal{U}} \langle \mu_m, x_n\rangle = \theta \qquad (m \in \mathbb{N}),$$

so that $\langle M, \Phi\rangle = \lim_{m\in\mathcal{U}} \langle \Phi, \mu_m\rangle = \theta$. Conversely, we have

$$\langle M, \kappa_E(x_n)\rangle = \lim_{m\in\mathcal{U}} \langle \mu_m, x_n\rangle = 0 \qquad (x \in \mathbb{N}),$$

so that if $M = \kappa_{E'}(\mu)$ for some $\mu \in E'$, we have

$$\langle M, \Phi\rangle = \langle \Phi, \mu\rangle = \lim_{n\in\mathcal{U}} \langle \mu, x_n\rangle = \lim_{n\in\mathcal{U}} \langle M, \kappa_E(x_n)\rangle = 0,$$

a contradiction. Thus E is not reflexive. $\square$

Recall (Theorem 2.5.2) that if $\mathcal{B}(E)$ is Arens regular, then E must be reflexive. Conversely, we have the following.

Proposition 4.1.2. *Let F be a non-reflexive Banach space and let $(M_n, \|\cdot\|_n)$ be a sequence of Banach spaces such that, for some $\varepsilon > 0$ and each $M \in \mathrm{FIN}(F)$, M is $(1+\varepsilon)$-isomorphic to some subspace of some M_n. Let $E = \bigoplus_{n=1}^\infty M_n$ as a linear space, and suppose that E admits a norm $\|\cdot\|$ which satisfies:*

1. *there exists $C > 0$ such that, when (x_n) and (y_n) are sequences in E with $\|y_n\|_n \leq \|x_n\|_n$ for all n, we have $\|(y_n)\| \leq C\|(x_n)\|$;*

2. *the natural inclusion maps $\iota_n : M_n \to E$ are uniformly bounded;*

3. *the natural projection maps $\pi_n : E \to M_n$ are uniformly bounded.*

Let E_0 be the norm-completion of E. Then $\mathcal{B}(E_0)$ is not Arens regular.

Proof. Condition (3) on the norm implies that for $N \in \mathbb{N}$ and $\lambda \in M'_N$, we can define $\lambda_0 \in E'$ by

$$\langle \lambda_0, (x_n)\rangle = \langle \lambda, x_N\rangle \qquad ((x_n) \in E),$$

with $\|\lambda_0\| \leq \|\pi_N\|\|\lambda\|$. We can hence view M'_N as a subspace of E' (note that $E' = E'_0$).

Use Theorem 4.1.1 to find sequences (x_n) in F and (μ_n) in F', for some $\theta \in (0,1)$. For each $i \in \mathbb{N}$, let $N_i = \mathrm{lin}\{x_1, \dots, x_i\} \in \mathrm{FIN}(F)$, so that there exists $n(i) \in \mathbb{N}$ with N_i being $(1+\varepsilon)$-isomorphic to a subspace of $M_{n(i)}$. For each $i, j \in \mathbb{N}$, we can regard μ_j as being in N'_i, by restriction. As N'_i is $(1+\varepsilon)$-isomorphic to a quotient of $M'_{n(i)}$, we can find $\lambda_j \in M'_{n(i)}$ such that λ_j maps to the image of μ_j under the quotient map, and such that, say, $1 \leq \|\lambda_j\| \leq (1+\varepsilon)\|\mu_j\| = 1+\varepsilon$.

We can hence find an increasing sequence $(n(i))_{i=1}^\infty$ such that, for each $i \in \mathbb{N}$, there exist vectors $(x_j^{(i)})_{j=1}^i \subseteq M_{n(i)}$ and $(\lambda_j^{(i)})_{j=1}^i \subseteq M'_{n(i)}$ such that

$$\langle \lambda_j^{(i)}, x_k^{(i)}\rangle = \begin{cases} \theta & : j \leq k, \\ 0 & : j > k. \end{cases} \qquad (1 \leq j, k \leq i).$$

We may also suppose that $(1+\varepsilon)^{-1} \leq \|x_j^{(i)}\|_{n(i)} \leq (1+\varepsilon)$ and $1 \leq \|\lambda_j^{(i)}\|_{n(i)} \leq 1+\varepsilon$, for each i and j.

For each $N \in \mathbb{N}$, define $T_N : E \to E$ by setting, for $(x_n) \in E$, $T_N(x_n) = (y_n)$, where

$$y_n = \begin{cases} \langle \lambda_N^{(i)}, x_{n(i)}\rangle x_N^{(i)} & : n = n(i), i \geq N \\ 0 & : \text{otherwise}. \end{cases} \qquad (n \in \mathbb{N})$$

For all $n \in \mathbb{N}$, we thus have $\|y_n\| = 0$, or that $n = n(i)$ for some $i \geq N$, and so $\|y_n\|_n \leq \|x_N^{(i)}\|\|\lambda_N^{(i)}\|\|x_n\|_n \leq (1+\varepsilon)^2\|x_n\|_n$. By condition (1) on the norm, for each $N \in \mathbb{N}$, T_N is continuous, and so T_N extends to a member of $\mathcal{B}(E_0)$. We also see that the family $(T_N)_{N=1}^\infty$ is bounded.

For $N, M \in \mathbb{N}$ and $(x_n) \in E$, let $T_N(x_n) = (y_n)$ and $T_M(y_n) = (z_n)$, so that we have

$$y_{n(i)} = \begin{cases} \langle \lambda_N^{(i)}, x_{n(i)} \rangle x_N^{(i)} & : i \geq N, \\ 0 & : \text{otherwise.} \end{cases} \qquad (i \in \mathbb{N})$$

$$z_n = \begin{cases} \langle \lambda_M^{(i)}, y_{n(i)} \rangle x_M^{(i)} & : n = n(i), i \geq M, \\ 0 & : \text{otherwise,} \end{cases}$$

$$= \begin{cases} \langle \lambda_N^{(i)}, x_{n(i)} \rangle \langle \lambda_M^{(i)}, x_N^{(i)} \rangle x_M^{(i)} & : n = n(i), i \geq M, i \geq N, \\ 0 & : \text{otherwise,} \end{cases}$$

$$= \begin{cases} \theta \langle \lambda_N^{(i)}, x_{n(i)} \rangle x_M^{(i)} & : n = n(i), i \geq N, M \leq N \\ 0 & : \text{otherwise,} \end{cases}$$

Thus we see that $T_M \circ T_N = 0$ for $M > N$.

Now suppose that $M \leq N$, so that $(T_M \circ T_N)(\iota_{n(i)}(x_i^{(i)})) = \theta^2 \iota_{n(i)}(x_M^{(i)})$, for $i \geq N$. Thus

$$\langle \pi'_{n(i)}(\lambda_1^{(i)}), (T_M \circ T_N)(\iota_{n(i)}(x_i^{(i)})) \rangle = \theta^3 \qquad (i \geq N).$$

By conditions (2) and (3) on the norm, we have can thus define

$$\langle \lambda, T \rangle = \lim_{i \in \mathcal{U}} \langle \pi'_{n(i)}(\lambda_1^{(i)}), T(\iota_{n(i)}(x_i^{(i)})) \rangle \qquad (T \in \mathcal{B}(E_0)),$$

where $\mathcal{U}$ is some non-principal ultrafilter on $\mathbb{N}$. Then $\langle \lambda, T_M \circ T_N \rangle = \theta^3$ for $M \leq N$, and 0 for $M > N$. We conclude that $\mathcal{B}(E_0)$ is not Arens regular by Theorem 1.7.2. □

Corollary 4.1.3. *Let $p \in (1, \infty)$ and $E = l^p(\bigoplus_{n=1}^\infty l_n^1)$. Then E is reflexive, and $\mathcal{B}(E)$ is not Arens regular.*

Proof. A direct calculation shows that $E' = l^q(\bigoplus_{n=1}^\infty l_n^\infty)$, where $p^{-1} + q^{-1} = 1$, and that E is reflexive. It is easy to see that if $M_n = l_n^1$, then for each finite-dimensional subspace M of l^1 and each $\varepsilon > 0$, M is $(1+\varepsilon)$-isomorphic to some subspace of some M_n. Clearly the l^p norm satisfies conditions (1), (2) and (3) in the above proposition, so that we are done. □

There are many other examples to which we could apply this idea. However, the key idea is that we "glue together" a sequence of (we may suppose) finite-dimensional Banach spaces which are asymptotically non-reflexive. Importantly, we suppose that we have operators which manipulate each of these finite-dimensional spaces in an independent way.

If we were to start with an arbitrary reflexive, but not super-reflexive, Banach space, then we could still find finite-dimensional subspaces which were asymptotically non-reflexive (from the definition of what it means to be not super-reflexive). However, there is no guarantee that we can define a (bounded) operator which can manipulate these finite-dimensional subspaces in an independent way. This is precisely what we need to do, however, if we try to reproduce the above proof.

The central problem here (and, to some extent, in previous sections) is that, for an entirely arbitrary Banach space E, the only operators which we can write down are approximable or a scalar multiple of the identity. This is not a failure of our approach, but rather, is an unsolved problem: does there exist a Banach space E such that every operator on E is the sum of an approximable (or compact) operator and a scalar multiple of the identity? For some recent progress on this, see, for example, [Schlumprecht, 2003]. Suppose that we have a Banach space E such that $\mathcal{B}(E) = \mathcal{K}(E) \oplus \mathbb{C}$. We then see that $\mathcal{B}(E)$ is Arens regular if and only if E is reflexive. It seems a slightly strong (and perhaps unreasonable) conjecture that a reflexive Banach space E with $\mathcal{B}(E)/\mathcal{K}(E) = \mathbb{C}$ is automatically super-reflexive.

Having said all of this, we now head on into the next section, where we shall show that an algebraic property of $\mathcal{B}(l^p)$ does indeed characterise the Hilbert space: namely when $\mathcal{B}(l^p)''$ is semi-simple.

4.2 Semi-simplicity of $\mathcal{B}(E)''$

Looking back to Section 1.5, we see that $\mathcal{B}(E)$ is trivially a semi-simple Banach algebra. When E is a Hilbert space, as $\mathcal{B}(E)''$ is a C*-algebra, it is also semi-simple. As such, it seems a natural question to ask whether or not $\mathcal{B}(E)''$ is semi-simple for other Banach spaces E: in particular, is $\mathcal{B}(l^p)''$ semi-simple?

The first surprise is that there are reasonably well-behaved spaces E for which $\mathcal{B}(E)''$ is not semi-simple. The main result here (Corollary 4.2.9) was first demonstrated by C.J. Read, by a direct construction of an element of the radical. We take a more algebraic approach here.

For this section, let E be a reflexive Banach space. Then, as before, $(E\widehat{\otimes}E')' = \mathcal{B}(E)$, so that $\mathcal{B}(E)$ is a dual Banach algebra (see Definition 1.6.5, and after Proposition 2.2.3).

Proposition 4.2.1. *Let E be a Banach space. Then $\mathcal{B}(E)$ is a dual Banach algebra if and only if E is reflexive.*

Proof. We need to show that when $\mathcal{B}(E)$ is a dual Banach algebra, E is reflexive. By Proposition 1.6.6, let F be a Banach $\mathcal{B}(E)$-bimodule and $\phi : \mathcal{B}(E) \to F'$ be a $\mathcal{B}(E)$-bimodule isomorphism.

Let (x_n) be a bounded sequence in E and (μ_n) be a bounded sequence in E'. Let $x_0 \in E$ and $\mu_0 \in E'$ be such that $\langle \mu_0, x_0 \rangle = 1$, let $T_n = \mu_n \otimes x_0$ and $S_n = \mu_0 \otimes x_n$, for each $n \in \mathbb{N}$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and define

$$\lambda_S = \operatorname*{weak^*-lim}_{n\in\mathcal{U}} \phi(S_n) \in F' \quad , \quad \lambda_T = \operatorname*{weak^*-lim}_{n\in\mathcal{U}} \phi(T_n) \in F'.$$

As ϕ is an isomorphism, let $T, S \in \mathcal{B}(E)$ be such that $\phi(S) = \lambda_S$ and $\phi(T) = \lambda_T$. For $y \in F$, as ϕ is a $\mathcal{B}(E)$-bimodule homomorphism, we have

$$\begin{aligned}
\lim_{n\in\mathcal{U}} \lim_{m\in\mathcal{U}} \langle \phi(T_n S_m), y \rangle &= \lim_{n\in\mathcal{U}} \lim_{m\in\mathcal{U}} \langle \mu_n, x_m \rangle \langle \phi(\mu_0 \otimes x_0), y \rangle \\
&= \lim_{n\in\mathcal{U}} \lim_{m\in\mathcal{U}} \langle \phi(S_m), y \cdot T_n \rangle = \lim_{n\in\mathcal{U}} \langle \phi(S), y \cdot T_n \rangle = \lim_{n\in\mathcal{U}} \langle \phi(T_n S), y \rangle \\
&= \lim_{n\in\mathcal{U}} \langle \phi(T_n), S \cdot y \rangle = \langle \phi(T), S \cdot y \rangle = \langle \phi(TS), y \rangle, \\
\lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \langle \phi(T_n S_m), y \rangle &= \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \langle \mu_n, x_m \rangle \langle \phi(\mu_0 \otimes x_0), y \rangle \\
&= \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \langle \phi(T_n), S_m \cdot y \rangle = \lim_{m\in\mathcal{U}} \langle \phi(T), S_m \cdot y \rangle = \lim_{m\in\mathcal{U}} \langle \phi(T S_m), y \rangle \\
&= \lim_{m\in\mathcal{U}} \langle \phi(S_m), y \cdot T \rangle = \langle \phi(S), y \cdot T \rangle = \langle \phi(TS), y \rangle.
\end{aligned}$$

Hence we have

$$\lim_{n\in\mathcal{U}} \lim_{m\in\mathcal{U}} \langle \mu_n, x_m \rangle = \lim_{m\in\mathcal{U}} \lim_{n\in\mathcal{U}} \langle \mu_n, x_m \rangle,$$

as $\phi(\mu_0 \otimes x_0) \neq 0$, as ϕ is an isomorphism. By Theorem 1.4.7, we see that E must be reflexive, as required. $\square$

Notice that $\kappa'_{E\widehat{\otimes}E'} : \mathcal{B}(E)'' \to \mathcal{B}(E)$ is a projection; that is, $\kappa'_{E\widehat{\otimes}E'} \circ \kappa_{\mathcal{B}(E)}$ is the identity on $\mathcal{B}(E)$. Recall that $E\widehat{\otimes}E'$ is a Banach $\mathcal{B}(E)$-bimodule. The following could be formulated for general dual Banach algebras (see after Theorem 1.14 in [Dales, Lau, 2004]), but we will only prove the limited version which we need.

Proposition 4.2.2. *Let E be a reflexive Banach space, and let $\kappa = \kappa_{E\widehat{\otimes}E'} : E\widehat{\otimes}E' \to \mathcal{B}(E)'$. Then we have the following:*

1. *κ is a $\mathcal{B}(E)$-bimodule homomorphism;*

2. *κ' is a $\mathcal{B}(E)$-bimodule homomorphism;*

3. *for $\Phi \in \mathcal{B}(E)''$ and $\tau \in E\widehat{\otimes}E'$, we have $\Phi \cdot \kappa(\tau) = \kappa(\kappa'(\Phi) \cdot \tau)$ and $\kappa(\tau) \cdot \Phi = \kappa(\tau \cdot \kappa'(\Phi))$;*

4. *κ' is a homomorphism for both Arens products on $\mathcal{B}(E)''$;*

5. *if we identify $\mathcal{B}(E)$ with its image in $\mathcal{B}(E)''$, then κ' is a projection onto $\mathcal{B}(E)$, and so we have $\mathcal{B}(E)'' = \mathcal{B}(E) \oplus \ker \kappa'$;*

6. *writing $\mathcal{B}(E)'' = \mathcal{B}(E) \oplus \ker \kappa'$, we have*

$$(T, \Gamma_1)\Box(S, \Gamma_2) = (TS, T \cdot \Gamma_2 + \Gamma_1 \cdot S + \Gamma_1 \Box \Gamma_2) \in \mathcal{B}(E) \oplus \ker \kappa',$$

for $(T, \Gamma_1), (S, \Gamma_2) \in \mathcal{B}(E) \oplus \ker \kappa'$, and similarly for the product $\Diamond$.

Proof. 1. For $S, T \in \mathcal{B}(E)$ and $\tau \in E'\widehat{\otimes}E$, we have

$$\langle \kappa(T \cdot \tau), S \rangle = \langle S, T \cdot \tau \rangle = \langle S \circ T, \tau \rangle = \langle \kappa(\tau), S \circ T \rangle = \langle T \cdot \kappa(\tau), S \rangle,$$

and similarly $\kappa(\tau \cdot T) = \kappa(\tau) \cdot T$.

2. This is now standard from (1).

3. For $T \in \mathcal{B}(E)$, we have

$$\begin{aligned}\langle \Phi \cdot \kappa(\tau), T \rangle &= \langle \Phi, \kappa(\tau) \cdot T \rangle = \langle \Phi, \kappa(\tau \cdot T) \rangle = \langle \kappa'(\Phi), \tau \cdot T \rangle \\ &= \langle T \circ \kappa'(\Phi), \tau \rangle = \langle T, \kappa'(\Phi) \cdot \tau \rangle = \langle \kappa(\kappa'(\Phi) \cdot \tau), T \rangle,\end{aligned}$$

and similarly $\kappa(\tau) \cdot \Phi = \kappa(\tau \cdot \kappa'(\Phi))$.

4. For $\Phi, \Psi \in \mathcal{B}(E)''$ and $\tau \in E\widehat{\otimes}E'$, we have

$$\langle \kappa'(\Phi \Box \Psi), \tau \rangle = \langle \Phi, \Psi \cdot \kappa(\tau) \rangle = \langle \Phi, \kappa(\kappa'(\Psi) \cdot \tau) \rangle = \langle \kappa'(\Phi) \circ \kappa'(\Psi), \tau \rangle,$$

and similarly,

$$\langle \kappa'(\Phi \Diamond \Psi), \tau \rangle = \langle \Psi, \kappa(\tau) \cdot \Phi \rangle = \langle \Psi, \kappa(\tau \cdot \kappa'(\Phi)) \rangle = \langle \kappa'(\Phi) \circ \kappa'(\Psi), \tau \rangle.$$

5. This is immediate.

6. We have $\kappa'((T+\Gamma_1)\Box(S+\Gamma_2)) = \kappa'(TS)+\kappa'(\Gamma_1)\cdot S+T\cdot\kappa'(\Gamma_2)+\kappa'(\Gamma_1)\circ\kappa'(\Gamma_2) = T \circ S$, the rest following immediately. $\Box$

We will continue to slightly abuse notation and treat $\mathcal{B}(E)$ as a subalgebra of $\mathcal{B}(E)''$, thus writing $T \in \mathcal{B}(E)''$ instead of the more correct $\kappa_{\mathcal{B}(E)}(T) \in \mathcal{B}(E)''$, for $T \in \mathcal{B}(E)$. In particular, Id_E is the identity of $\mathcal{B}(E)''$, for either Arens product.

Proposition 4.2.3. *Let E and κ be as before. Let $\Phi \in \mathcal{B}(E)''$, and suppose that $\kappa'(\Phi) \neq 0$. Then $\Phi \notin \operatorname{rad} \mathcal{B}(E)''$ for either Arens product.*

Proof. Pick $x \in E$ and $\mu \in E'$ with $\kappa'(\Phi)(x) \neq 0$ and $\langle \mu, \kappa'(\Phi)(x) \rangle = 1$. Then let $T = \mu \otimes x \in \mathcal{B}(E)$, so that $\kappa'(T\Box\Phi)(x) = T(\kappa'(\Phi)(x)) = x$, and hence $\kappa'(\mathrm{Id}_E - T\Box\Phi)$ has non-trivial kernel and so cannot be invertible. Thus $\mathrm{Id}_E - T\Box\Phi$ is not invertible in $\mathcal{B}(E)''$, so that $\Phi \notin \operatorname{rad}\mathcal{B}(E)''$ (by Theorem 1.5.5). The same holds for the second Arens product. $\square$

Note that Proposition 4.2.2(6) shows that $\ker\kappa'$ is an ideal of $\mathcal{B}(E)''$ for either Arens product. Thus, by Proposition 1.5.6, $\operatorname{rad}\ker\kappa' = \ker\kappa' \cap \operatorname{rad}\mathcal{B}(E)''$. However, the last proposition tells us that $\operatorname{rad}\mathcal{B}(E)'' \subseteq \ker\kappa'$, so that $\operatorname{rad}\mathcal{B}(E)'' = \operatorname{rad}\ker\kappa'$. Thus we can concentrate on $\ker\kappa' \subseteq \mathcal{B}(E)''$ when considering the radical of $\mathcal{B}(E)''$.

We will now consider the case where $E = F \oplus G$ (so that F and G are required to be reflexive). We can regard $\mathcal{B}(E)$ as an algebra of two-by-two matricies with entries from $\mathcal{B}(F)$, $\mathcal{B}(F, G)$ etc. Indeed, we have

$$\mathcal{B}(E) = \left\{ \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} : \begin{matrix} A_{11} \in \mathcal{B}(F), A_{21} \in \mathcal{B}(G, F), \\ A_{12} \in \mathcal{B}(F, G), A_{22} \in \mathcal{B}(G) \end{matrix} \right\},$$

and so

$$\mathcal{B}(E)'' = \left\{ \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} : \begin{matrix} \Phi_{11} \in \mathcal{B}(F)'', \Phi_{12} \in \mathcal{B}(G, F)'', \\ \Phi_{21} \in \mathcal{B}(F, G)'', \Phi_{22} \in \mathcal{B}(G)'' \end{matrix} \right\}.$$

Lemma 4.2.4. *Let $\mathcal{A}$ be a unital Banach algebra, and let $p, q \in \mathcal{A}$ be orthogonal idempotents (that is, by definition, $p^2 = p, q^2 = q$ and $pq = qp = 0$) such that $p + q = e_{\mathcal{A}}$. Then*

$$\mathcal{A} = \begin{pmatrix} p\mathcal{A}p & p\mathcal{A}q \\ q\mathcal{A}p & q\mathcal{A}q \end{pmatrix}.$$

Let $\mathfrak{A}$ be a subalgebra of $\mathcal{A}$, and let $\mathfrak{B}$ be an ideal in $\mathfrak{A}$, such that

$$\mathfrak{A} \subseteq \begin{pmatrix} p\mathcal{A}p & 0 \\ q\mathcal{A}p & q\mathcal{A}q \end{pmatrix} \quad , \quad \mathfrak{B} \subseteq \begin{pmatrix} 0 & 0 \\ q\mathcal{A}p & 0 \end{pmatrix}.$$

Then $\mathfrak{B}$ lies in the radical of $\mathfrak{A}$.

Proof. Firstly note that for $a \in \mathcal{A}$, we have $a = e_{\mathcal{A}} a e_{\mathcal{A}} = pap + paq + qap + qaq$, so that $\mathcal{A}$ does have the form of a two-by-two matrix algebra, at least linearly. Furthermore, for

$a, b \in \mathcal{A}$, we have

$$
\begin{aligned}
ab &= \Big(pap + paq + qap + qaq\Big)\Big(pbp + pbq + qbp + qbq\Big) \\
&= papbp + papbq + paqbp + paqbq + qapbp + qapbq + qaqbp + aqabq \\
&= \begin{pmatrix} p(apb + aqb)p & p(apb + aqb)q \\ q(apb + aqb)p & q(apb + qab)q \end{pmatrix} = \begin{pmatrix} pap & paq \\ qap & qaq \end{pmatrix} \begin{pmatrix} pbp & pbq \\ qbp & qbq \end{pmatrix},
\end{aligned}
$$

so that the algebra structure on $\mathcal{A}$ also gives rise to a two-by-two matrix algebra.

Pick $b \in \mathfrak{B}$ and $a \in \mathfrak{A}$. Then

$$
e_{\mathfrak{A}} + ba = \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ qbp & 0 \end{pmatrix} \begin{pmatrix} pap & 0 \\ qap & qaq \end{pmatrix} = \begin{pmatrix} p & 0 \\ qbpap & q \end{pmatrix},
$$

which has inverse $\left(\begin{smallmatrix} p & 0 \\ -qbpap & q \end{smallmatrix}\right)$. Thus, as $a \in \mathfrak{A}$ was arbitrary, $b \in \operatorname{rad} \mathfrak{A}$. □

We can certainly apply this lemma to $\mathcal{A} = \mathcal{B}(F \oplus G)'' = \mathcal{B}(E)''$, with either of the Arens products (with p and q being the projections onto F and G, respectively). However, we have no hope of getting $\mathcal{B}(F \oplus G)''$ into the correct form (i.e. that of $\mathfrak{A}$) in which to apply the above lemma. However, with reference to the comment after Proposition 4.2.3, we can work with $\ker \kappa'$. Hence we wish to impose conditions on F and G so that $\ker \kappa'$ has the form of an algebra of lower-triangular matrices.

We need some results on when the space $\mathcal{B}(F, G)$ is reflexive.

Theorem 4.2.5. *Let F and G be reflexive Banach spaces. If $\mathcal{B}(F, G) = \mathcal{K}(F, G)$, then $\mathcal{B}(F, G)$ is reflexive. Suppose that at least one of F and G has the approximation property. Then $\mathcal{B}(F, G)$ is reflexive if and only if $\mathcal{B}(F, G) = \mathcal{K}(F, G)$.*

Proof. The first part is [Ryan, 2002, Theorem 4.19], which is a direct calculation that the unit ball of $\mathcal{B}(F, G)$ is weakly sequentially compact. The second part is [Ryan, 2002, Theorem 4.20], which we will now sketch. As F and G are reflexive and one has the approximation property, we know that $\mathcal{K}(F, G) = \mathcal{A}(F, G) = F' \check{\otimes} G$, and that $(F' \check{\otimes} G)' = \mathcal{I}(F', G') = \mathcal{N}(F', G') = F'' \widehat{\otimes} G' = F \widehat{\otimes} G'$. Hence we have $\mathcal{K}(F, G)'' = (F \widehat{\otimes} G')' = \mathcal{B}(F, G)$, so that $F \widehat{\otimes} G'$ is reflexive when $\mathcal{B}(F, G)$ is reflexive.

We wish to show that when $\mathcal{B}(F, G)$ is reflexive, we have that $\mathcal{B}(F, G) = \mathcal{K}(F, G)$. Towards a contradiction, suppose that $T \in \mathcal{B}(F, G)$ is not compact, so that for some sequence (x_n) in $F_{[1]}$, $(T(x_n))$ has no convergent subsequence. By moving to a subsequence if necessary, we may suppose that (x_n) is weakly convergent to $x \in F_{[1]}$, as the unit ball of F is weakly sequentially compact. Then, taking a suitable subsequence of $(x_n - x)$, we obtain a bounded sequence (w_n) which converges weakly to 0, but for which there exists $\delta > 0$ with $\|T(w_n)\| \geq \delta$ for each $n \in \mathbb{N}$. Then let (μ_n) be a sequence in $G'_{[1]}$ with $\langle \mu_n, T(w_n)\rangle = \|T(w_n)\|$ for each n. As $F\widehat{\otimes}G'$ is reflexive, by moving to a subsequence, we may suppose that $(w_n \otimes \mu_n)$ is weakly convergent to $u \in F\widehat{\otimes}G'$. Thus we have

$$\langle T, u\rangle = \lim_{n\to\infty} \langle \mu_n, T(w_n)\rangle = \lim_{n\to\infty} \|T(w_n)\| \geq \delta.$$

However, for $S = \lambda \otimes y \in F'\check{\otimes}G$, we have

$$\langle u, S\rangle = \lim_{n\to\infty} \langle \mu_n, S(w_n)\rangle = \lim_{n\to\infty} \langle \mu_n, y\rangle\langle \lambda, w_n\rangle = 0,$$

as (w_n) tends weakly to zero. By linearity and continuity, we have that $\langle u, S\rangle = 0$ for each $S \in F'\check{\otimes}G$, so that $u = 0$ as $F\widehat{\otimes}G' = (F'\check{\otimes}G)'$. This contradiction completes the proof. □

Lemma 4.2.6. *If every bounded linear map from G to F is compact, then $\ker\kappa'$ has the form of $\mathfrak{A}$, as in the above lemma.*

Proof. We wish to show that for $\Phi \in \mathcal{B}(G,F)''$ with $\kappa'\left(\begin{smallmatrix} 0 & \Phi \\ 0 & 0 \end{smallmatrix}\right) = 0$, we actually have $\Phi = 0$. Then the required result follows by linearity. Now, $\kappa'\left(\begin{smallmatrix} 0 & \Phi \\ 0 & 0 \end{smallmatrix}\right) = 0$ if and only if $\langle \Phi, \lambda\rangle = 0$ for each $\lambda \in G\widehat{\otimes}F'$ (noting that $(G\widehat{\otimes}F')' = \mathcal{B}(G,F)$). It is thus enough to show that $\kappa_{G\widehat{\otimes}F'} : G\widehat{\otimes}F' \to \mathcal{B}(G,F)'$ is surjective, that is, $G\widehat{\otimes}F'$ is reflexive. We are thus done, by the above theorem. □

Finally, we would like $\mathfrak{B}$ to not be the zero space.

Lemma 4.2.7. *With F, G and κ as above, there is a non-zero $\Psi \in \ker\kappa' \cap \mathcal{B}(F,G)''$ if and only if $\mathcal{B}(F,G)$ is not reflexive.*

Proof. As κ' restricts to a projection of $\mathcal{B}(F,G)''$ onto $\mathcal{B}(F,G)$, this is clear. □

Theorem 4.2.8. *Let F and G be reflexive Banach spaces such that at least one has the approximation property, such that $\mathcal{B}(F,G) = \mathcal{K}(F,G)$, and such that $\mathcal{B}(G,F) \neq \mathcal{K}(G,F)$. Then $\mathcal{B}(F \oplus G)''$, with either Arens product, is not semi-simple.*

Proof. This follows directly from the above results. □

Corollary 4.2.9. *Choose p and q with $1 < p < q < \infty$. Then $\mathcal{B}(l^p \oplus l^q)''$ is not semi-simple.*

Proof. We shall see later (Theorem 5.1.7) that $\mathcal{B}(l^q, l^p) = \mathcal{K}(l^q, l^p)$. By considering the formal identity map from l^p to l^q we see that $\mathcal{B}(l^p, l^q) \neq \mathcal{K}(l^p, l^q)$. □

4.3 Semi-simplicity of $\mathcal{B}(l^p)''$

We shall now show that $\mathcal{B}(l^p)''$ is not semi-simple for $p \neq 2$. This is joint work with C.J Read; the main idea of the construction is Read's, although the current proof is simplified in the case of Proposition 4.3.6, and is entirely my own approach in the case of Lemma 4.3.7 through Proposition 4.3.10.

Firstly, we note that we can concentrate on the case where $p \in (1, 2)$. For a Banach algebra $\mathcal{A}$, let $\mathcal{A}^{\mathrm{op}}$ be the Banach algebra obtained from $\mathcal{A}$ by reversing the product. Then, letting $p^{-1} + q^{-1} = 1$, we have that $\mathcal{B}(l^p)^{\mathrm{op}}$ is isometrically isomorphic to $\mathcal{B}(l^p)^a = \mathcal{B}(l^q)$ by the map $T \mapsto T'$. We can then check that, at least when $\mathcal{A}$ is Arens regular, $(\mathcal{A}'')^{\mathrm{op}}$ is isometrically isomorphic to $(\mathcal{A}^{\mathrm{op}})''$, and thus conclude that $\mathcal{B}(l^p)''$ is semi-simple if and only if $\mathcal{B}(l^q)''$ is semi-simple. Throughout this section, we shall suppose that $p \in (1, 2)$.

The following construction is motivated by the previous section, where we proved, in particular, that $\mathcal{B}(l^p, l^2)''$ is not semi-simple. We cannot find a copy of l^2 in l^p, but we can find copies of l^2_n in l^p, by a deep result of Dvoretsky (for example, [Figiel et al., 1977]). In particular, for each $n \in \mathbb{N}$, each $\varepsilon > 0$ and each infinite-dimensional Banach space E, we can find a $(1 + \varepsilon)$-isomorphism $T : l^2_n \to M$ for some $M \in \mathrm{FIN}(E)$. In fact, we can have that $E \in \mathrm{FIN}$, as long as the dimension of E is sufficiently large (where there are estimates on how large this must be). Furthermore, in the case where $E = l^p$, we can give an explicit construction of T, at least for some values of $\varepsilon > 0$ (see, for example, [Ryan, 2002, Section 2.5]). We shall not need this, however.

4.3.1 Use of ultrapowers

Recall, from Section 3.4, the definition of when a super-reflexive Banach space E has a 1-unconditional basis (e_n) of block p-type. When E is such a space, we can apply Theorem 3.4.18 to show that, for a suitable ultrafilter $\mathcal{U}$, the map $\phi_1 : (E)_\mathcal{U} \widehat{\otimes} (E')_\mathcal{U} \to \mathcal{B}(E)'$ is a quotient map, and thus that $\theta_1 : \mathcal{B}(E)'' \to \mathcal{B}((E)_\mathcal{U})$ is an isometry onto its range. By the discussion after Proposition 2.6.3 (or by a brief, direct calculation), we have shown that, if $\Phi \in \mathcal{B}(E)''$ is the weak*-limit of a bounded net (T_α) in $\mathcal{B}(E)$, then, for $x \in (E)_\mathcal{U}$, we have

$$\theta_1(\Phi)(x) = \operatorname*{weak\text{-}lim}_\alpha T_\alpha(x).$$

This makes sense as $(T_\alpha(x))$ is a bounded family in $(E)_\mathcal{U}$, and is thus relatively weakly-compact, as $(E)_\mathcal{U}$ is reflexive.

Henceforth, when we mention the maps ϕ_1 and θ_1 or the ultrafilter $\mathcal{U}$, we shall assume that $\mathcal{U}$ is such that ϕ_1 is a quotient operator.

Lemma 4.3.1. *Let E be as above. Then, for each $\Phi \in \mathcal{B}(E)$, each $\varepsilon > 0$, and each $x \in (E)_{\mathcal{U}}$, there exists $T \in \mathcal{B}(E)$ with $\|T(x) - \theta_1(\Phi)(x)\| < \varepsilon$ and $\|T\| \leq \|\Phi\|$.*

Proof. By the above discussion, and by, for example, Helley's Lemma, we know that $\theta_1(\Phi)(x)$ is in the weak closure of $\{T(x) : T \in \mathcal{B}(E), \|T\| \leq \|\Phi\|\}$. As this set is convex and bounded, its weak closure coincides with its norm closure, and hence we are done. $\square$

As before, $\mathcal{B}(E)$ is Arens regular and Id_E (or, more accurately, $\kappa_{\mathcal{B}(E)}(\mathrm{Id}_E)$) is the identity of $\mathcal{B}(E)''$.

Proposition 4.3.2. *Let E be as above, and suppose that $\Phi \notin \operatorname{rad} \mathcal{B}(E)''$. Then, for some $\Psi \in \mathcal{B}(E)''$, the operator $\mathrm{Id}_{(E)_{\mathcal{U}}} - \theta_1(\Psi) \circ \theta_1(\Phi)$ is not bounded below on $(E)_{\mathcal{U}}$.*

Proof. As $\Phi \notin \operatorname{rad} \mathcal{B}(E)''$, there exists $\Psi \in \mathcal{B}(E)''$ with the spectrum of $\Psi \Box \Phi$ containing a non-zero complex number. By scaling Ψ, we may suppose that 1 lies in the boundary of the spectrum of $\Psi \Box \Phi$. Thus, there exist (λ_n), a sequence in $\mathbb{C}$ such that $\lim_n \lambda_n = 1$, with $\lambda_n \mathrm{Id}_E - \Psi \Box \Phi \in \operatorname{Inv} \mathcal{B}(E)''$ for each n. Let $\Lambda_n = (\lambda_n \mathrm{Id}_E - \Psi \Box \Phi)^{-1}$ for each n, and suppose that (Λ_n) is a bounded sequence. Then we have

$$\begin{aligned}\|\mathrm{Id}_E - \Lambda_n \Box (\mathrm{Id}_E - \Psi \Box \Phi)\| &= \|\Lambda_n \Box (\lambda_n \mathrm{Id}_E - \Psi \Box \Phi) - \Lambda_n \Box (\mathrm{Id}_E - \Psi \Box \Phi)\| \\ &= \|\Lambda_n \Box (\lambda_n \mathrm{Id}_E - \mathrm{Id}_E)\| \leq |1 - \lambda_n| \sup_n \|\Lambda_n\|,\end{aligned}$$

so that, by Proposition 1.6.1, $\Lambda_n \Box (\mathrm{Id}_E - \Psi \Box \Phi)$ is invertible for some $n \in \mathbb{N}$, and as Λ_n is invertible, this contradicts the fact that $\mathrm{Id}_E - \Psi \Box \Phi \notin \operatorname{Inv} \mathcal{B}(E)''$. Note that we have actually shown that no subsequence of (Λ_n) can be bounded.

Let $T_n = \theta_1(\Lambda_n) \|\theta_1(\Lambda_n)\|^{-1}$, so as θ_1 is an isomorphism onto its range, we certainly have that $\|\theta_1(\Lambda_n)\|^{-1} \to 0$. Then we have

$$\begin{aligned}&\left\| T_n \circ \Big(\mathrm{Id}_{(E)_{\mathcal{U}}} - \theta_1(\Psi) \circ \theta_1(\Phi) \Big) \right\| \\ &\quad = \|\theta_1(\Lambda_n)\|^{-1} \left\| \theta_1 \Big(\Lambda_n \Box (\lambda_n \mathrm{Id}_E - \Psi \Box \Phi) \Big) + \theta_1(\Lambda_n)(1 - \lambda_n) \right\| \\ &\quad \leq \|\theta_1(\Lambda_n)\|^{-1} \|\Lambda_n \Box (\lambda_n \mathrm{Id}_E - \Psi \Box \Phi)\| + \|\theta_1(\Lambda_n)\|^{-1} \|\theta_1(\Lambda_n)\| |1 - \lambda_n| \\ &\quad = \|\theta_1(\Lambda_n)\|^{-1} + |1 - \lambda_n|,\end{aligned}$$

which tends to 0 as $n \to \infty$. Thus $\mathrm{Id}_{(E)_{\mathcal{U}}} - \theta_1(\Psi) \circ \theta_1(\Phi)$ cannot be bounded below. $\square$

For completeness, we have the following.

Proposition 4.3.3. *Let E be as above, and $\Phi \in \mathcal{B}(E)''$. Suppose that there exists $\Psi \in \mathcal{B}(E)''$ such that for each $\varepsilon > 0$, there exists $x \in (E)_{\mathcal{U}}$ with $\|x\| = 1$ and $\|x - \theta_1(\Psi)(\theta_1(\Phi)(x))\| < \varepsilon$. Then $\Phi \notin \operatorname{rad} \mathcal{B}(E)''$.*

Proof. Towards a contradiction, suppose that $\Phi \in \operatorname{rad} \mathcal{B}(E)''$. Then, in particular, we have $\mathrm{Id}_E - \Psi \Box \Phi \in \operatorname{Inv} \mathcal{B}(E)''$. Let $T = \theta_1((\mathrm{Id}_E - \Psi\Box\Phi)^{-1})$, so that, for $x \in (E)_{\mathcal{U}}$ with $\|x\| = 1$, we have

$$1 = \|x\| = \|T(x) - T(\theta_1(\Psi)\theta_1(\Phi)(x))\| \leq \|T\|\|x - \theta_1(\Psi)\theta_1(\Phi)(x)\|,$$

which is a contradiction. $\square$

4.3.2 Construction of an element in the radical

We shall now construct a non-zero element in $\operatorname{rad} \mathcal{B}(l^p)''$, for a, from now on, fixed $p \in (1, 2)$ (although much of what is below will work for any $p \in (1, \infty)$.)

As usual, let $(e_n)_{n=1}^\infty$ be the standard unit vector basis for l^p. Choose an increasing sequence of integers (n_k), and let $N_0 = 0$, $N_1 = n_1$, $N_{i+1} = N_i + n_{i+1}$ and $A_k = \{i : N_{k-1} < i \leq N_k\}$. Then we can find a linear map $T : l^p \to l^p$ which maps $\operatorname{lin}\{e_i : i \in A_k\}$ to a $(1 + \frac{1}{k})$-isomorphic copy of $l^2_{n_k}$, say $w_i = T(e_i)$. By this, we mean that if $(a_i)_{i\in A_k}$ is a sequence of scalars, then

$$\frac{k-1}{k}\left(\sum_{i\in A_k}|a_i|^2\right)^{1/2} \leq \left\|\sum_{i\in A_k} a_i w_i\right\|_{l^p} \leq \frac{k+1}{k}\left(\sum_{i\in A_k}|a_i|^2\right)^{1/2}.$$

Further, we may suppose that, when $k \neq l$, the sets $\{w_i : i \in A_k\}$ and $\{w_i : i \in A_l\}$ are disjointly supported in l^p. That is, for $i \in A_l$ and $j \in A_k$, we have $\operatorname{supp}(w_i) \cap \operatorname{supp}(w_j) = \emptyset$.

Thus, when (a_k) is a sequence of scalars, we have

$$\begin{aligned} \left\|T\left(\sum_k a_k e_k\right)\right\| &= \left\|\sum_k \sum_{i\in A_k} a_i w_i\right\| = \left(\sum_k \left\|\sum_{i\in A_k} a_i w_i\right\|^p\right)^{1/p} \\ &\leq \left(\sum_k \left(\frac{k+1}{k}\right)^p \left(\sum_{i\in A_k}|a_i|^2\right)^{p/2}\right)^{1/p} \leq 2\|(a_k)\|_p, \end{aligned} \tag{4.1}$$

so that $T \in \mathcal{B}(l^p)$ with $\|T\| \leq 2$.

For a sequence of positive reals (a_n), recall that

$$\limsup_{k\to\infty} a_k := \inf_{k\in\mathbb{N}}\left(\sup_{j\geq k} a_j\right) = \lim_{k\to\infty}\left(\sup_{j\geq k} a_j\right),$$

where $(\sup_{j\geq k} a_j)_{k=1}^\infty$ is clearly a decreasing sequence bounded below by 0. For each $A \subseteq \mathbb{N}$, let

$$\mathrm{ud}(A) = \limsup_{k\to\infty} \frac{|A \cap A_k|}{|A_k|},$$

and let $\mathcal{F} = \{A \subseteq \mathbb{N} : \mathrm{ud}(\mathbb{N} \setminus A) = 0\}$. We claim that $\mathcal{F}$ is a filter on $\mathbb{N}$; the only non-trivial thing to check is that, for $A, B \in \mathcal{F}$, we have $A \cap B \in \mathcal{F}$, but this follows, as

$$\begin{aligned}\mathrm{ud}(\mathbb{N} \setminus (A \cap B)) &= \limsup_{k\to\infty} \frac{|A_k \setminus (A \cap B)|}{|A_k|} \\ &\leq \limsup_{k\to\infty} \frac{|A_k \setminus A|}{|A_k|} + \limsup_{k\to\infty} \frac{|A_k \setminus B|}{|A_k|} = 0.\end{aligned}$$

Let $\mathcal{W}$ be an ultrafilter on $\mathbb{N}$ refining $\mathcal{F}$. As $\mathcal{W}$ is a filter, $\mathcal{W}$ becomes a directed set if we partially order $\mathcal{W}$ by reverse inclusion. Thus let $\mathcal{V}$ be an ultrafilter on $\mathcal{W}$ which refines the order filter. In particular, for $A \in \mathcal{W}$, the set $V_A = \{B \in \mathcal{W} : B \subseteq A\}$ is in $\mathcal{V}$. Define

$$\Phi_0 = \operatorname*{weak^*\text{-}lim}_{A\in\mathcal{V}} TP_A \in \mathcal{B}(l^p)'',$$

where P_A is the projection of l^p onto those vectors with support in $A \subseteq \mathbb{N}$. This Φ_0 will turn out to be our radical element.

Define $\psi \in \mathcal{B}((l^p)_\mathcal{U})$ by

$$\psi(x) = \operatorname*{weak\text{-}lim}_{A\in\mathcal{V}} P_A(x) \qquad (x \in (l^p)_\mathcal{U}).$$

Lemma 4.3.4. *The map ψ is a projection onto the subspace*

$$\{x \in (l^p)_\mathcal{U} : P_A(x) = x \ (A \in \mathcal{W})\} \subseteq (l^p)_\mathcal{U}.$$

Furthermore, for each $x \in (l^p)_\mathcal{U}$, the limit $\lim_{A\in\mathcal{V}} P_A(x)$ exists (we only know a priori that the limit exists in the weak-topology, not the norm topology).

Proof. For $\mu \in (l^q)_\mathcal{U}$ and $B \in \mathcal{W}$, we have

$$\begin{aligned}\langle \mu, P_B(\psi(x))\rangle &= \lim_{A\in\mathcal{V}} \langle P_B'(\mu), P_A(x)\rangle = \lim_{A\in\mathcal{V}} \langle \mu, P_{B\cap A}(x)\rangle \\ &= \lim_{A\in\mathcal{V}} \langle \mu, P_A(x)\rangle = \langle \mu, \psi(x)\rangle,\end{aligned}$$

so that $P_B \circ \psi = \psi$, and hence $\psi \circ \psi = \psi$. For $x \in (l^p)_\mathcal{U}$ with $P_A(x) = x$ for each $A \in \mathcal{W}$, we clearly have $\psi(x) = x$, as required.

Let C be the convex hull of $\{P_A(x) : A \in \mathcal{W}\}$, so that the norm and weak closures of C coincide. Thus, for each $\varepsilon > 0$, we can find a convex combination $S = \sum_{i=1}^n \lambda_i P_{A_i}$ such that $\|S(x) - \psi(x)\| < \varepsilon$. Let $A = A_1 \cap \cdots \cap A_n$, so that $A \in \mathcal{W}$, and $P_A(S(x)) = \sum_{i=1}^n \lambda_i P_A P_{A_i}(x) = P_A(x)$. Then

$$\|P_A(x) - \psi(x)\| = \|P_A(S(x)) - P_A(\psi(x))\| < \|P_A\|\varepsilon = \varepsilon.$$

Hence, for each $B \in V_A$, we have

$$\|P_B(x) - \psi(x)\| = \|P_B(P_A(x)) - P_B(\psi(x))\| \leq \|P_A(x) - \psi(x)\| < \varepsilon.$$

Thus $\{B \in \mathcal{W} : \|P_B(x) - \psi(x)\| < \varepsilon\} \supseteq V_A \in \mathcal{V}$, so that $\psi(x) = \lim_{A\in\mathcal{V}} P_A(x)$. $\square$

Lemma 4.3.5. *We have $\theta_1(\Phi_0) = T \circ \psi$, and $\Phi_0 \neq 0$.*

Proof. Choose $x \in (l^p)_\mathcal{U}$ and $\mu \in (l^q)_\mathcal{U}$, so that

$$\begin{aligned}\langle \mu, \theta_1(\Phi_0)(x)\rangle &= \lim_{A\in\mathcal{V}} \langle \mu, TP_A(x)\rangle = \lim_{A\in\mathcal{V}} \langle T'(\mu), P_A(x)\rangle \\ &= \langle T'(\mu), \psi(x)\rangle = \langle \mu, T(\psi(x))\rangle.\end{aligned}$$

Thus $\theta_1(\Phi_0) = T \circ \psi$.

Let $\phi_1^\mathcal{V} : (l^p)_\mathcal{V}\widehat{\otimes}(l^q)_\mathcal{V} \to \mathcal{B}(l^p)'$ be as usual (though using the ultrafilter $\mathcal{V}$ and not $\mathcal{U}$) and let $\theta_1^\mathcal{V} = (\phi_1^\mathcal{V})'$. We have then actually also shown that $\theta_1^\mathcal{V}(\Phi_0) = T \circ \psi_\mathcal{V}$, where $\psi_\mathcal{V} \in \mathcal{B}((l^p)_\mathcal{V})$ is defined as for $\psi \in \mathcal{B}((l^p)_\mathcal{U})$.

Now let $\alpha : \mathcal{W} \to \mathbb{N}$ be such that $\alpha(A) \in A$ for each $A \in \mathcal{W}$. Then let $x_A = e_{\alpha(A)}$ so that $x = (x_A) \in (l^p)_\mathcal{V}$. For each $B \in \mathcal{W}$, we have

$$\{A \in \mathcal{W} : P_B(x_A) = x_A\} = \{A \in \mathcal{W} : \alpha(A) \in B\} \supseteq \{A \in \mathcal{W} : A \subseteq B\} \in \mathcal{V},$$

and so $\lim_{A\in\mathcal{V}} \|P_B(x_A) - x_A\| = 0$. Thus $P_B(x) = x$. So, by the proof of Lemma 4.3.4, $\psi_\mathcal{V}(x) = x$, and clearly $T(x) \neq 0$, so that $\theta_1^\mathcal{V}(\Phi_0)(x) \neq 0$, and hence $\Phi_0 \neq 0$. $\square$

We aim to show by contradiction that $\Phi_0 \in \operatorname{rad}\mathcal{B}(l^p)''$. Let $B \in \mathbb{N}$, and let us say that $C \subset \mathbb{N}$ is *B-reasonable* if $|C \cap A_k| \leq B$ for every k. For any $r \in (1,\infty)$, a vector $x \in l^r$ is *B-reasonable* if $\operatorname{supp}(x)$ is *B-reasonable*. For an ultrafilter $\mathcal{U}$, we say that $x \in (l^r)_\mathcal{U}$ is *B-reasonable* if, for some representative (x_i) of x, x_i is B-reasonable for each i.

Proposition 4.3.6. *Suppose that $\Phi_0 \notin \operatorname{rad}\mathcal{B}(l^p)''$. Then there exists $\Psi \in \mathcal{B}(l^p)''$, $B \in \mathbb{N}$ and a B-reasonable $z \in (l^p)_\mathcal{U}$ with the following properties:*

1. *$\|z\| \leq 1$;*
2. *$P_A(z) = z$ for each $A \in \mathcal{W}$;*
3. *if $\mu^z \in (l^q)_\mathcal{U}$ with $\langle \mu^z, z\rangle = \|z\|$ and $\|\mu^z\| = 1$, then*

$$|\langle \mu^z, \theta_1(\Psi)(T(z))\rangle| > \frac{1}{2}\|\Psi\|^{-1}.$$

Proof. By Proposition 4.3.2, we can find $\Psi \in \mathcal{B}(l^p)''$ and $x \in (l^p)_{\mathcal{U}}$ with $\|x\| = 1$ and

$$\|\theta_1(\Psi \Box \Phi)(x) - x\| = \|(\theta_1(\Psi) \circ T \circ \psi)(x) - x\| < \varepsilon,$$

where $\varepsilon > 0$ is to be chosen later. With reference to Lemma 4.3.4, set $y = \lim_{A \in \mathcal{V}} P_A(x)$, so that $\|y\| \leq 1$ and $\|\theta_1(\Psi)(T(y)) - x\| < \varepsilon$. In particular, we thus also have

$$\|\theta_1(\Psi)(T(y))\| > 1 - \varepsilon. \tag{4.2}$$

Choose a representative (y_i) of y with, for each $i \in I$, $\|y_i\| = \|y\|$ and $y_i = \sum_j y_{i,j} e_j$. Then let $\gamma_{i,k} = \left(\sum_{j \in A_k} |y_{i,j}|^p\right)^{1/p}$, and let $\delta_{i,k} = \max_{j \in A_k} |y_{i,j}|$. Then, for each k and i, we have

$$\begin{aligned}\left(\sum_{j \in A_k} |y_{i,j}|^2\right)^{1/2} &= \gamma_{i,k} \left(\sum_{j \in A_k} \frac{|y_{i,j}|^2}{|\gamma_{i,k}|^2}\right)^{1/2} \leq \gamma_{i,k} \left(\sum_{j \in A_k} \frac{|y_{i,j}|^p}{|\gamma_{i,k}|^p} \delta_{i,k}^{2-p} \gamma_{i,k}^{p-2}\right)^{1/2} \\ &= \delta_{i,k}^{1-p/2} \gamma_{i,k}^{p/2} \left(\sum_{j \in A_k} \frac{|y_{i,j}|^p}{|\gamma_{i,k}|^p}\right)^{1/2} = \delta_{i,k}^{1-p/2} \gamma_{i,k}^{p/2}.\end{aligned}$$

Hence, by equation (4.1), we have

$$\|T(y_i)\| \leq \left(\sum_k \frac{(k+1)^p}{k^p} \Big(\sum_{j \in A_k} |y_{i,j}|^2\Big)^{p/2}\right)^{1/p} \leq \left(\sum_k \frac{(k+1)^p}{k^p} \delta_{i,k}^{p(1-p/2)} \gamma_{i,k}^{p^2/2}\right)^{1/p}. \tag{4.3}$$

Pick $K \in \mathbb{N}$ and choose $B \in \mathbb{N}$ so that $B \geq |A_k|$ for $k \leq K$, and $B^{1/p-1/2} > (K+1)/K\varepsilon$. For each $i \in \mathbb{N}$ choose a B-reasonable set $D_i \subset \mathbb{N}$ so that $\sum_{j \in D_i} |y_{i,j}|^p$ is maximal. For each i let $\hat{y}_i = P_{\mathbb{N} \setminus D_i}(y_i)$, and define $\hat{\gamma}_{i,k}$ and $\hat{\delta}_{i,k}$ for $\hat{y}_i$ in an analogous manner to the definitions of $\gamma_{i,k}$ and $\delta_{i,k}$. Note that, if $B \geq |A_k|$, then $\hat{\gamma}_{i,k} = 0$ for each i. For each i and k, $\hat{\gamma}_{i,k} \leq \gamma_{i,k}$, and we have that either $A_k \cap D_i = A_k$, so that $\hat{\delta}_{i,k} = 0$, or we have that

$$\sum_{j \in A_k \cap D_i} |y_{i,j}|^p \geq B \max_{j \in A_k \setminus D_i} |y_{i,j}|^p = B \hat{\delta}_{i,k}^p,$$

so that

$$\gamma_{i,k}^p = \sum_{j \in A_k \cap D_i} |y_{i,j}|^p + \sum_{j \in A_k \setminus D_i} |y_{i,j}|^p \geq B \hat{\delta}_{i,k}^p,$$

and hence $\hat{\delta}_{i,k} \leq B^{-1/p} \gamma_{i,k}$. Thus, by equation (4.3),

$$\begin{aligned}\|T(\hat{y}_i)\| &\leq \left(\sum_{k > K} \frac{(k+1)^p}{k^p} \hat{\delta}_{i,k}^{p(1-p/2)} \hat{\gamma}_{i,k}^{p^2/2}\right)^{1/p} \leq \left(\sum_{k > K} \frac{(k+1)^p}{k^p} B^{p/2-1} \gamma_{i,k}^p\right)^{1/p} \\ &= B^{1/2-1/p} \left(\sum_{k > K} \frac{(k+1)^p}{k^p} \gamma_{i,k}^p\right)^{1/p} \leq \frac{K+1}{K} B^{1/2-1/p} \|y_i\| < \varepsilon,\end{aligned}$$

by our choice of B.

Let $z = y - \hat{y} = (P_{D_i}(y_i))$, so that z is B-reasonable, and $\|z\| \leq 1$. For each $A \in \mathcal{W}$, we have $y = P_A(y)$, and so

$$\begin{aligned}\|P_A(z) - z\| &= \lim_{i\in\mathcal{U}} \|P_A(P_{D_i}(y_i)) - P_{D_i}(y_i)\| \\ &\leq \lim_{i\in\mathcal{U}} \|P_A(y_i) - y_i\| = \|P_A(y) - y\| = 0.\end{aligned}$$

Now let $\mu^z = (\mu_i^z) \in (l^q)_{\mathcal{U}}$ be such that $\|\mu_i^z\| = 1$ and $\langle \mu_i^z, z_i\rangle = \|z_i\|$ for each i. Then, for each i, $\operatorname{supp}(z_i) = \operatorname{supp}(\mu_i^z)$ so that

$$\langle \mu_i^z, y_i - z_i\rangle = \langle P_{D_i}(\mu_i^z), P_{\mathbb{N}\setminus D_i}(y_i)\rangle = 0.$$

Thus $\langle \mu^z, z\rangle = \langle \mu^z, y\rangle$. For $A \in \mathcal{W}$, as $P_A(z) = z$, we have $P_A(\mu^z) = \mu^z$, and so

$$\|z\| = \langle \mu^z, z\rangle = \langle \mu^z, y\rangle = \lim_{A\in\mathcal{V}} \langle \mu^z, P_A(x)\rangle = \lim_{A\in\mathcal{V}} \langle P_A(\mu^z), x\rangle = \langle \mu^z, x\rangle.$$

Let T_K be T restricted to the subspace of vectors in l^p whose support is contained in $\bigcup_{k>K} A_k$. Then we have $T(z) = T(y - \hat{y}) = T_K(z)$ and $\|T_K\| \leq (K+1)/K$. As $\|T(\hat{y})\| < \varepsilon$, and by equation 4.2, we have

$$\begin{aligned}\|z\| &\geq \|T_K\|^{-1}\|T_K(z)\| \geq K(K+1)^{-1}(\|T(y)\| - \|T(y-z)\|) \\ &\geq K(K+1)^{-1}(\|\theta_1(\Psi)(T(y))\|\|\Psi\|^{-1} - \varepsilon) \\ &\geq K(K+1)^{-1}((1-\varepsilon)\|\Psi\|^{-1} - \varepsilon).\end{aligned}$$

So finally we have

$$\begin{aligned}|\langle \mu^z, \theta_1(\Psi)(T(z))\rangle| &\geq |\langle \mu^z, \theta_1(\Psi)(T(y))\rangle| - \|\mu^z\|\|\Psi\|\|T(z-y)\| \\ &\geq |\langle \mu^z, x\rangle| - |\langle \mu^z, x - \theta_1(\Psi)(T(y))\rangle| - \varepsilon\|\Psi\| \\ &\geq \|z\| - \varepsilon - \varepsilon\|\Psi\|.\end{aligned}$$

Thus, for each $\delta > 0$, we can, by a choice of $\varepsilon > 0$ and $K \in \mathbb{N}$, ensure that

$$|\langle \mu^z, \theta_1(\Psi)(T(z))\rangle| \geq \|\Psi\|^{-1}(1-\delta).$$

We thus have conclusions (1) and (2), and, setting $\delta = 1/2$, we get conclusion (3). □

We shall now study maps from l^2 to l^p, and show how this gives rise to a contradiction with the above proposition. The following, suggested by Béla Bollobás, are an improvement upon the original proofs which appeared in [Daws, Read, 2004].

Lemma 4.3.7. *Let*

$$\varepsilon(k,p) = \max\{\varepsilon > 0 : \exists T \in \mathcal{B}(l_k^2, l_k^p), \|T\| \leq 1, |\langle T(e_j), e_j\rangle| \geq \varepsilon \ (1 \leq j \leq k)\}.$$

Then we have $\varepsilon(k,p) = k^{1/2-1/p}$.

Proof. Let $I_k : l_k^2 \to l_k^p$ be the formal identity. We abuse notation, and write e_j for the jth standard unit vector basis element, in each l_k^p space.

Let $a, b \in \mathbb{C}$, and let $c = d = 2^{-1/2}(|a|^2 + |b|^2)^{1/2}$, so that $|c|^2 + |d|^2 = |a|^2 + |b|^2$. Then, as the function $[0,\infty) \to [0,\infty); t \mapsto t^{p/2}$ is strictly convex, we have

$$\left(\tfrac{1}{2}t + \tfrac{1}{2}s\right)^{p/2} < \tfrac{1}{2}t^{p/2} + \tfrac{1}{2}s^{p/2} \qquad (s,t \geq 0, t \neq s),$$

so that, if $|a| < |b|$, we have

$$|c|^p + |d|^p = 2(\tfrac{1}{2}|a|^2 + \tfrac{1}{2}|b|^2)^{p/2} < |a|^p + |b|^p.$$

As l_k^2 has compact unit ball, we can find $x \in l_k^2$ such that $\|x\| = 1$ and $\|I_k(x)\| = \|I_k\|$. The above argument shows that, if $x = \sum_{j=1}^k x_j e_j$, then we must have $|x_j| = |x_i|$ for each i and j, or else I_k would not obtain its norm on x. Thus we have $1 = \sum_{j=1}^k |x_j|^2 = k|x_1|^2$, so that

$$\|I_k\| = \|I_k(x)\| = \Big(\sum_{j=1}^k |x_j|^p\Big)^{1/p} = k^{1/p}|x_1| = k^{1/p-1/2}.$$

Thus we have $\varepsilon(k,p) \geq k^{1/2-1/p}$.

We now show that $\varepsilon(k,p) \leq k^{1/2-1/p}$. By convexity, and an induction argument, we have that

$$\sum_{i=1}^m p_i|x_i|^p \geq \Big(\sum_{i=1}^m p_i|x_i|\Big)^p,$$

for $m \in \mathbb{N}$, $(x_i)_{i=1}^m \subseteq \mathbb{C}$ and $(p_i)_{i=1}^m \subseteq [0,1]$ with $\sum_{i=1}^m p_i = 1$. Then, for $(a_j)_{j=1}^k \subseteq \mathbb{C}$, we have

$$\begin{aligned}
|a_1|^p &= \Big|\sum_{\pm} 2^{1-k}\Big(a_1 + \sum_{j=2}^k \pm a_j\Big)\Big|^p \leq \Big(\sum_{\pm} 2^{1-k}\Big|a_1 + \sum_{j=2}^k \pm a_j\Big|\Big)^p \\
&\leq \sum_{\pm} 2^{1-k}\Big|a_1 + \sum_{j=2}^k \pm a_j\Big|^p \\
&= 2^{-k}\Big(\sum_{\pm}\Big|a_1 + \sum_{j=2}^k \pm a_j\Big|^p + \sum_{\pm}\Big| - a_1 + \sum_{j=2}^k \pm a_j\Big|^p\Big) = 2^{-k}\sum_{\pm}\Big|\sum_{j=1}^k \pm a_j\Big|^p.
\end{aligned}$$

Hence we have shown that

$$2^{-k}\sum_{\pm}\Big|\sum_{j=1}^k \pm a_j\Big|^p \geq \max_{1\leq j\leq k}|a_j|^p \qquad ((a_j)_{j=1}^k \subseteq \mathbb{C}).$$

Consequently, for $T \in \mathcal{B}(l_k^2, l_k^p)$ with $\|T\| \leq 1$ and $|\langle T(e_j), e_j\rangle| \geq \varepsilon$ for each $1 \leq j \leq k$, we have

$$2^{-k}\sum_{\pm}\Big\|T\Big(\sum_{j=1}^{k}\pm e_j\Big)\Big\|^p = 2^{-k}\sum_{\pm}\sum_{i=1}^{k}\Big|\langle T\Big(\sum_{j=1}^{k}\pm e_j\Big), e_i\rangle\Big|^p$$
$$= \sum_{i=1}^{k} 2^{-k}\sum_{\pm}\Big|\sum_{j=1}^{k}\pm\langle T(e_j), e_i\rangle\Big|^p \geq \sum_{i=1}^{k}|\langle T(e_i), e_i\rangle|^p \geq k\varepsilon^p.$$

For any choice of signs, we have $\|\sum_{j=1}^{k}\pm T(e_j)\| \leq \|T\|\|\sum_{j=1}^{k}\pm e_j\|_2 = k^{1/2}$. Thus $k^{p/2} \geq k\varepsilon^p$, so that $\varepsilon \leq k^{1/2-1/p}$, and hence $\varepsilon(k,p) = k^{1/2-1/p}$. □

Lemma 4.3.8. *Let $S \in \mathcal{B}(l^2, l^p)$, let $(x_i)_{i=1}^n$ be an orthonormal set in l^2, and let $(A_i)_{i=1}^n$ be a pairwise disjoint family of subsets of $\mathbb{N}$. Suppose that $\|P_{A_i}(S(x_i))\| \geq \varepsilon$ for each $i \in \mathbb{N}$. Then $\varepsilon \leq \|S\|n^{1/2-1/p}$.*

Proof. For each $i \in \mathbb{N}$, choose $\mu_i \in l^q$ with $\|\mu_i\| = 1$ and $\langle \mu_i, S(x_i)\rangle = \|P_{A_i}(S(x_i))\|$, so (we may suppose, although it is automatic) that $\operatorname{supp}(\mu_i) \subseteq A_i$. Choose $U \in \mathcal{B}(l^2)$ with $\|U\| = 1$ and $U(e_i) = x_i$ for $1 \leq i \leq n$. Choose $V \in \mathcal{B}(l^q)$ with $\|V\| = 1$ and $V(e_i) = \mu_i$ for $1 \leq i \leq n$, which we may do, as the (μ_i) have disjoint support. Let $R = V'\circ S\circ U \in \mathcal{B}(l^2, l^p)$. Then we have, for $1 \leq i \leq n$, $|\langle R(e_i), e_i\rangle| = |\langle \mu_i, S(x_i)\rangle| \geq \varepsilon$.

For each $q \in (1,\infty)$, we identify l_n^q as the subspace of l^q consisting of vectors supported on $\{1, 2, \ldots, n\}$. Then let $R_n \in \mathcal{B}(l_n^2, l_n^p)$ be the restriction of R to $l_n^2 \subseteq l^2$, followed by the projection of l^p onto l_n^p. Then $|\langle R_n(e_i), e_i\rangle| \geq \varepsilon$ for $1 \leq i \leq n$, so by Lemma 4.3.7, $\varepsilon \leq \|S\|n^{1/2-1/p}$. □

Recall that the sequence of integers (n_k) was used to define the sets (A_k), and hence the operator T, which in turn was used to define Φ_0.

Lemma 4.3.9. *If the sequence (n_k) is such that $n_k \to \infty$, then, for each $S \in \mathcal{B}(l^p)$, each $B \in \mathbb{N}$ and each $\varepsilon > 0$, we can find $A \in \mathcal{F} \subset \mathcal{W}$ such that, for any B-reasonable $x \in l^p$ and $\mu \in l^q$ with $\langle \mu, x\rangle = \|\mu\| = \|x\| = 1$, we have*

$$\sum_{k=1}^{\infty}|\langle \mu, P_{A_k\cap A}STP_{A_k\cap A}(x)\rangle| < \varepsilon.$$

Proof. Note that if x is B-reasonable and $\langle \mu, x\rangle = \|x\| = \|\mu\| = 1$, then μ is automatically B-reasonable.

For $k \in \mathbb{N}$, let $T_k = T \circ P_{A_k}$ so, as $l_{n_k}^p$ is canonically isomorphic to $l^p(A_k)$, the image of P_{A_k}, we can view T_k as a map from $l_{n_k}^p$ to l^p. Then, for $x \in l_{n_k}^p$, we have

$$\frac{k-1}{k}\|x\|_2 \leq \|T_k(x)\| \leq \frac{k+1}{k}\|x\|_2, \tag{4.4}$$

so we can view T_k as an isomorphism from $l^2_{n_k}$ onto its image in l^p. Thus, for each k, let $S_k = S \circ T \circ P_{A_k} : l^2_{n_k} \to l^p$, so that

$$\|S_k\|_{\mathcal{B}(l^2_{n_k},l^p)} \leq \|S\|\|T \circ P_{A_k}\|_{\mathcal{B}(l^2_{n_k},l^p)} \leq \frac{k+1}{k}\|S\| \leq 2\|S\|,$$

by Equation (4.4). Let $m \in \mathbb{N}$ be maximal so that we can find $(x_i)_{i=1}^m$, a set of B-reasonable norm one vectors in $l^2_{n_k}$ with pairwise-disjoint support, and $(B_i)_{i=1}^m$, a set of B-reasonable pairwise-disjoint subsets of A_k, with $\|P_{B_i}(S_k(x_i))\| \geq \varepsilon$, for each i. By Lemma 4.3.8, we have $\varepsilon \leq 2\|S\|m^{1/2-1/p}$, so that $m \leq (2\|S\|\varepsilon^{-1})^{2p/(2-p)}$.

Let

$$C_k = \bigcup_{i=1}^m \operatorname{supp}(x_i) \cup \bigcup_{i=1}^m B_i \subseteq A_k,$$

so that $|C_k| \leq 2Bm \leq 2B(2\|S\|\varepsilon^{-1})^{2p/(2-p)}$. Let $A = \mathbb{N} \setminus \bigcup_{k=1}^\infty C_k$, so that for each k, we have

$$|(\mathbb{N} \setminus A) \cap A_k||A_k|^{-1} = |C_k||A_k|^{-1} \leq 2Bn_k^{-1}(2\|S\|\varepsilon^{-1})^{2p/(2-p)},$$

and thus $\limsup_{k\to\infty} |(\mathbb{N} \setminus A) \cap A_k||A_k|^{-1} = 0$, so that $A \in \mathcal{F}$.

Suppose we have a B-reasonable $x \in l^2_{n_k}$ with $C_k \cap \operatorname{supp}(x) = \emptyset$, and we have a B-reasonable $\mu \in l^q$ with $\operatorname{supp}(\mu) \cap C_k = \emptyset$. Then, by the maximality of m,

$$|\langle \mu, S_k(x)\rangle| \leq \|\mu\|\|P_{\operatorname{supp}(\mu)}(S_k(x))\| < \varepsilon\|\mu\|\|x\|.$$

Thus, for a B-reasonable $x \in l^p$, and $\mu \in l^q$, with $1 = \langle \mu, x\rangle = \|x\| = \|\mu\|$, we have

$$\begin{aligned}
\sum_{k=1}^\infty |\langle \mu, P_{A_k\cap A} STP_{A_k\cap A}(x)\rangle| &= \sum_{k=1}^\infty |\langle \mu, P_{A_k\cap A} S_k P_{A_k\cap A}(x)\rangle| \\
&< \varepsilon \sum_{k=1}^\infty \|P_{A_k\cap A}(\mu)\|\|P_{A_k\cap A}(x)\| \\
&\leq \varepsilon \left(\sum_{k=1}^\infty \|P_{A_k\cap A}(\mu)\|^q\right)^{1/q} \left(\sum_{k=1}^\infty \|P_{A_k\cap A}(x)\|^p\right)^{1/p} \leq \varepsilon,
\end{aligned}$$

as required. □

The following is based upon an idea of C.J. Read, although the preceding lemmas remove the need for a long calculation which was used originally.

Proposition 4.3.10. *If the sequence (n_k) increases fast enough, then for $S \in \mathcal{B}(l^p)$, $B \in \mathbb{N}$ and $\varepsilon > 0$, we can find $A \in \mathcal{F}$ so that for any B-reasonable $x \in l^p$ and $\mu \in l^q$ with $\langle \mu, x\rangle = \|x\|$ and $\|\mu\| = 1$, we have $|\langle \mu, P_ASTP_A(x)\rangle| < \varepsilon\|x\|$.*

Proof. First note that it is enough to prove the result in the case where $\|x\| = 1$, for otherwise let $y = \|x\|^{-1}x$, so that $\|y\| = 1$ and $\langle \mu, y\rangle = \|x\|^{-1}\langle \mu, x\rangle = 1$, so that $|\langle \mu, P_A STP_A(x)\rangle| = \|x\||\langle \mu, P_A STP_A(y)\rangle| < \varepsilon\|x\|$, as required. Hence we shall suppose that $\|x\| = 1$.

By "(n_k) increasing fast enough", we mean that

$$\lim_{k\to\infty} \frac{2^{1+k+n_1+\ldots+n_{k-1}}}{n_k} = 0.$$

We can clearly construct such a sequence by induction.

If $x = \sum_{i=1}^\infty x_i e_i$ and $\mu = \sum_{i=1}^\infty \mu_i e_i$ then, for each $i \in \mathbb{N}$, $\mu_i = \overline{x_i}|x_i|^{p-2}$. We then have

$$\begin{aligned} |\langle \mu, P_A STP_A(x)\rangle| &= \left| \sum_{i,j\in A} \overline{x_j}|x_j|^{p-2} x_i \langle e_j, ST(e_i)\rangle \right| \\ &\leq \sum_{k=1}^\infty \left| \sum_{l=1}^\infty \sum_{i\in A\cap A_k} \sum_{j\in A\cap A_l} \overline{x_j}|x_j|^{p-2} x_i \langle e_j, ST(e_i)\rangle \right| \leq \alpha_1 + \alpha_2 + \alpha_3, \end{aligned}$$

where we shall define α_1, α_2 and α_3 below. Note that, if we can find $A_i \in \mathcal{F}$ so that, with $A = A_1$, α_1 is small, and similarly for A_2 and A_3, then setting $A = A_1 \cap A_2 \cap A_3 \in \mathcal{F}$ will ensure that $|\langle \mu, P_A STP_A(x)\rangle|$ is small.

We first ensure that α_1 can be made as small as we like by a choice of $A \in \mathcal{F}$. Indeed,

$$\begin{aligned} \alpha_1 &= \sum_{k=1}^\infty \left| \sum_{l=k+1}^\infty \sum_{i\in A\cap A_k} \sum_{j\in A\cap A_l} \overline{x_j}|x_j|^{p-2} x_i \langle e_j, ST(e_i)\rangle \right| \\ &\leq B^2 \sum_{k=1}^\infty \sum_{l=k+1}^\infty \max_{i\in A\cap A_k, j\in A\cap A_l} |x_j|^{p-1}|x_i||\langle e_j, ST(e_i)\rangle| \\ &\leq B^2 \sum_{k=1}^\infty \sum_{l=k+1}^\infty \max_{i\in A\cap A_k, j\in A\cap A_l} |\langle e_j, ST(e_i)\rangle|, \end{aligned} \tag{4.5}$$

as both x and μ are B-reasonable. Let C be chosen later to be much larger than B. For each $k \in \mathbb{N}$ and $i \in A_k$, let $E_i \subset A_{k+1} \cup A_{k+2} \cup \cdots$ be chosen so that, for each $l > k$, $|E_i \cap A_l| \leq 2^{i+l}C$ and $\sum_{j\in E_i} |\langle e_j, ST(e_i)\rangle|^p$ is maximal. Let $A = \mathbb{N} \setminus \bigcup_{i=1}^\infty E_i$, so for each k,

$$|(\mathbb{N}\setminus A)\cap A_k| = \left| \bigcup_{i=1}^{N_{k-1}} E_i \cap A_k \right| \leq \sum_{i=1}^{N_{k-1}} |E_i \cap A_k| \leq C \sum_{i=1}^{N_{k-1}} 2^{i+k} \leq C2^{N_{k-1}+k+1},$$

and so $|(\mathbb{N}\setminus A)\cap A_k||A_k|^{-1} \leq C2^{1+k+n_1+\cdots+n_{k-1}}/n_k$. By the assumption on (n_k), we thus have $|(\mathbb{N}\setminus A)\cap A_k||A_k|^{-1} \to 0$ as $n \to \infty$, so that $\mathrm{ud}(\mathbb{N}\setminus A) = 0$, and so $A \in \mathcal{F}$.

Now, for each $k \in \mathbb{N}$, $l > k$, $i \in A\cap A_k$ and $j \in A\cap A_l$, we have that $j \in A_l \setminus \bigcup_{r=1}^{N_{l-1}} E_r$, so that certainly $j \in A_l \setminus E_i$. We hence see that

$$\begin{aligned}(2\|S\|)^p \geq \|ST(e_i)\|^p &= \sum_{s=1}^{\infty} |\langle e_s, ST(e_i)\rangle|^p \\ &= \sum_{s\in A_l\cap E_i} |\langle e_s, ST(e_i)\rangle|^p + \sum_{s\in A_l\setminus E_i} |\langle e_s, ST(e_i)\rangle|^p \\ &\geq \sum_{s\in A_l\cap E_i} |\langle e_s, ST(e_i)\rangle|^p \geq |A_l\cap E_i||\langle e_j, ST(e_i)\rangle|^p,\end{aligned}$$

so that $|\langle e_j, ST(e_i)\rangle| \leq 2\|S\|(2^{i+l}C)^{-1/p}$. Thus

$$\begin{aligned}\alpha_1 &\leq B^2 \sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \max_{i\in A\cap A_k, j\in A\cap A_l} 2\|S\|(2^{i+l}B')^{-1/p} \\ &\leq 2\|S\|B^2C^{-1/p}\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} 2^{-(N_k+l)/p} \\ &\leq DB^2\|S\|C^{-1/p}\end{aligned}$$

for some constant D depending only on $(n_k)_{k=1}^{\infty}$. Thus, by choosing C sufficiently large, we can make α_1 arbitrarily small, independently of x and μ.

Now we will look at α_2, which is

$$\begin{aligned}\alpha_2 &= \sum_{k=1}^{\infty}\left|\sum_{l=1}^{k-1}\sum_{i\in A\cap A_k}\sum_{j\in A\cap A_l} \overline{x_j}|x_j|^{p-2}x_i\langle e_j, ST(e_i)\rangle\right| \\ &\leq B^2\sum_{k=1}^{\infty}\sum_{l=k+1}^{\infty} \max_{i\in A\cap A_k, j\in A\cap A_l} |\langle T'S'(e_i), e_j\rangle|.\end{aligned}$$

Comparing this to Equation (4.5), and we see that we can use exactly the same argument as above to ensure that α_2 is arbitrarily small.

Finally, we need to show that α_3 can be made small, where

$$\alpha_3 = \sum_{k=1}^{\infty}\left|\sum_{i,j\in A\cap A_k} \overline{x_j}|x_j|^{p-2}x_i\langle e_j, ST(e_i)\rangle\right| = \sum_{k=1}^{\infty}|\langle \mu, P_{A\cap A_k}STP_{A\cap A_k}(x)\rangle|.$$

So by Lemma 4.3.9, we are done. □

We now put Propositions 4.3.6 and 4.3.10 together.

Theorem 4.3.11. *The element Φ_0 lies in the radical of $\mathcal{B}(l^p)''$. In particular, $\mathcal{B}(l^p)''$ is not semi-simple, for $p \in (1,\infty)$, $p \neq 2$.*

Proof. Choose and fix (n_k) so that Proposition 4.3.10 can be applied. Suppose towards a contradiction that $\Phi_0 \notin \operatorname{rad}\mathcal{B}(l^p)''$, so that by Proposition 4.3.6, there exists $\Psi \in \mathcal{B}(l^p)''$ and $z \in (l^p)_{\mathcal{U}}$ with

$$|\langle \mu^z, \theta_1(\Psi)(T(z))\rangle| > \tfrac{1}{2}\|\Psi\|^{-1}.$$

By Lemma 4.3.1, we can find $S \in \mathcal{B}(l^p)$ with $\|S\| \leq \|\Psi\|$ and $\|\theta_1(\Psi)(T(z)) - ST(z)\| < \varepsilon$, so that $|\langle \mu^z, ST(z)\rangle| > 1/2\|\Psi\|$, if $\varepsilon > 0$ is sufficiently small. As z is such that $P_A(z) = z$ for every $A \in \mathcal{W}$, we also have $P_A(\mu^z) = \mu^z$ for every $A \in \mathcal{W}$. Thus we have

$$\lim_{A\in\mathcal{V}} |\langle \mu^z, P_A STP_A(z)\rangle| \geq \tfrac{1}{2}\|\Psi\|^{-1}.$$

However, by Proposition 4.3.10, for every $\delta > 0$ we can find $A \in \mathcal{F} \subset \mathcal{W}$ so that, for each i, we have $|\langle \mu_i^z, P_A STP_A(z_i)\rangle| < \delta$. Thus we have

$$|\langle \mu^z, P_A STP_A(z)\rangle| \leq \delta,$$

and as $\delta > 0$ was arbitrary, we have

$$\lim_{A\in\mathcal{V}} |\langle \mu^z, P_A STP_A(z)\rangle| = 0.$$

This contradiction shows that $\Phi_0 \in \operatorname{rad} \mathcal{B}(l^p)''$ and so $\mathcal{B}(l^p)''$ is not semi-simple. □

4.3.3 A generalisation

We can use the same idea as in Lemma 4.2.4 to find further examples of Banach spaces E such that $\mathcal{B}(E)''$ is not semi-simple. This idea was first suggested by H.G. Dales, and uses Theorem 1.6.2.

Proposition 4.3.12. *Let $\mathcal{A}$ be a unital Banach algebra, and let $p, q \in \mathcal{A}$ be orthogonal idempotents such that $p + q = e_{\mathcal{A}}$. If the subalgebra $p\mathcal{A}p$ is not semi-simple, then $\mathcal{A}$ is not semi-simple.*

Proof. As in Lemma 4.2.4, we can view $\mathcal{A}$ as a matrix algebra. Let $c \in \operatorname{rad} p\mathcal{A}p$ be non-zero, let $a = pcp \in \mathcal{A}$, and pick $b \in \mathcal{A}$. Then

$$ab = \begin{pmatrix} pcp & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} pbp & pbq \\ qbp & qbq \end{pmatrix} = \begin{pmatrix} pcpbp & pcpbq \\ 0 & 0 \end{pmatrix},$$

so that

$$(ab)^n = \begin{pmatrix} (pcpbp)^n & (pcpbp)^{n-1}(pcpbq) \\ 0 & 0 \end{pmatrix}.$$

As $c \in \operatorname{rad} p\mathcal{A}p$, we see that $\lim_{n\to\infty} \|(pcpbp)^n\|^{1/n} = \lim_{n\to\infty} \|(cbp)^n\|^{1/n} = 0$. We then have

$$\begin{aligned}\|(ab)^n\|^{1/n} &= \|(pcpbp)^n + (pcpbp)^{n-1}(pcpbq)\|^{1/n} \\ &\leq (\|(pcpbp)^n\| + \|(pcpbp)^{n-1}\|\|pcpbq\|)^{1/n} \to 0\end{aligned}$$

as $n \to 0$. Thus, as b was arbitrary, $a \in \operatorname{rad} \mathcal{A}$, and so $\mathcal{A}$ is not semi-simple. □

Let F and G be Banach spaces, and let $E = F \oplus G$. Then

$$\mathcal{B}(E)'' = \left\{ \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix} : \Phi_{11} \in \mathcal{B}(F)'', \Phi_{12} \in \mathcal{B}(G, F)'' \text{ etc.} \right\}.$$

We can thus apply the above proposition to see that if E is a Banach space with complemented subspace F such that $\mathcal{B}(F)''$ is not semi-simple with respect to one of the Arens products, then $\mathcal{B}(E)''$ is not semi-simple with respect to the same Arens product.

We now set out some results about general $L^p(\nu)$-spaces, with the aim of showing that $\mathcal{B}(L^p(\nu))''$ is semi-simple if and only if $L^p(\nu)$ is isomorphic to a Hilbert space.

Proposition 4.3.13. *Let $\varepsilon > 0$, $p \in (2, \infty)$ and ν be an arbitrary measure, and let (x_n) be a normalised sequence in $L^p(\nu)$ equivalent to the canonical basis of l^p. Then there exists a subsequence $(x_{n(i)})$ which is $(1+\varepsilon)$-equivalent to the basis of l^p, and whose closed linear span is $(1+\varepsilon)$-complemented in $L^p(\nu)$.*

Proof. This follows from the proof of [Kadec, Pełczyński, 1962, Theorem 2]; see also the proof of [Heinrich, 1980, Theorem 10]. □

Proposition 4.3.14. *Let $p \in [1, \infty)$, and let E be a separable subspace of $L^p(\nu)$ for some measure ν. Then E is isometrically isomorphic to a subspace of $L^p[0,1]$.*

Proof. This is [Guerre-Delabriére, 1992, Theorem IV.1.7]. □

Proposition 4.3.15. *Let $p \in [2, \infty)$, and let E be an infinite-dimensional subspace of $L^p[0,1]$. Then either E is isomorphic to l^2 or, for each $\varepsilon > 0$, E contains a subspace which is $(1+\varepsilon)$-isomorphic to l^p.*

Proof. This is [Guerre-Delabriére, 1992, Corollary IV.4.4]. □

Theorem 4.3.16. *Let $p \in (2, \infty)$, ν be an arbitrary measure, and let E be a subspace of $L^p(\nu)$ such that E is not isomorphic to a Hilbert space. Then $\mathcal{B}(E)''$ is not semi-simple.*

Proof. Choose a separable subspace F of E, so that, by Theorem 4.3.14, F is isometrically isomorphic to a subspace of $L^p[0,1]$. Then by Proposition 4.3.15, either F is isomorphic to l^2, or F contains an isomorphic copy of l^p. If the latter, then E contains an isomorphic copy of l^p, so that by Proposition 4.3.13, E contains a complemented copy of l^p, and so, by an application of Proposition 4.3.12, $\mathcal{B}(E)''$ is not semi-simple.

So the only case left to consider is when every separable subspace of E is isomorphic to l^2. By the next lemma, E is then itself isomorphic to a Hilbert space, a contradiction. □

Lemma 4.3.17. *Let E be a Banach space such that each separable subspace of E is isomorphic to a Hilbert space. Then E itself is isomorphic to a Hilbert space.*

Proof. Every separable Hilbert space is isometric to l^2. Let $F \subseteq E$ be a separable subspace and $T : F \to l^2$ be an isomorphism. Let $[\cdot,\cdot]$ be the inner-product on l^2, and define

$$[x,y]_F = [T(x),T(y)] \qquad (x,y \in F).$$

Then $[\cdot,\cdot]_F$ is an inner product on F, and the norm $\|\cdot\|_F$ on F induced by this inner product satisfies

$$\|T^{-1}\|^{-1}\|x\| \leq \|x\|_F \leq \|T\|\|x\| \qquad (x \in F).$$

Suppose that (F_n) is a sequence of separable subspace of E such that, for each n, every isomorphism $T : F_n \to l^2$ satisfies $\|T\|\|T^{-1}\| \geq n$. Let F be the closure of the linear span of F_n, so that F is separable (see Lemma 5.2.4 for an indication as to why this is true). Let $T : F \to l^2$ be an isomorphism. For each n, $T(F_n)$ is a closed subspace of l^2 and is hence isometrically isomorphic to l^2. Consequently, there is an isomorphism $S : F_n \to l^2$ such that $\|S\|\|S^{-1}\| \leq \|T\|\|T^{-1}\|$, a contradiction for sufficiently large n.

We hence see that there is some constant K such that each separable subspace F of E can be given an inner product $[\cdot,\cdot]_F$ which satisfies

$$K^{-1}\|x\| \leq [x,x]_F^{1/2} \leq K\|x\| \qquad (x \in F).$$

Let I be the collection of separable subspaces of E, partially ordered by inclusion, and let $\mathcal{U}$ be an ultrafilter refining the order filter on I. For $x,y \in E$, we can hence define

$$[x,y] = \lim_{F\in\mathcal{U}} [x,y]_F.$$

Then $[\cdot,\cdot]$ is an inner product; everything is trivial to check except that $[x,x]^{1/2}$ is a norm and not a semi-norm: this fact follows as

$$[x,x]^{1/2} = \lim_{F\in\mathcal{U}} [x,x]_F^{1/2} \geq \lim_{F\in\mathcal{U}} K^{-1}\|x\| \qquad (x \in E).$$

By a similar argument, we see that $[\cdot,\cdot]$ induces a norm on E which is equivalent to the existing norm on E; that is, E is isomorphic to a Hilbert space. $\square$

An examination of the above proof shows that if E is a Banach space such that each finite-dimensional subspace of E is isomorphic to a Hilbert space, with uniform control on the isomorphism-constant, then E is isomorphic to a Hilbert space.

The class of $\mathfrak{L}^g_{p,\lambda}$ spaces are defined in [Defant, Floret, 1993, Section 3.13], for $1 \leq p \leq \infty$, $1 \leq \lambda < \infty$, to be Banach spaces E such that for each finite-dimensional subspace M of E, and each $\varepsilon > 0$, we can find $R \in \mathcal{B}(M, l^p_m)$ and $S \in \mathcal{B}(l^p_m, E)$ for some $m \in \mathbb{N}$, such that $SR(x) = x$ for each $x \in M$, and $\|S\|\|R\| \leq \lambda + \varepsilon$. Then E is an $\mathfrak{L}^g_p$-space if it is an $\mathfrak{L}^g_{p,\lambda}$-space for some λ. In [Defant, Floret, 1993, Section 23.2], it is shown that for $1 < p < \infty$, E is an $\mathfrak{L}^g_p$-space if and only if E is isomorphic to a complemented subspace of some $L^p(\nu)$ space. Thus we have the following.

Corollary 4.3.18. *Let E be an $\mathfrak{L}^g_p$-space. Then $\mathcal{B}(E)''$ is semi-simple if and only if E is isomorphic to a Hilbert space.* □

Summing up our results, we have the following.

Theorem 4.3.19. *Let E be a Banach space such that at least one of the following holds:*

1. *E is reflexive and $E = F \oplus G$ with one of F and G having the approximation property, $\mathcal{B}(F, G) = \mathcal{K}(F, G)$ and $\mathcal{B}(F, G) \neq \mathcal{K}(F, G)$;*
2. *E is a complemented subspace of $L^p(\nu)$, for some measure ν and $1 < p < \infty$, such that E is not isomorphic to a Hilbert space;*
3. *E is a closed subspace of $L^p(\nu)$, for some measure ν and $2 < p < \infty$, such that E is not isomorphic to a Hilbert space;*
4. *E contains a complemented subspace F such that F has property (1), (2) or (3).*

Then $\mathcal{B}(E)''$ is not semi-simple. □

In particular, at present the only Banach spaces E for which $\mathcal{B}(E)''$ is semi-simple are those isomorphic to a Hilbert space. It is tempting to conjecture that $\mathcal{B}(E)''$ is semi-simple only if E is isomorphic to a Hilbert space, at least when E is super-reflexive. However, given the remarks about when $\mathcal{B}(E)$ is Arens regular, it seems possible that some pathological Banach space E could have $\mathcal{B}(E)''$ semi-simple, but E not being isomorphic to a Hilbert space.

Chapter 5

Closed ideals in $\mathcal{B}(E)$

In this chapter we shall study the closed ideal structure of $\mathcal{B}(E)$. While operator ideals and tensor norms give us a way of generating ideals in $\mathcal{B}(E)$, they are rarely closed in $\mathcal{B}(E)$, with $\mathcal{A}(E) = E'\check{\otimes}E$ being the obvious exception. As each ideal of $\mathcal{B}(E)$ must contain $\mathcal{F}(E)$, we see that $\mathcal{A}(E)$ is the smallest closed ideal in $\mathcal{B}(E)$, and as the unit ball of E is compact if and only if E is finite-dimensional, we see that $\mathcal{A}(E)$ is always a proper ideal for infinite-dimensional Banach spaces E.

We have seen a number of other obvious closed ideals, namely $\mathcal{K}(E)$ and $\mathcal{W}(E)$. When E has the approximation property, $\mathcal{A}(E) = \mathcal{K}(E)$, and when E is reflexive, $\mathcal{B}(E) = \mathcal{W}(E)$. As shown originally by Calkin, $\mathcal{A}(H) = \mathcal{K}(H)$ is the unique proper, closed ideal in $\mathcal{B}(H)$ for a separable Hilbert space H. In [Gohberg et al., 1967], Gohberg, Markus and Feldman showed that this result actually holds for l^p, $1 \leq p < \infty$ and c_0. In another direction, in [Luft, 1968] and [Gramsch, 1967], Gramsch and Luft independently classified the closed ideals in $\mathcal{B}(H)$ for any Hilbert space H (see Theorem 5.3.9). Until recently (see [Laustsen et al., 2004] for recent progress) this was the complete list of Banach spaces for which we knew the full closed ideal structure of $\mathcal{B}(E)$. See [Laustsen, Loy, 2003] for a good survey of known results.

We shall sketch a modern approach to the Gohberg, Markus and Feldman result (which will provide proofs for some results stated in previous chapters). We will then give a generalisation of the Gramsch and Luft results to the spaces $l^p(I)$, for $1 \leq p < \infty$ and $c_0(I)$, for arbitrary infinite sets I. This presentation is heavily based upon the preprint [Daws(2), 2004].

5.1 Perfectly homogeneous bases

Recall the notions of a basis, unconditional basis and block-basis from Section 3.4. As before, we write $\operatorname{supp}(x)$ for the support, with respect to a basis (e_n), of a vector x. We use P_A and P_n for the canonical projections associated with a basis.

Definition 5.1.1. Let E and F be Banach spaces with bases (e_n) and (f_n) respectively. Suppose that, for each sequence of scalars (a_n), $\sum_{n=1}^\infty a_n e_n$ converges if and only if $\sum_{n=1}^\infty a_n f_n$ converges. Then (e_n) and (f_n) are *equivalent*.

Lemma 5.1.2. *Let E and F be Banach spaces with bases (e_n) and (f_n) respectively. Then (e_n) and (f_n) are equivalent if and only if there is an isomorphism $T : E \to F$ such that $T(e_n) = f_n$ for each n.*

Proof. This is an exercise involving the closed graph theorem, detailed in, for example, [Megginson, 1998, Proposition 4.3.2]. □

Proposition 5.1.3. *Let $E = l^p$, for $1 \leq p < \infty$, or $E = c_0$, let (e_n) be the standard unit vector basis for E, and let (x_n) be a normalised block-basis of (e_n). Let F be the closed linear span of (x_n) (so that F has (x_n) as a basis). Then the map $T : E \to F$, defined by $T(e_n) = x_n$ for each n, is an isometry, and there is a norm-one projection $P : E \to F$.*

Proof. This is, for example, [Lindenstrauss, Tzafriri, 1977, Proposition 2.a.1]. We do the l^p case here, the c_0 case following similarly. Let $a = (a_n) \in l^p$, so that

$$\Big\| \sum_{n=1}^\infty a_n x_n \Big\| = \left(\sum_{n=1}^\infty |a_n|^p \|x_n\|^p \right)^{1/p} = \|a\|_p = \Big\| \sum_{n=1}^\infty a_n e_n \Big\|,$$

where the first equality is because the (x_n) have pairwise-disjoint support. Thus T is an isometry.

Let (μ_n) be a sequence in l^q (where, as usual, $p^{-1} + q^{-1} = 1$) such that $\langle \mu_n, x_n \rangle = \|x_n\| = \|\mu_n\| = 1$, for each n. A simple calculation shows that this actually uniquely defines (μ_n), and that, for each n, $\operatorname{supp}(x_n) = \operatorname{supp}(\mu_n)$, so that (μ_n) is a block-basis in l^q (except, of course, in the $p = 1$ case). For $x \in l^p$, define

$$P(x) = \sum_{n=1}^\infty \langle \mu_n, x \rangle x_n,$$

so that we have

$$\|P(x)\| = \left(\sum_{n=1}^\infty |\langle \mu_n, x \rangle|^p \right)^{1/p} \leq \left(\sum_{n=1}^\infty \|P_{\operatorname{supp}(x_n)}(x)\|^p \right)^{1/p} \leq \|x\|_p.$$

Thus $\|P\| \leq 1$, and we have $P(x_n) = x_n$ for each n, so that P is a projection from E onto F. □

We say that a basis (e_n) for a Banach space E is *perfectly homogeneous* if every (normalised) block-basis (x_n) of (e_n) is equivalent to (e_n). Thus the standard bases of l^p, $1 \leq p < \infty$ and c_0 are perfectly homogeneous. Actually, as shown by Zippin in [Zippin, 1966], these are the only Banach spaces with perfectly homogeneous bases.

Our argument below is essentially that given in [Pietsch, 1980, Chapter 5]. It can also be viewed as a more constructive version of the argument given in [Herman, 1968].

Lemma 5.1.4. *Let E be l^p, for $1 \leq p < \infty$, or c_0, let (e_n) be the standard basis for E, and let $T \in \mathcal{B}(E)$. Suppose that, for each bounded sequence (x_n) in E such that $\lim_n P_m(x_n) = 0$ and $\lim_n P_m T(x_n) = 0$ for each $m \in \mathbb{N}$, we have $\lim_n T(x_n) = 0$. Then T is compact.*

Proof. Suppose, towards a contradiction, that $T \in \mathcal{B}(E)$ satisfies the hypotheses but is not compact. As $\mathcal{K}(E) = \mathcal{A}(E)$, as in Lemma 3.4.8, we must have

$$\delta := \inf_{n\in\mathbb{N}} \|T - P_n T\| > 0,$$

as T is not approximable. Choose a sequence (x_n) in E such that

$$\|x_n\| = 1 \quad , \quad \|T(x_n) - P_n T(x_n)\| \geq \delta/2 \qquad (n \in \mathbb{N}).$$

For each m, the image of P_m is finite-dimensional, so we can move to a subsequence such that $P_m(x_n)$ and $P_m T(x_n)$ converge. By a diagonal argument, we can find a subsequence $(n_k)_{k=1}^\infty$ such that $P_m(x_{n_k})$ and $P_m T(x_{n_k})$ are Cauchy sequences, for each $m \in \mathbb{N}$. Suppose that $(T(x_{n_k}))$ is not a Cauchy sequence, so that, for some $\varepsilon > 0$, we can find increasing sequences of integers $(j(i))$ and $(k(i))$ such that

$$\|T(x_{n_{j(i)}}) - T(x_{n_{k(i)}})\| \geq \varepsilon > 0 \qquad (i \in \mathbb{N}).$$

Then set $y_i = x_{n_{j(i)}} - x_{n_{k(i)}}$ for each i, so that as $P_m(x_{n_k})$ and $P_m T(x_{n_k})$ are Cauchy-sequences, we have

$$\lim_{i\to\infty} P_m(y_i) = \lim_{i\to\infty} P_m T(y_i) = 0.$$

By the hypotheses, we must have $\lim_{i\to\infty} T(y_i) = 0$, which is a contradiction. We conclude that $(T(x_{n_k}))$ is a Cauchy-sequence.

Let $y = \lim_{k\to\infty} T(x_{n_k})$. We then have

$$\|P_m T(x_{n_k}) - T(x_{n_k})\| \geq \|P_{n_k} T(x_{n_k}) - T(x_{n_k})\| \geq \delta/2 \qquad (k \in \mathbb{N}, m \leq n_k),$$

so that $\|y - P_m(y)\| \geq \delta/2$ for each $m \in \mathbb{N}$. This is a contradiction, which completes the proof. $\square$

Proposition 5.1.5. *Let E and F be Banach spaces of the form l^p, for $1 \leq p < \infty$, or c_0. Let $T \in \mathcal{B}(E, F)$ be such that T is not compact. Then there exists $\delta > 0$ such that, for each $\varepsilon > 0$, we can find a block-basis (x_n) in E and a block-basis (y_n) in F such that $\sum_{n=1}^{\infty} \|T(x_n) - \delta y_n\| < \varepsilon$.*

Proof. This is a variant of the Bessaga-Pełcyński Selection Principle (see, for example, [Megginson, 1998, Theorem 4.3.19]). As T is not compact, by the preceding lemma, we can find a sequence of unit vectors (z_n) such that $\lim_n P_m(z_n) = \lim_n P_m T(z_n) = 0$ for each $m \in \mathbb{N}$, and such that $T(z_n)$ does not converge to zero. By perturbing each z_n and moving to a subsequence, we may suppose that (z_n) is actually a block-basis, and that for some $\delta > 0$, we have $\lim_n \|T(z_n)\| = \delta^{-1}$. Then, for $\varepsilon > 0$, by again moving to subsequence, we may suppose that $\delta^{-1} \leq \|T(z_n)\| \leq \delta^{-1} + \varepsilon\delta^{-1}2^{-n-1}$ for each n.

Let $x_1 = z_1$ and pick n_1 such that $\|T(x_1) - P_{n_1}T(x_1)\| < \varepsilon\delta^{-1}2^{-3}$. As $0 = \lim_n P_{n_1}T(z_n)$, we can find m_1 such that $\|P_{n_1}T(z_n)\| < \varepsilon\delta^{-1}2^{-3}$ for each $n \geq m_1$. Let $x_2 = z_{m_1}$ and pick $n_2 > n_1$ such that $\|T(x_2) - P_{n_2}T(x_2)\| < \varepsilon\delta^{-1}2^{-4}$. We can then pick $m_2 > m_1$ such that $\|P_{n_2}T(z_n)\| < \varepsilon\delta^{-1}2^{-4}$ for each $n \geq m_2$. By induction, we choose increasing sequences (n_k) and (m_k) such that, for each $k \in \mathbb{N}$ and $j > k$, we have

$$x_k = z_{m_{k-1}} \quad , \quad \|T(x_k) - P_{n_k}T(x_k)\| < \varepsilon\delta^{-1}2^{-k-2} \quad , \quad \|P_{n_k}T(x_j)\| < \varepsilon\delta^{-1}2^{-k-2},$$

where we set $n_0 = 0$ and $P_0 = 0$.

Then (x_k) is clearly a block-basis. For $k \in \mathbb{N}$, let

$$\hat{y}_k = (P_{n_k} - P_{n_{k-1}})T(x_k) \quad , \quad y_k = \hat{y}_k\|\hat{y}_k\|^{-1},$$

so that (y_k) is a block-basis in F. Then we have

$$\|\hat{y}_k - T(x_k)\| \leq \|P_{n_k}T(x_k) - T(x_k)\| + \|P_{n_{k-1}}T(x_k)\| < \varepsilon\delta^{-1}2^{-k-1}.$$

Thus we have

$$\|\hat{y}_k\| \leq \|\hat{y}_k - T(x_k)\| + \|T(x_k)\| \leq \varepsilon\delta^{-1}2^{-k-1} + \delta^{-1} + \varepsilon\delta^{-1}2^{-k-1},$$
$$\|\hat{y}_k\| \geq \|T(x_k)\| - \|T(x_k) - \hat{y}_k\| \geq \delta^{-1} - \varepsilon\delta^{-1}2^{-k-1},$$

so that $-\varepsilon 2^{-k-1} \leq \delta\|\hat{y}_k\| - 1 \leq \varepsilon 2^{-k}$. Hence

$$\begin{aligned}\|y_k - \delta T(x_k)\| &\leq \|y_k - \delta\hat{y}_k\| + \delta\|\hat{y}_k - T(x_k)\| < |1 - \delta\|\hat{y}_k\|| + \delta 2^{-k} \\ &\leq \varepsilon 2^{-k} + \varepsilon 2^{-k-1} \leq \varepsilon 2^{1-k}.\end{aligned}$$

So in conclusion, we have

$$\sum_{k=1}^{\infty} \|y_k - \delta^{-1} T(x_k)\| < 2\varepsilon.$$

$\square$

Proposition 5.1.6. *Let E and F be Banach spaces of the form l^p, for $1 \leq p < \infty$, or c_0. Let $T \in \mathcal{B}(E, F)$ be such that T is not compact. Let (e_n) and (f_n) be the standard bases for E and F respectively. Then there exists $R \in \mathcal{B}(E)$ and $S \in \mathcal{B}(F)$ such that $STR(e_n) = f_n$ for each n.*

Proof. Use the preceding lemma to pick block-bases (x_n) and (y_n) for some $\varepsilon \in (0, 1)$. Let $R \in \mathcal{B}(E)$ be given by Proposition 5.1.3, so that $R(e_n) = x_n$ for each $n \in \mathbb{N}$. Let G be the closed linear span of the (y_n). Using Proposition 5.1.3 again, let P be a norm-one projection of F onto G, and let $S_0 \in \mathcal{B}(F)$ be such that $S_0(f_n) = y_n$ for each n. Then the operator $S_0^{-1} \circ P \in \mathcal{B}(F)$ is well-defined, and satisfies $S_0^{-1} P(y_n) = f_n$ for each n.

Define $S_1 : G \to \text{lin}(T(x_n))$ by $S_1(y_n) = T(x_n)$ for each n. Then, for a sequence of scalars (a_n), let $y = \sum_{n=1}^{\infty} a_n y_n \in F$, so that

$$\|S_1(y)\| = \Big\| \sum_{n=1}^{\infty} a_n T(x_n) \Big\| \leq \|T\| \Big\| \sum_{n=1}^{\infty} a_n x_n \Big\| = \|T\| \|y\|,$$

again appealing to Proposition 5.1.3, as (x_n) and (y_n) are block-bases. Hence S_1 is bounded. For y as before, we also have

$$\begin{aligned} \|P(y) - PS_1(y)\| \leq \|y - S_1(y)\| &= \Big\| \sum_{n=1}^{\infty} a_n (y_n - T(x_n)) \Big\| \\ &\leq \|(a_n)\|_\infty \sum_{n=1}^{\infty} \|y_n - T(x_n)\| < \varepsilon \|y\|. \end{aligned}$$

Hence we see that $\text{Id}_G - PS_1 \in \mathcal{B}(G)$ is such that $\|\text{Id}_G - PS_1\| < \varepsilon < 1$, so we can let $S_2 = (PS_1)^{-1} \in \mathcal{B}(G)$.

In conclusion, let $S = S_0^{-1} PS_2 P$, so that for each $n \in \mathbb{N}$, we have

$$STR(e_n) = S_0^{-1} PS_2 PT(x_n) = S_0^{-1} P(PS_1)^{-1} PS_1(y_n) = S_0^{-1} P(y_n) = f_n. \qquad \square$$

Theorem 5.1.7. *Let p and q be such that $1 \leq p < q < \infty$. Then $\mathcal{B}(l^q, l^p) = \mathcal{K}(l^q, l^p)$.*

Proof. Suppose we have $T \in \mathcal{B}(l^q, l^p) \setminus \mathcal{K}(l^q, l^p)$. Let (e_n) and (f_n) be the standard bases for l^q and l^p respectively. By the above, we can find $R \in \mathcal{B}(l^q)$ and $S \in \mathcal{B}(l^p)$ so that $STR(e_n) = f_n$ for each n. In particular, for each $n \in \mathbb{N}$, we have

$$n^{1/p} = \Big\| \sum_{i=1}^{n} f_i \Big\| = \Big\| \sum_{i=1}^{n} STR(e_i) \Big\| \leq \|S\| \|T\| \|R\| \Big\| \sum_{i=1}^{n} e_i \Big\| = \|S\| \|T\| \|R\| n^{1/q},$$

which is impossible, as we would then have $\|S\|\|T\|\|R\| \geq \sup_n n^{1/p-1/q} = \infty$. $\square$

Theorem 5.1.8. *Let $E = l^p$ for $1 \leq p < \infty$, or $E = c_0$. If J is a non-trivial closed ideal in $\mathcal{B}(E)$, then $J = \mathcal{K}(E)$.*

Proof. It suffices to show that if $T \in \mathcal{B}(E)$ is not compact, then the (closed) ideal generated by T is $\mathcal{B}(E)$. This follows immediately from the above, as we can find $R, S \in \mathcal{B}(E)$ such that $STR = \mathrm{Id}_E$. $\square$

In [Herman, 1968], the author proves a little more than the above, but only for spaces with a perfectly homogeneous basis. Thus, by the Zippin result, the above theorem is all we can say. Indeed, there are still no further examples known of Banach spaces E for which $\mathcal{A}(E)$ is the unique closed ideal in $\mathcal{B}(E)$.

5.2 Ordinal and cardinal numbers

We want to have a way of measuring the "size" of a set which is not countable. The tools we need are ordinal and cardinal numbers. Following a "naive" approach to set theory (see [Halmos, 1960]) we formalise the usual natural numbers by setting $0 = \emptyset$ and, for a natural number n, we set $n^+ = n \cup \{n\}$, the *successor* to n. We see that then $n^+ = \{n, n-1, n-2, \ldots, 1, 0\}$. By the axiom of infinity (see [Halmos, 1960, Section 11]) there is a set which contains $\emptyset$ and the successor of each of its elements. We define ω to be the smallest set which contains 0 and which contains the successor of each of its elements. Thus ω is the set of natural numbers (well, to be precise, the natural numbers and 0, as we have been following to the convention that 0 is not a natural number). Notice that each natural number n satisfies the condition

$$n = \{m : m \in n\} = \{m : m < n\}$$

where the order $<$ is induced by the binary relation $\in$.

Definition 5.2.1. Let $(S, \leq)$ be a totally ordered set. Then $(S, \leq)$ is *well-ordered* if each non-empty subset of S has a smallest element.

Let $(S, \leq_S)$ and $(T, \leq_T)$ be partially ordered sets and $f : S \to T$ be a function. Then f is *order-preserving* if $f(a) \leq_T f(b)$ whenever $a, b \in S$ with $a \leq_S b$. Then S and T are *similar* if there is an order-preserving bijection $f : S \to T$.

Let $(S, \leq)$ be a well-ordered set. Then an *initial segment* of S if a subset of S of the form $\{a \in S : a < b\}$ for some $b \in S$.

Let S and T be well-ordered sets. Then S is a *continuation* of T if T is an initial segment of S and the partial order on T is the restriction of the partial order on S.

We can show that given two well-ordered sets S and T, either S and T are similar, or S is similar to an initial segment of T, or T is similar to an initial segment of S.

Definition 5.2.2. An *ordinal* is a well-ordered set $(\alpha, \leq)$ such that

$$\{\eta \in \alpha : \eta < \xi\} = \xi \qquad (\xi \in \alpha).$$

We see that each natural number is an ordinal, as given $n \in \mathbb{N}$, we have $n = \{n-1, n-2, \ldots, 1, 0\}$ so that if $m \in n$ then $\{k \in n : k < m\} = \{m-1, m-2, \ldots, 1, 0\} = m$. We also see that ω is an ordinal number, and that if α is an ordinal number, then so is α^+.

We see that for an ordinal number α and $\eta, \xi \in \alpha$, we have that $\eta < \xi$ if and only if $\eta \in \xi$, so that α is (well-) ordered by $\in$. We also see that each initial segment of α is equal to an element of α. Suppose that β is also an ordinal number, so that α and β are well-ordered sets. Suppose that α is similar to an initial segment of β, so that α is similar to an element of β, that is, we can treat α as an element of β. In a similar manner, we see that either $\alpha = \beta$ or $\alpha \in \beta$ or $\beta \in \alpha$.

A Zorn's Lemma argument can be used to show that each set S admits a partial order $\leq$ such that $(S, \leq)$ is well-ordered (the *well-ordering theorem*). We can then show that each well-ordered set S is similar to some (necessarily unique) ordinal number. In this way, we can "count" the elements of any set.

We have a problem here though. There can be many different ways in which we can well-order a set, and hence many different ways to "count" the elements of a set. For example, given a countable set I, we can well-order I as ω, or as ω^+. This example is a little circular: what do we mean exactly by "countable"?

The Schröder-Bernstein theorem tells us that, when we have two sets S and T such that there are injective maps $f : S \to T$ and $g : T \to S$, we can find a bijective map $h : S \to T$. This gives us a way of comparing the size of sets. We then say that a set I is *countable* (or *countably-infinite*) if there is a bijection $I \to \omega$.

We pick ω out in the definition because of the following fact: if n is an ordinal number with $n < \omega$, then there is no bijection between n and ω. We use this idea to make a definition.

Definition 5.2.3. Let α be an ordinal number such that whenever $\beta \in \alpha$, there is no bijection between α and β. Then α is a *cardinal number*.

We see that each natural number is a cardinal number, and that each cardinal number is an ordinal number. Given a set S, the *cardinality* of S is the (unique) cardinal number κ such that there is a bijection $\kappa \to S$. We write $|S| = \kappa$. As already mentioned, ω is a cardinal number; clearly it is the smallest infinite cardinal number, which we denote by $\aleph_0$.

Some care is required when talking about sets of ordinal or cardinal numbers, as the collection of all ordinal numbers does not form a set. However, for example, we can show that when given a cardinal number κ, always find an ordinal number β such that $\kappa \in \beta$ but such that there is no bijection $\kappa \to \beta$. As β is a set, it thus makes sense to talk about the smallest cardinal greater than κ, denoted by κ^+ (which should not be confused with the ordinal successor to κ which, in general, will have the same cardinality as κ). In this way, we can form the cardinal numbers $\aleph_1, \aleph_2, \ldots, \aleph_n, \ldots$. We define $\aleph_\omega$ to be the smallest cardinal greater than anything in the set $\{\aleph_n : n \in \omega\}$. Then $\aleph_\omega$ is not the successor of any cardinal: it is a *limit cardinal*. Similarly, $\aleph_0$ is a limit cardinal. We see that $\aleph_1$, as an ordinal, is the smallest ordinal which is not countable.

We can perform arithmetic on cardinal numbers. Given two cardinal numbers κ, σ, we define $\kappa + \sigma$ to be the cardinality of the set $S \cup T$ where S and T are sets such that $S \cap T = \emptyset$, $|S| = \kappa$ and $|T| = \sigma$. We can, of course, check that this does not depend upon the choice of S or T. With S and T as above, we also have $\kappa.\sigma = \kappa \times \sigma = |S \times T|$. We extend this to arbitrary sums and products in the obvious way.

We can show that

$$\kappa + \sigma = \kappa \times \sigma = \max(\kappa, \sigma),$$

when at least one of κ and σ is infinite.

We have skipped many technical details in this sketch. A rigourous approach is taken in, for example, citeHJ. Note that this book uses slightly different definitions to those given above, but that, in the cases we are interested in, these boil down to being the same thing.

Lemma 5.2.4. *Let V be a vector space and $X \subseteq V$ be some subset. Let $\mathbb{Q}[\imath]$ be the subfield of $\mathbb{C}$ consisting of complex numbers with rational real and imaginary parts. Let*

$$Y = \Big\{ \sum_{k=1}^{n} a_k x_k : n \in \mathbb{N}, (a_k)_{k=1}^{n} \subseteq \mathbb{Q}[\imath], (x_k)_{k=1}^{n} \subseteq X \Big\},$$

that is, the $\mathbb{Q}[\imath]$-linear span of X. When X is infinite, we have $|Y| = |X|$.

Proof. As $X \subseteq Y$, we clearly have $|Y| \geq |X|$. We can write Y as a countable union of sets (Y_n), where

$$Y_n = \left\{ \sum_{k=1}^{n} a_k x_k : (a_k)_{k=1}^n \subseteq \mathbb{Q}[\imath], (x_k)_{k=1}^n \subseteq X \right\} \qquad (n \in \mathbb{N}).$$

Then $|Y_1| \leq |\mathbb{Q}[\imath] \times X| = |\mathbb{Q}[\imath]| \times |X| = \max(|X|, \aleph_0) \leq |X|$, as $|X| \geq \aleph_0$. We also have $|Y_2| \leq |Y_1 \times Y_1| = |Y_1| \times |Y_1| = |X| \times |X| = |X|$. We see that, for each $n \in \mathbb{N}$, we have $|Y_n| \leq |Y_1 \times Y_{n-1}| = |X|$, by induction. Then $|Y| \leq \sum_{n=1}^{\infty} |Y_n| = \sum_{n=1}^{\infty} |X| = |\mathbb{N} \times X| = \aleph_0 \times |X| = |X|$, as required. $\square$

5.3 Non-separable Banach spaces

We shall sketch the theory of unconditional bases in non-separable Banach spaces (called *extended unconditional bases* in [Singer, 1981, Chapter 17]). The proofs of these results follow in a simple way from the standard theory of unconditional bases which was presented in Section 3.4.

When X is a topological vector space (that is, a vector space with a topology which makes addition and scalar-multiplication continuous) and $(x_\alpha)_{\alpha \in I}$ is a family in X, we say that (x_α) *sums unconditionally* to $x \in X$, written $x = \sum_{\alpha \in I} x_\alpha$, if, for each open neighbourhood U of x, there is a finite $A \subseteq I$ such that, if $B \subseteq I$ is finite and $A \subseteq B$, then $\sum_{\alpha \in B} x_\alpha \in U$. This definition agrees with the usual one for sequences.

For a Banach space E, a family of vectors $(e_\alpha)_{\alpha \in I}$ is an *unconditional basis* for E if, for each $x \in E$, there is a unique family of scalars (a_α) such that

$$x = \sum_{\alpha \in I} a_\alpha e_\alpha,$$

with summation interpreted as above. Again, if I is countable, then E is separable, and this definition agrees with the usual one of an unconditional basis (c.f. Theorem 3.4.5).

As in the separable case, we can define bounded linear functionals $e_\alpha^* \in E'$ such that

$$\langle e_\alpha^*, e_\beta \rangle = \begin{cases} 1 & : \alpha = \beta, \\ 0 & : \alpha \neq \beta. \end{cases}$$

If $\|e_\alpha\| = 1$ for each α, then the unconditional basis (x_α) is *normalised*. In this case, the family (e_α^*) is bounded.

For each $A \subseteq I$, we can define a map $P_A : E \to E$,

$$P_A(x) = \sum_{\alpha \in A} \langle e_\alpha^*, x \rangle e_\alpha.$$

A closed-graph argument shows that P_A is bounded, so that P_A is a projection onto the subspace $P_A(E)$. For $x \in E$, we again define the *support* of x to be

$$\mathrm{supp}(x) = \{\alpha \in I : \langle e_\alpha^*, x\rangle \neq 0\}.$$

Thus $P_A(E)$ is the subspace of vectors in E with support contained in A. From our meaning of summation, we can see that the support of x is always a countable subset of I, for each $x \in E$.

A uniform boundedness argument shows that the family $(P_A)_{A\subseteq I}$ is bounded, and by a standard re-norming, we may suppose that $\|P_A\| = 1$ for each $A \subseteq I$ (and so, in particular, that $\|e_\alpha^*\| = 1$ for each $\alpha \in I$). Henceforth we shall suppose that an unconditional basis is normalised and that $\|P_A\| = 1$ for each $A \subseteq I$.

The family $(e_\alpha^*)_{\alpha\in I}$ forms an unconditional basis for the closure of its span in E'. When this closure is the whole of E', we say that (e_α) is *shrinking*. We can show that (e_α) is shrinking if and only if

$$\inf\{\|P_A'(\mu)\| : A \subseteq I, |I \setminus A| < \infty\} = 0 \tag{5.1}$$

for each $\mu \in E'$, where $|A|$ denotes the cardinality of A.

Let I be an infinite set and recall the definitions of $c_0(I)$ and $l^p(I)$, for $1 \leq p < \infty$. In a more explicit form than that given in Chapter 1, we have

$$c_0(I) = \{(x_i)_{i\in I} \subseteq \mathbb{C} : \forall\, \varepsilon > 0, |\{i \in I : |x_i| \geq \varepsilon\}| < \infty\}.$$

Similarly, for $1 \leq p < \infty$, we have

$$l^p(I) = \left\{ (x_i)_{i\in I} \subseteq \mathbb{C} : \|(x_i)\|_p := \left(\sum_{i\in I} |x_i|^p\right)^{1/p} < \infty \right\}.$$

Then the family of vectors $(e_i)_{i\in I}$, defined by $e_i = (\delta_{ij})_{j\in I}$, is an unconditional basis for $c_0(I)$ and $l^p(I)$. Here δ_{ij} denotes the Kronecker delta. In all cases except $p = 1$, this basis is also shrinking; when $p = 1$, (e_i^*) spans $c_0(I) \subseteq l^\infty(I) = l^1(I)'$.

The *density character* of a Banach space E is the least cardinality of a dense subset of E. Thus E is separable if and only if E has density character $\aleph_0$.

5.3.1 Generalisation of compact operators

Let E and F be Banach spaces. For an infinite cardinal κ and $T \in \mathcal{B}(E, F)$, we say that T is *κ-compact* if, for each $\varepsilon > 0$, we can find a subset X of $E_{[1]}$ with $|X| < \kappa$, and such that

$$\inf\{\|T(x-y)\| : y \in X\} \leq \varepsilon \qquad (x \in E_{[1]}).$$

This is clearly equivalent to the condition that for each $\varepsilon > 0$, we can find a subset Y of F with $|Y| < \kappa$, and such that

$$\inf\{\|T(x) - y\| : y \in Y\} \leq \varepsilon \qquad (x \in E_{[1]}).$$

We write $\mathcal{K}_\kappa(E,F)$ for the set of κ-compact operators. In [Luft, 1968] it is shown that $\mathcal{K}_\kappa(E,F)$ is a closed operator ideal; that is, we have the following.

Proposition 5.3.1. *Let E and F be Banach spaces. Then $\mathcal{K}_\kappa(E,F)$ is a closed subspace of $\mathcal{B}(E,F)$. Let D and G be Banach spaces, and $T \in \mathcal{K}_\kappa(E,F)$, $S \in \mathcal{B}(D,E)$ and $R \in \mathcal{B}(F,G)$. Then $RTS \in \mathcal{K}_\kappa(D,G)$.* □

In particular, $\mathcal{K}_\kappa(E)$ is a closed ideal in $\mathcal{B}(E)$. The $\aleph_0$-compact operators are just the usual compact operators, so that $\mathcal{K}_{\aleph_0}(E,F) = \mathcal{K}(E,F)$. For higher cardinals, there is an easier description of κ-compact operators, subject to a technicality. For a cardinal κ, the *cofinality* of κ, $\mathrm{cf}(\kappa)$, is the least ordinal $\sigma \leq \kappa$ such that there is an order-preserving map $f : \sigma \to \kappa$ which is not bounded above. See, for example, [Hrbacek, Jech, 1999, Chapter 9, Section 2]. Then $\mathrm{cf}(\kappa)$ is a cardinal; if $\mathrm{cf}(\kappa) = \kappa$ we say that κ is *regular*, otherwise κ is *singular*. In particular, if κ is singular, then κ is a limit cardinal. Notice that $\mathrm{cf}(\aleph_0) = \aleph_0$, as if σ is an ordinal with $\sigma < \aleph_0$, then σ is finite, and so any $f : \sigma \to \aleph_0$ will be bounded above.

Lemma 5.3.2. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \aleph_0$ (so that $\kappa > \aleph_0$). Then, if (A_n) is a sequence of sets, each of cardinality less than κ, then $|\bigcup_n A_n| < \kappa$.*

Proof. For each $n \in \mathbb{N}$, let $B_n = \bigcup_{m \leq n} A_m$ so that $(|B_n|)$ is an increasing sequence of cardinals with, for each n, $|B_n| \leq \sum_{m=1}^n |A_m| < \kappa$. As $\mathrm{cf}(\kappa) > \aleph_0$, $(|B_n|)$ is bounded above by some $\sigma < \kappa$, for otherwise, we could define $f : \aleph_0 \to \kappa$ by $f(n) = |B_n|$, and then f would be order-preserving and unbounded, showing that $\mathrm{cf}(\kappa) \leq \aleph_0$, a contradiction. Thus $|\bigcup_n A_n| \leq \sup_n |B_n| \leq \sigma < \kappa$ as required. □

Notice that we can write $\aleph_\omega$ as the countable union of sets strictly smaller than $\aleph_\omega$, namely $\aleph_\omega = \bigcup_{n \in \omega} \aleph_n$.

Lemma 5.3.3. *Let κ be a cardinal with $\mathrm{cf}(\kappa) > \aleph_0$, and let E and F be Banach spaces. Then $T \in \mathcal{B}(E,F)$ is κ-compact if and only if there is a set $A \subseteq E$ with $|A| < \kappa$ and such that $T(A) := \{T(x) : x \in A\}$ is dense in $T(E)$.*

Proof. For $T \in \mathcal{K}_\kappa(E,F)$ and $n \in \mathbb{N}$, let $A_n \subset E_{[1]}$ be a set with $|A_n| < \kappa$ and

$$\inf\{\|T(x-y)\| : y \in A_n\} \leq n^{-1} \qquad (x \in E_{[1]}).$$

Then let $B = \bigcup_n A_n$, so, by Lemma 5.3.2, $|B| < \kappa$, and $T(B)$ is dense in $T(E_{[1]})$. Then we can let $A = \bigcup_{n\in\mathbb{N}} nB$, so that $|A| = |B|$ and $T(A)$ is dense in $T(E)$.

The converse statement is clear. □

Following Pietsch, we write $\mathfrak{X}(E,F)$ for the closed operator-ideal of $\mathcal{B}(E,F)$ formed by those operators with separable image. Thus $\mathfrak{X}(E,F) = \mathcal{K}_{\aleph_1}(E,F)$.

Example 5.3.4. Consider $l^1(\aleph_\omega)$, noting that $\mathrm{cf}(\aleph_\omega) = \aleph_0$. As $\aleph_\omega$ is an ordinal, we have $\aleph_\omega = \{\alpha \text{ is an ordinal } : \alpha < \aleph_\omega\}$ and thus, if $\alpha \in \aleph_\omega$, either α is finite, or $\aleph_{n-1} \leq |\alpha| < \aleph_n$ for some $n \geq 1$. Define $T \in \mathcal{B}(l^1(\aleph_\omega))$ by

$$T(e_\alpha) = \begin{cases} e_\alpha & : \alpha < \omega, \\ n^{-1}e_\alpha & : \aleph_{n-1} \leq |\alpha| < \aleph_n. \end{cases} \qquad (\alpha \in \aleph_\omega).$$

Then T is clearly $\aleph_\omega$-compact, but if A is a dense subset of $T(E_{[1]})$, then $|A| = \aleph_\omega$. Thus the preceding lemma does not hold more generally. □

Lemma 5.3.5. *Let E be a Banach space with density character κ. Then $\mathcal{B}(E) = \mathcal{K}_{\kappa^+}(E)$, and, if $\mathrm{cf}(\kappa) > \aleph_0$, then $\mathcal{K}_\kappa(E) \subsetneq \mathcal{B}(E)$.*

Proof. As E contains a dense subset of cardinality κ, clearly every operator on T is κ^+-compact. If, further, $\mathrm{cf}(\kappa) > \aleph_0$, then by Lemma 5.3.3, if Id_E is κ-compact, then for some $A \subseteq E$ with $|A| < \kappa$, we have that A is dense in E, a contradiction. Thus $\mathcal{K}_\kappa(E)$ is a proper ideal in $\mathcal{B}(E)$. □

Recall (Theorem 2.2.8) that for a Banach space E and $T \in \mathcal{B}(E)$, we have that $T \in \mathcal{K}(E)$ if and only if $T' \in \mathcal{K}(E')$. This does not generalise to κ-compact operators, as the identity on l^1 has separable range, but its adjoint is the identity on l^∞, which does not have separable range. The relation between T being κ-compact and T' being κ-compact is considered only for the Hilbert space case in [Luft, 1968].

Proposition 5.3.6. *Let E and F be Banach spaces, κ be an infinite cardinal, and $T \in \mathcal{B}(E,F)$. If $T' \in \mathcal{K}_\kappa(F',E')$ then $T \in \mathcal{K}_\kappa(E,F)$.*

Proof. We may suppose that $\kappa > \aleph_0$. Fix $\varepsilon > 0$. As $T' \in \mathcal{K}_\kappa(F',E')$, there exists $Y \subset F'_{[1]}$ with $|Y| < \kappa$ such that,

$$\inf\{\|T'(\mu-\lambda)\| : \lambda \in Y\} < \varepsilon \qquad (\mu \in F'_{[1]}).$$

For each $\lambda \in Y$, pick $x_\lambda \in E_{[1]}$ with $|\langle T'(\lambda), x_\lambda\rangle| > (1-\varepsilon)\|T'(\lambda)\|$. Let $\mathbb{Q}[\imath]$ be the subfield of $\mathbb{C}$ comprising of complex numbers with rational real and imaginary parts, so that $\mathbb{Q}[\imath]$ is dense in $\mathbb{C}$. Then let

$$X = \left\{ \sum_{i=1}^n a_i x_{\lambda_i} : n \in \mathbb{N}, (a_i)_{i=1}^n \subseteq \mathbb{Q}[\imath], (\lambda_i)_{i=1}^n \subseteq Y \right\}$$

so that X is dense in $\operatorname{lin}(x_\lambda)_{\lambda\in Y}$. By Lemma 5.2.4, $|X| = |Y| < \kappa$.

Suppose $y \in E_{[1]}$ is such that $\|T(x-y)\| \geq \delta$ for every $x \in X$. Then $\|T(x-y)\| \geq \delta$ for every $x \in \operatorname{lin}(x_\lambda)_{\lambda\in Y}$. By Hahn-Banach, we can find $\mu \in F'$ with $\langle \mu, T(x)\rangle = 0$ for each $x \in X$, and such that $\langle \mu, T(y)\rangle = \delta$ and $\|\mu\| \leq 1$. We can then find $\lambda \in Y$ with $\|T'(\mu - \lambda)\| < \varepsilon$. Then

$$(1-\varepsilon)\|T'(\lambda)\| \leq |\langle \lambda, T(x_\lambda)\rangle| = |\langle \lambda - \mu, T(x_\lambda)\rangle| = |\langle T'(\lambda-\mu), x_\lambda\rangle| < \varepsilon,$$

so that $\|T'(\lambda)\| < \varepsilon/(1-\varepsilon)$. Hence

$$\delta = |\langle \mu, T(y)\rangle| \leq \|T'(\mu)\| \leq \|T'(\mu-\lambda)\| + \|T'(\lambda)\| < \varepsilon + \varepsilon/(1-\varepsilon) < 3\varepsilon,$$

if ε is sufficiently small. Thus we have

$$\inf\{\|T(x-y)\| : x \in X\} \leq 3\varepsilon \qquad (y \in E_{[1]}),$$

so as $\varepsilon > 0$ was arbitrary, we are done. □

We now restrict ourselves to spaces with an unconditional basis.

Lemma 5.3.7. *Let E have an unconditional basis $(e_i)_{i\in I}$, and let κ be an infinite cardinal. If $A \subseteq I$ with $|A| = \kappa$, then $P_A(E)$ has density character κ, and $P_A \in \mathcal{K}_{\kappa^+}(E) \setminus \mathcal{K}_\kappa(E)$.*

Proof. Let

$$X = \left\{ \sum_{i=1}^n a_i e_{\alpha_i} : n \in \mathbb{N}, (a_i)_{i=1}^n \subseteq \mathbb{Q}[\imath], (\alpha_i)_{i=1}^n \subseteq A \right\},$$

so that, again, $|X| = \kappa$. If $x \in P_A(E)$, then $x = \sum_{i\in A} x_i e_i$ say, where, for $\varepsilon > 0$, we can find a finite $B \subseteq A$ with

$$\left\| x - \sum_{i\in B} x_i e_i \right\| < \varepsilon.$$

We can clearly approximate $\sum_{i\in B} x_i e_i$, to any required accuracy, by a member of X, and thus we see that X is dense in $P_A(E)$. Hence $P_A \in \mathcal{K}_{\kappa^+}(E)$ by Lemma 5.3.3, and $P_A(E)$ has density character κ.

If $\kappa = \aleph_0$ then $P_A \in \mathcal{K}_\kappa(E)$ means that P_A is compact, and thus that $P_A(E)$ is finite-dimensional, which in turn means that A is finite, a contradiction. Thus, if $P_A \in \mathcal{K}_\kappa(E)$, then $\kappa > \aleph_0$, and we can find a set $Y \subseteq P_A(E_{[1]})$ such that $|Y| < \kappa$ and such that

$$\inf\{\|P_A(x) - y\| : y \in Y\} \leq 1/2 \qquad (x \in E_{[1]}).$$

Then let $B = \bigcup_{y\in Y} \operatorname{supp}(y) \subseteq I$, so that $|B| \leq \aleph_0 \times |Y| < \kappa$. As $|B| < \kappa = |A|$, we can find $\alpha \in A \setminus B$. Then $e_\alpha \in P_A(E)$, and, for each $y \in Y$, $P_B(y) = y$, so that $P_{A\setminus B}(y) = 0$. Thus, for $y \in Y$, we have $1 = \|e_\alpha\| = \|P_{A\setminus B}(e_\alpha)\| = \|P_{A\setminus B}(e_\alpha - y)\| \leq \|e_\alpha - y\|$, a contradiction which shows that $P_A \notin \mathcal{K}_\kappa(E)$. □

Proposition 5.3.8. *Let E be a Banach space with an unconditional basis $(e_i)_{i\in I}$. For cardinals $\kappa, \sigma \leq |I|$, we have that $\mathcal{K}_\kappa(E) \neq \mathcal{K}_\sigma(E)$ if $\kappa \neq \sigma$. Furthermore, $\mathcal{K}_{|I|}(E) \neq \mathcal{B}(E)$.*

Proof. We may suppose that $\kappa < \sigma$, so that $\mathcal{K}_\kappa(E) \subseteq \mathcal{K}_\sigma(E)$. By Lemma 5.3.7, we can find $T \in \mathcal{K}_{\kappa^+}(E) \setminus \mathcal{K}_\kappa(E)$ (indeed, we can have $T = P_A$ for a suitable set $A \subseteq I$), as $\kappa \leq |I|$. Then, as $\kappa^+ \leq \sigma$, $T \in \mathcal{K}_\sigma(E)$ but $T \notin \mathcal{K}_\kappa(E)$.

By Lemma 5.3.7, applied with $A = I$, we see that Id_E is $|I|^+$-compact, but not $|I|$-compact, so that $\mathcal{K}_{|I|}(E) \neq \mathcal{B}(E)$. □

Note that this is an improvement on Lemma 5.3.5, in the case where our Banach space has an unconditional basis.

Thus, when E has an unconditional basis $(e_i)_{i\in I}$, we have a chain of closed ideals in $\mathcal{B}(E)$,

$$\{0\} \subsetneq \mathcal{K}(E) \subsetneq \mathcal{K}_{\aleph_1}(E) \subsetneq \cdots \subsetneq \mathcal{K}_{|I|}(E) \subsetneq \mathcal{K}_{|I|^+}(E) = \mathcal{B}(E).$$

Let E be a Banach space such that every closed ideal J in $\mathcal{B}(E)$ has the form $J = \mathcal{K}_\kappa(E)$ for some cardinal κ. Then we say that $\mathcal{B}(E)$ has *compact ideal structure*. If, further, when κ and σ are infinite cardinals less than or equal to the density character of E, we have $\mathcal{K}_\kappa(E) = \mathcal{K}_\sigma(E)$ only when $\kappa = \sigma$, and that $\mathcal{B}(E) = \mathcal{K}_{\tau^+}(E)$ where τ is the density character of E, then $\mathcal{B}(E)$ has *perfect compact ideal structure*.

Luft and Gramsch proved the following. Recall that the density character of an infinite-dimensional Hilbert space is the same as its Hilbert space dimension, that is, the cardinality of a complete orthonormal system.

Theorem 5.3.9. *Let H be an infinite-dimensional Hilbert space with density character κ. Then $\mathcal{B}(H)$ has perfect compact ideal structure. Thus, the closed ideals in $\mathcal{B}(H)$ form an* *ordered chain*

$$\{0\} \subsetneq \mathcal{K}(H) \subsetneq \mathcal{K}_{\aleph_1}(H) \subsetneq \cdots \subsetneq \mathcal{K}_\kappa(H) \subsetneq \mathcal{K}_{\kappa^+}(H) = \mathcal{B}(H). \qquad \square$$

We will show that this result holds for $l^p(I)$, $1 \leq p < \infty$ and $c_0(I)$, the $l^2(I)$ case being precisely the theorem above.

As a foot-hold towards this goal, Theorem 5.1.8 is enough to show the following.

Proposition 5.3.10. *Let I be an uncountable set, let $E = l^p(I)$, for $1 \leq p < \infty$, or $E = c_0(I)$, and let $T \in \mathcal{B}(E)$ have separable range, but not be compact. Then the ideal generated by T is $\mathfrak{X}(E) = \mathcal{K}_{\aleph_1}(E)$.*

Proof. Note that the ideal generated by T is certainly contained in $\mathfrak{X}(E)$, as T has separable range. Let $(x_n)_{n=1}^\infty$ be a dense sequence in $T(E)$, and let $A = \bigcup_n \operatorname{supp}(x_n) \subseteq I$, so that A is countable and $P_A T = T$. Since T is not compact, we can choose a sequence (y_n) in $E_{[1]}$ such that $(T(y_n))$ has no convergent subsequence. Let $B = \bigcup_n \operatorname{supp}(y_n)$, so that B is countable, and, as $P_B(y_n) = y_n$ for each n, $P_A T P_B$ cannot be compact.

We can view $P_A T P_B$ as an operator on $l^p(\mathbb{N})$ or $c_0(\mathbb{N})$, as appropriate. Thus, by Theorem 5.1.8, the ideal generated by T contains an isomorphism from $P_B(E)$ to $P_A(E)$ of the form

$$S(e_{\beta(n)}) = e_{\alpha(n)} \qquad (n \in \mathbb{N}),$$

where we have enumerations $A = \{\alpha(n) : n \in \mathbb{N}\}$ and $B = \{\beta(n) : n \in \mathbb{N}\}$. We thus see that, if $C \subseteq I$ is countable, then the ideal generated by T contains P_C.

Then, as above, for $R \in \mathfrak{X}(E)$, we can find a countable $C \subseteq I$ with $P_C R = R$, and thus R is in the ideal generated by T. $\square$

5.4 Closed ideal structure of $\mathcal{B}(l^p(I))$ and $\mathcal{B}(c_0(I))$

For the moment we can work with Banach spaces E which merely have an unconditional basis. Our aim is to classify the closed ideals of $\mathcal{B}(E)$, but first we need some lemmas, which can be seen as a generalisation of Proposition 3.4.9.

Lemma 5.4.1. *Let E have an unconditional basis $(e_i)_{i \in I}$. Let $\kappa \geq \aleph_0$ be a cardinal and $T \in \mathcal{K}_\kappa(E)$. Then we have:*

1. *if $\operatorname{cf}(\kappa) > \aleph_0$, then there exists $A \subseteq I$ with $|A| < \kappa$ and $P_A T = T$;*
2. *if $\operatorname{cf}(\kappa) = \aleph_0$, then, for each $\varepsilon > 0$, there exists $A \subseteq I$ with $|A| < \kappa$ and $\|P_A T - T\| < \varepsilon$.*

Proof. 1. We have $\kappa > \aleph_0$. By Lemma 5.3.3, we can find $X \subseteq E$ with $|X| < \kappa$ and with $T(X)$ dense in $T(E)$. Let $A = \bigcup_{x\in X} \operatorname{supp}(T(x))$, so that $|A| \leq |X| \times \aleph_0 < \kappa$ and $P_A(T(x)) = T(x)$ for each $x \in X$. Thus we see that $P_AT = T$, as required.

2. This result is clearly true for compact operators, so we may suppose that $\kappa > \aleph_0$. Let $\varepsilon > 0$. Then we can find $Y \subseteq E$ with $|Y| < \kappa$ and such that

$$\inf\{\|T(x) - y\| : y \in Y\} \leq \varepsilon/4 \qquad (x \in E_{[1]}).$$

Then let $A = \bigcup_{y\in Y} \operatorname{supp}(y)$, so that $|A| \leq \aleph_0 \times |Y| < \kappa$ as $\kappa > \aleph_0$. Then $P_A(y) = y$ for each $y \in Y$, so that, for $x \in E_{[1]}$ and $y \in Y$, we have

$$\begin{aligned}\|T(x) - y\| &= \|T(x) - P_A(y)\| = \|P_{I\setminus A}T(x) + P_A(T(x) - y)\| \\ &\geq \|P_{I\setminus A}T(x)\| - \|P_A(T(x) - y)\| \geq \|P_{I\setminus A}T(x)\| - \|T(x) - y\|.\end{aligned}$$

We can find $y \in Y$ with $\|T(x) - y\| \leq \varepsilon/3$ so that $\varepsilon/3 \geq \|P_{I\setminus A}T(x)\| - \varepsilon/3$, and thus $\|P_{I\setminus A}T(x)\| < \varepsilon$. As $x \in E_{[1]}$ was arbitrary, we see that $\|T - P_AT\| = \|P_{I\setminus A}T\| < \varepsilon$, as required. ☐

For an infinite cardinal κ, as $\mathcal{K}_\kappa(E)$ is a closed ideal in $\mathcal{B}(E)$, we can form the quotient $\mathcal{B}(E)/\mathcal{K}_\kappa(E)$, which is a Banach algebra for the norm

$$\|T + \mathcal{K}_\kappa(E)\| = \inf\{\|T + S\| : S \in \mathcal{K}_\kappa(E)\} \qquad (T \in \mathcal{B}(E)).$$

Lemma 5.4.2. *Let E have an unconditional basis $(e_i)_{i\in I}$. For $T \in \mathcal{B}(E)$, we have*

$$\|T + \mathcal{K}_\kappa(E)\| = \inf\{\|P_{I\setminus A}T\| : A \subseteq I, |A| < \kappa\}.$$

Further, if $\operatorname{cf}(\kappa) > \aleph_0$, then we can find $A \subseteq I$ with $|A| < \kappa$ and $\|T + \mathcal{K}_\kappa(E)\| = \|P_{I\setminus A}T\|$.

Proof. For any κ, if $A \subseteq I$ with $|A| < \kappa$, then $P_A \in \mathcal{K}_\kappa(E)$, so that

$$\|P_{I\setminus A}T\| = \|T - P_AT\| \geq \|T + \mathcal{K}_\kappa(E)\|.$$

Suppose we have $\varepsilon > 0$ and $S \in \mathcal{K}_\kappa(E)$ such that $\|T + S\| + \varepsilon \leq \|T - P_AT\|$ for each $A \subseteq I$ with $|A| < \kappa$. By Lemma 5.4.1, we can find $A \subseteq I$ with $|A| < \kappa$ and $\|S - P_AS\| < \varepsilon/2$. Then

$$\begin{aligned}\|P_{I\setminus A}T\| &= \|P_{I\setminus A}(T + P_AS)\| \leq \|T + P_AS\| \leq \|T + S\| + \|P_AS - S\| \\ &< \varepsilon/2 + \|P_{I\setminus A}T\| - \varepsilon,\end{aligned}$$

a contradiction. This completes the first part of the proof.

If $\mathrm{cf}(\kappa) > \aleph_0$, then, for each $n \in \mathbb{N}$, choose $A_n \subseteq I$ with $|A_n| < \kappa$ and

$$\|P_{I \setminus A_n} T\| < \|T + \mathcal{K}_\kappa(E)\| + n^{-1}.$$

Let $A = \bigcup_n A_n$, so that $|A| < \kappa$ and, for each $n \in \mathbb{N}$,

$$\|T + \mathcal{K}_\kappa(E)\| \leq \|P_{I \setminus A} T\| = \|P_{I \setminus A} P_{I \setminus A_n} T\| \leq \|P_{I \setminus A_n} T\| < \|T + \mathcal{K}_\kappa(E)\| + n^{-1}.$$

Thus we must have $\|T + \mathcal{K}_\kappa(E)\| = \|P_{I \setminus A} T\|$. □

We can now prove a converse to Proposition 5.3.6, at least when E has a shrinking, unconditional basis.

Proposition 5.4.3. *Let E have a shrinking, unconditional basis $(e_i)_{i \in I}$, let κ be an infinite cardinal, and let $T \in \mathcal{K}_\kappa(E)$. Then $T' \in \mathcal{K}_\kappa(E')$.*

Proof. As $(e_i^*)_{i \in I}$ is a basis for E', let $Q_A \in \mathcal{B}(E')$ be the analogue of $P_A \in \mathcal{B}(E)$. Then a quick calculation shows that $Q_A = P_A'$. Pick $\varepsilon > 0$, and let $T \in \mathcal{K}_\kappa(E)$. By Lemma 5.4.1, for some $A \subseteq I$ with $|A| < \kappa$, we have $\|T - P_A T\| < \varepsilon$. Thus $\|T' - T' Q_A\| = \|(T - P_A T)'\| < \varepsilon$, so as $Q_A \in \mathcal{K}_\kappa(E')$, we have $\|T' + \mathcal{K}_\kappa(E')\| < \varepsilon$. As $\varepsilon > 0$ was arbitrary, we conclude that $T' \in \mathcal{K}_\kappa(E')$, as required. □

For $T \in \mathcal{B}(E)$, let $\mathrm{ideal}(T)$ be the algebraic ideal generated by T in $\mathcal{B}(E)$, and $\overline{\mathrm{ideal}}(T)$ be its closure.

Lemma 5.4.4. *Let E be a Banach space with an unconditional basis $(e_i)_{i \in I}$. Suppose that for each cardinal $\kappa \geq \aleph_0$ and each $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$, we have $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$. Then for each closed ideal J in $\mathcal{B}(E)$, we have $J = \mathcal{K}_\kappa(T)$ for some cardinal κ.*

Proof. If $J \subseteq \mathcal{K}(E)$, then $J = \mathcal{K}(E)$. Thus we may suppose that $\mathcal{K}(E) = \mathcal{K}_{\aleph_0}(E) \subsetneq J$. Let

$$X = \{\sigma : J \setminus \mathcal{K}_\sigma(E) \neq \emptyset\}$$

so that, by our assumption, if $\sigma \in X$, then $\mathcal{K}_{\sigma^+}(E) \subseteq J$. Suppose that X contains a maximal element κ, so that $\kappa \geq \aleph_1$. Then, as $\kappa \in X$, we have $\mathcal{K}_{\kappa^+}(E) \subseteq J$, and as κ is maximal in X, we have $J \setminus \mathcal{K}_{\kappa^+}(E) = \emptyset$, so that $J \subseteq \mathcal{K}_{\kappa^+}(E)$. Thus $J = \mathcal{K}_{\kappa^+}(E)$ as required.

If X does not contain a maximum element, then, for some limit cardinal κ, $X = \{\sigma : \sigma < \kappa\}$, and so $\kappa \notin X$, meaning that $J \subseteq \mathcal{K}_\kappa(E)$. Choose $T \in \mathcal{K}_\kappa(E)$ and $\varepsilon > 0$.

Then, by Lemma 5.4.1, we can find $A \subseteq I$ with $|A| < \kappa$ and $\|P_A T - T\| < \varepsilon$. As $P_A \in \mathcal{K}_{|A|^+}(E)$, $P_A T \in J$. As $\varepsilon > 0$ was arbitrary and J is closed, $T \in J$. Thus $J = \mathcal{K}_\kappa(E)$. $\square$

We thus wish to show that, for each $T \in \mathcal{B}(E)$ and each cardinal κ, if T is not κ-compact, then $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$. At this point we have to restrict ourselves to considering $E = l^p(I)$, for $1 \leq p < \infty$, or $E = c_0(I)$. Then, by the structure of E, if $A, B \subseteq I$ with $|A| = |B|$, then $P_B \in \mathrm{ideal}(P_A)$.

Lemma 5.4.5. *Let $E = l^p(I)$, for $1 \leq p < \infty$, or $E = c_0(I)$, let $\kappa \geq \aleph_0$ be a cardinal, and let $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$. Then $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$ if and only if, for some $A \subseteq I$ with $|A| = \kappa$, $P_A \in \overline{\mathrm{ideal}}(T)$.*

Proof. If $P_A \in \overline{\mathrm{ideal}}(T)$ for some $A \subseteq I$ with $|A| = \kappa$, then $P_B \in \mathrm{ideal}(P_A)$ for every $B \subseteq I$ with $|B| \leq |A|$. For $S \in \mathcal{K}_{\kappa^+}(E)$, by Lemma 5.4.1, there exists $B \subseteq I$ with $|B| \leq \kappa$ and $P_B S = S$. Thus $S \in \mathrm{ideal}(P_B) \subseteq \mathrm{ideal}(P_A) \subseteq \overline{\mathrm{ideal}}(T)$, so we see that $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$.

Conversely, if $\mathcal{K}_{\kappa^+}(E) \subseteq \overline{\mathrm{ideal}}(T)$, then for $A \subseteq I$ with $|A| = \kappa$, we have $P_A \in \mathcal{K}_{\kappa^+}(E)$, so that $P_A \in \overline{\mathrm{ideal}}(T)$. $\square$

Proposition 5.4.6. *Let $E = l^p(I)$, for $1 \leq p < \infty$, or $E = c_0(I)$. Suppose that for each cardinal $\kappa \geq \aleph_0$ and each $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$, there exists $A \subseteq I$ with $|A| = \kappa$ and $P_A \in \overline{\mathrm{ideal}}(T)$. Then $\mathcal{B}(E)$ has perfect compact ideal structure.*

Proof. Use Proposition 5.3.8, and Lemma 5.4.5 applied with Lemma 5.4.4. $\square$

5.5 When E has a shrinking basis

For the moment, we shall suppose only that E has a shrinking basis $(e_\alpha)_{\alpha \in I}$.

Proposition 5.5.1. *Let E have a shrinking basis $(e_\alpha)_{\alpha \in I}$, let $\kappa > \aleph_0$ be a cardinal, and let $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$. Then we can find a family $(x_i)_{i \in \kappa}$ of vectors in E such that for some $\delta > 0$ we have:*

1. *for $i \in \kappa$, we have $\|x_i\| = 1$ and $\|T(x_i)\| \geq \delta$;*
2. *for each $i, j \in \kappa$ with $i \neq j$, we have $\operatorname{supp} T(x_i) \cap \operatorname{supp} T(x_j) = \operatorname{supp}(x_i) \cap \operatorname{supp}(x_j) = \emptyset$.*

Proof. As $T \notin \mathcal{K}_\kappa(E)$, let $2\delta = \|T + \mathcal{K}_\kappa(E)\| > 0$. For $A \subseteq I$ with $|A| < \kappa$, as $P_A \in \mathcal{K}_\kappa(E)$, we have that $2\delta = \|T + \mathcal{K}_\kappa(E)\| \leq \|T - TP_A\| = \|TP_{I\setminus A}\|$.

A simple Zorn's Lemma argument shows that we can find a maximal family of vectors X in E such that conditions (i) and (ii) hold.

If $|X| \geq \kappa$, then we are done. Suppose, towards a contradiction, that $|X| < \kappa$, so that if we set $A = \bigcup_{x\in X} \operatorname{supp}(x)$ and $B = \bigcup_{x\in X} \operatorname{supp} T(x)$, then $|A| \leq |X| \times \aleph_0 = \max(|X|, \aleph_0) < \kappa$ and, similarly, $|B| < \kappa$. As E has a shrinking basis, we may set

$$C = \bigcup_{i\in B} \operatorname{supp} T'(e_i^*),$$

so that, again, $|C| < \kappa$. For $y \in E$, we have that $B \cap \operatorname{supp} T(y) \neq \emptyset$ if and only if, for some $i \in B$, we have $0 \neq \langle e_i^*, T(y)\rangle = \langle T'(e_i^*), y\rangle$, which implies that $C \cap \operatorname{supp}(y) \neq \emptyset$. Thus, for each $y \in E$, we have $\operatorname{supp} TP_{I\setminus C}(y) \subseteq I \setminus B$. Finally, let $D = A \cup C$, so that $|D| < \kappa$, and if $y \in E$ with $P_{I\setminus D}(y) = y$, then $\operatorname{supp} T(y) \subseteq I \setminus B$, so that by the maximality of X, we must have $\|T(y)\| < \delta\|y\|$. This implies that $\|TP_{I\setminus D}\| \leq \delta$, which is a contradiction by our choice of δ. □

We can then certainly apply this proposition to $E = c_0(I)$ or $E = l^p(I)$, for $1 < p < \infty$.

Theorem 5.5.2. *If $E = c_0(I)$ or $E = l^p(I)$ for $1 < p < \infty$, then for a closed ideal J in $\mathcal{B}(E)$, we have $J = \mathcal{K}_\kappa(E)$ for some cardinal κ.*

Proof. We use Proposition 5.4.6, so let $\kappa \geq \aleph_0$ be a cardinal and $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$. If $\kappa = \aleph_0$, we need to show that, if T is not compact, then $P_A \in \overline{\operatorname{ideal}}(T)$ for some countable $A \subseteq I$. This follows directly from Proposition 5.3.10. Thus we may suppose that $\kappa > \aleph_0$. We can then apply Proposition 5.5.1 to find a family $(x_i)_{i\in\kappa}$ and $\delta > 0$ with properties as in the proposition.

As $(T(x_i))_{i\in\kappa}$ is a family of vectors with pairwise-disjoint support, we can find a family $(\mu_i)_{i\in\kappa} \subseteq E'$ with pairwise-disjoint support (recall that E has a shrinking basis) and such that $\langle \mu_i, T(x_j)\rangle = \delta_{ij}$, the Kronecker delta. As $\|T(x_i)\| \geq \delta$ for each $i \in \kappa$, we may suppose that $\|\mu_i\| \leq \delta^{-1}$ for each $i \in \kappa$. Let $K \subseteq I$ be some subset with $|K| = \kappa$, and let $\phi : K \to \kappa$ be a bijection. We can then define $Q, S \in \mathcal{B}(E)$ by

$$Q(x) = \sum_{j\in K} T(x_{\phi(j)})\langle \mu_{\phi(j)}, x\rangle \qquad S(x) = \sum_{j\in K} x_{\phi(j)}\langle \mu_{\phi(j)}, Q(x)\rangle \qquad (x \in E).$$

A calculation shows that, in all cases for E, $\|Q\| \leq \delta^{-1}\|T\|$ and $\|S\| \leq \delta^{-1}\|Q\| \leq \delta^{-2}\|T\|$. For $i \in \kappa$, we then have $Q(T(x_i)) = T(x_i)$, and so $ST(x_i) = x_i$.

Similarly, we can find a family $(\lambda_i)_{i\in K} \subseteq E'$ with pairwise-disjoint support and such that $\langle \lambda_i, x_{\phi(j)}\rangle = \delta_{ij}$, and $\|\lambda_i\| = 1$ for each $i \in K$. Then we may define $R, U \in \mathcal{B}(E)$ by

$$U\Big(\sum_{i\in I} a_i e_i\Big) = \sum_{i\in K} a_i x_{\phi(i)} \qquad R(x) = \sum_{j\in K} e_j \langle \lambda_j, x\rangle \qquad (x \in E),$$

and again a calculation yields that $\|R\| = 1$ and that U is an isometry onto its range. Then, for each $j \in K$, we have $R(x_{\phi(j)}) = e_j$, so that $RSTU(e_j) = RST(x_{\phi(j)}) = R(x_{\phi(j)}) = e_j$. Thus $RSTU = P_K$, and as $|K| = \kappa$, we are done. □

5.6 When $E = l^1(I)$

As $l^1(I)$ does not have a shrinking basis, the method of proof used in Proposition 5.5.1 does not adapt to the l^1 case. However, we shall see that the l^1-norm has properties meaning that we want to prove something different.

Lemma 5.6.1. *For an infinite set I, $T \in \mathcal{B}(l^1(I))$ and $A \subseteq I$, we have*

$$\|TP_A\| = \sup\{\|T(e_i)\| : i \in A\}.$$

Proof. Just note that, if $x = \sum_{i\in A} a_i e_i$, then

$$\|T(x)\| = \left\|\sum_{i\in A} a_i T(e_i)\right\| \leq \sum_{i\in A} |a_i| \|T(e_i)\| \leq \|x\| \sup\{\|T(e_i)\| : i \in A\}.$$

□

Proposition 5.6.2. *Let $E = l^1(I)$ for some index set I. Let $\kappa > \aleph_0$ be a cardinal, let $\varepsilon \in (0, 1/2)$, and let $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$ be such that*

$$1 \geq \|T\| \geq \|T + \mathcal{K}_\kappa(E)\| \geq 1 - \varepsilon.$$

Then there exists $K \subseteq I$ with $|K| \geq \kappa$, and a family $(A_i)_{i\in K}$ of subsets of I, such that:

1. *for $i \in K$, A_i is countable and $\|P_{A_i} T(e_i)\| \geq 1 - 2\varepsilon$;*
2. *for $i, j \in K$ with $i \neq j$, $A_i \cap A_j = \emptyset$.*

Proof. For $L \subseteq I$ and $B = (B_i)_{i\in L}$ a family of subsets of I, we say that (L, B) is *admissible* if conditions (1) and (2) are satisfied. Let X be the collection of admissible pairs; since $\|T\| \geq 1 - \varepsilon > 0$, the set X is not empty by Lemma 5.6.1. Partially order X be setting $(L, (B_i^L)_{i\in L}) \leq (J, (B_i^J)_{i\in J})$ if and only if $L \subseteq J$ and, for each $i \in L$, $B_i^J = B_i^L$.

Let $Y \subseteq X$ be a chain, and let $L_0 = \bigcup_{(L,B^L)\in Y} L \subseteq I$. Then, for $i \in L_0$, we have $i \in L$ for some $(L, B^L) \in Y$. Set $B_i = B_i^L$. This is well-defined, for if $i \in J$ for some $(J, B^J) \in Y$, then either $(L, B^L) \leq (J, B^J)$, so that $B_i^L = B_i^J$, or $(J, B^J) \leq (L, B^L)$ and $B_i^J = B_i^L$. Let $B = (B_i)_{i\in L_0}$, so, if $i \in L_0$, B_i is countable, and $\|P_{B_i}T(e_i)\| \geq 1 - 2\varepsilon$. Similarly, we can show that $(L_0, B) \in X$ and that (L_0, B) is an upper bound for Y. We can thus apply Zorn's Lemma to find a maximal admissible pair $(K, (A_i)_{i\in K})$.

If $|K| \geq \kappa$ then we are done. Otherwise, let $B = \bigcup_{i\in K} A_i$ so that $|B| \leq \aleph_0 \times |K| < \kappa$. As (K, A) is maximal, suppose that for some $i \in I \setminus K$ we have $\|P_{I\setminus B}T(e_i)\| \geq 1 - 2\varepsilon$. Then set $C = (I \setminus B) \cap \operatorname{supp} T(e_i)$, so that C is countable and $\|P_C T(e_i)\| \geq 1 - 2\varepsilon$. This contradicts the maximality of (K, A). Hence we see that

$$\|P_{I\setminus B}T(e_i)\| < 1 - 2\varepsilon \qquad (i \in I \setminus K).$$

By Lemma 5.6.1, we conclude that $\|P_{I\setminus B}TP_{I\setminus K}\| \leq 1 - 2\varepsilon$. By Lemma 5.3.7, P_B and P_K are κ-compact, so that

$$1 - 2\varepsilon \geq \|P_{I\setminus B}TP_{I\setminus K}\| = \|T - TP_K - P_B TP_{I\setminus K}\| \geq \|T + \mathcal{K}_\kappa(E)\| \geq 1 - \varepsilon.$$

This contradiction shows that $|K| \geq \kappa$, as required. ☐

Theorem 5.6.3. *Let $E = l^1(I)$, and let J be a closed ideal in $\mathcal{B}(E)$. Then $J = \mathcal{K}_\kappa(E)$ for some cardinal κ.*

Proof. We use Proposition 5.4.6, so let $\kappa \geq \aleph_0$ be a cardinal and $T \in \mathcal{B}(E) \setminus \mathcal{K}_\kappa(E)$. As in the proof of Theorem 5.5.2, we may suppose that $\kappa > \aleph_0$. Fix $\varepsilon > 0$. By Lemma 5.4.2, we can find $A \subseteq I$ with $|A| < \kappa$ and $\|P_{I\setminus A}T\| \geq \|T + \mathcal{K}_\kappa(E)\| \geq \|P_{I\setminus A}T\|(1 - \varepsilon/4)$. Then, since $P_A \in \mathcal{K}_\kappa(E)$, we have

$$\|P_{I\setminus A}T + \mathcal{K}_\kappa(E)\| = \|T + \mathcal{K}_\kappa(E)\| \geq (1 - \varepsilon/4)\|P_{I\setminus A}T\|.$$

Let $T_0 = P_{I\setminus A}T\|P_{I\setminus A}T\|^{-1}$, so that

$$1 = \|T_0\| \geq \|T_0 + \mathcal{K}_\kappa(E)\| = \|P_{I\setminus A}T + \mathcal{K}_\kappa(E)\|\|P_{I\setminus A}T\|^{-1} \geq 1 - \varepsilon/4.$$

Apply Proposition 5.6.2 to T_0 to find $K \subseteq I$ with $|K| = \kappa$, and a family $(A_k)_{k\in K}$ of subsets of I, such that:

1. for $k \in K$, A_k is countable, and $\|P_{A_k}T_0(e_k)\| \geq 1 - \varepsilon/2$;
2. for $j, k \in K$ with $j \neq k$, $A_j \cap A_k = \emptyset$.

For $k \in K$ let $v_k = P_{A_k}T_0(e_k)\|P_{A_k}T_0(e_k)\|^{-1}$, so that $\|v_k\| = 1$ and, recalling that $\|T_0\| = 1$, we also have

$$\begin{aligned}\|T_0(e_k) - v_k\| &= \Big\|P_{I\setminus A_k}T_0(e_k) + P_{A_k}T_0(e_k)\Big(1 - \|P_{A_k}T_0(e_k)\|^{-1}\Big)\Big\| \\ &= \|P_{I\setminus A_k}T_0(e_k)\| + \Big|\|P_{A_k}T_0(e_k)\| - 1\Big| \\ &= \|P_{I\setminus A_k}T_0(e_k)\| + 1 - \|P_{A_k}T_0(e_k)\| \\ &= \|T_0(e_k)\| + 1 - 2\|P_{A_k}T_0(e_k)\| \leq 1 + 1 - 2(1-\varepsilon/2) = \varepsilon.\end{aligned}$$

Let $F = \overline{\text{lin}}(v_k)_{k\in K}$, and define $U : F \to \overline{\text{lin}}(T_0(e_k))_{k\in K}$ by $U(v_k) = T_0(e_k)$. Then, for $x = \sum_{k\in K} a_k v_k$, we have, noting that the (v_k) have pairwise-disjoint support,

$$\|U(x)\| = \Big\|\sum_{k\in K} a_k T_0(e_k)\Big\| \leq \|T_0\|\Big\|\sum_{k\in K} a_k e_k\Big\| = \|T_0\|\|x\|,$$

so that U is bounded. As the (v_k) have pairwise disjoint support, we can find a projection $P : E \to F$ with $\|P\| = 1$. Then, with $x = \sum_{k\in K} a_k v_k \in F$, we have

$$\begin{aligned}\|x - PU(x)\| = \|P(x - U(x))\| \leq \|x - U(x)\| &= \Big\|\sum_{k\in K} a_k(v_k - T_0(e_k))\Big\| \\ &\leq \Big(\sup_{k\in K}\|v_k - T_0(e_k)\|\Big)\sum_{k\in K}|a_k| \leq \varepsilon\|x\|.\end{aligned}$$

Thus, if $\varepsilon < 1$, noting that $\text{Id}_F - PU \in \mathcal{B}(F)$, we have $\|\text{Id}_F - PU\| < 1$, so that PU is invertible in $\mathcal{B}(F)$.

Then we have, for $k \in K$, $PU(v_k) = PT_0(e_k)$, so that $v_k = (PU)^{-1}PT_0(e_k)$. Define $V : F \to P_K(E)$ by, for $k \in K$, $V(v_k) = e_k$, so that V is an isometry. Thus, letting $S = V(PU)^{-1}P$, we have $ST_0P_K = P_K$. Thus

$$P_K = ST_0P_K = \|P_{I\setminus A}T\|^{-1}SP_{I\setminus A}TP_K,$$

so that $P_K \in \text{ideal}(T)$, as required. □

This proof is the correct analogue of Theorem 5.5.2, for above we showed that

$$\sup_{k\in K}\|T(e_k) - v_k\| < 1,$$

whereas for the l^p and c_0 cases we would need to show that

$$\sum_{k\in K}\|T(e_k) - v_k\|^q < 1,$$

where $q^{-1} + p^{-1} = 1$ (or $q = 1$ in the c_0 case). However, if K is uncountable, then such a sum must contain all but countably many terms which are actually zero. As we are free to remove such terms (and still have K being of the same cardinality) we arrive at precisely the conclusions of Proposition 5.5.1 (as least if we replace (e_i) be a sequence of disjointly-supported unit vectors (x_i)).

To sum up, we have shown the following generalisation of the Gohberg, Markus and Feldman theorem.

Theorem 5.6.4. *Let I be an infinite set, and let $E = l^p(I)$ for $1 \leq p < \infty$, or $E = c_0(I)$. Then $\mathcal{B}(E)$ has perfect compact ideal structure. That is, the closed ideals in $\mathcal{B}(E)$ form an ordered chain*

$$\{0\} \subsetneq \mathcal{K}(E) \subsetneq \mathcal{K}_{\aleph_1}(E) \subsetneq \cdots \subsetneq \mathcal{K}_{|I|}(E) \subsetneq \mathcal{K}_{|I|^+}(E) = \mathcal{B}(E).$$

□

5.7 Generalisation

An immediate question is whether there are any other Banach spaces E such that $\mathcal{B}(E)$ has (perfect) compact ideal structure. However, even for non-separable spaces, we are hampered by our lack of knowledge in the separable case.

Proposition 5.7.1. *Let E be a Banach space such that $\mathcal{B}(E)$ has compact ideal structure. Suppose that F is a complemented subspace of E. Then $\mathcal{B}(F)$ has compact ideal structure.*

Proof. Let F be complemented in E with projection $P : E \to F$. Let J be a closed ideal in $\mathcal{B}(F)$, and define

$$J_0 = \overline{\text{lin}}\{RST : S \in J, T \in \mathcal{B}(E, F), R \in \mathcal{B}(F, E)\} \subseteq \mathcal{B}(E).$$

Clearly J_0 is a closed ideal in $\mathcal{B}(E)$, so that $J_0 = \mathcal{K}_\kappa(E)$ for some cardinal κ.

If $S \in J$, $T \in \mathcal{B}(E, F)$ and $R \in \mathcal{B}(F, E)$, then $PR \in \mathcal{B}(F)$ and $T|_F \in \mathcal{B}(F)$ so that $PRST|_F \in J$, as J is an ideal. Thus if $U \in J_0$ then $PU|_F \in J$. Let $\iota : F \to E$ be the inclusion map. Clearly, if $V \in J$, then $\iota VP \in J_0$.

We thus claim that $J = \mathcal{K}_\kappa(F)$, for if $V \in J$ then $\iota VP \in J_0$ so ιVP is κ-compact, and thus V is κ-compact. Conversely, if $W \in \mathcal{K}_\kappa(F)$ then ιWP is κ-compact, so that $\iota WP \in J_0$, and thus $P\iota WP|_F = W \in J$. □

Thus, in practical terms, if we exhibit a Banach space E with $\mathcal{B}(E)$ having compact ideal structure, we need separable complemented subspaces of E to be isomorphic to l^p or c_0. If we look at spaces with an unconditional basis, then such spaces have a plethora of separable complemented subspaces. Indeed, in some special cases, we can show that such spaces are trivial.

For Banach spaces E and F, the *Banach-Mazur distance* between E and F is

$$d(E,F) = \inf\{\|T\|\|T^{-1}\| : T : E \to F \text{ is an isomorphism}\}.$$

Proposition 5.7.2. *Let E be a Banach space with an unconditional basis $(e_i)_{i\in I}$ such that every subspace $P_A(E)$, for countably infinite $A \subseteq I$, is isomorphic to some l^p space $(1 \leq p < \infty)$, or to c_0. Then each separable, complemented subspace of E is isomorphic to a fixed l^p space, or c_0. Furthermore, if this fixed space is c_0, l^1 or l^2, then E is isomorphic to $c_0(I)$, $l^1(I)$ or $l^2(I)$, respectively.*

Proof. Throughout this proof, we shall write l^∞ for c_0. Then suppose that for countably infinite $A_i \subseteq I$, $P_{A_i}(E)$ is isomorphic to l^{p_i}, for $i = 1, 2$. Then let $A = A_1 \cup A_2$ so that $P_A(E)$ is isomorphic to l^p say. Then $P_{A_i}(E)$ is isomorphic to a complemented subspace of l^p, and thus must be isomorphic to l^p by [Lindenstrauss, Tzafriri, 1977, Theorem 2.a.3], as every complemented, infinite-dimensional subspace of l^p is isomorphic to l^p. Thus l^{p_i} is isomorphic to l^p, and so $p_i = p$, for $i = 1, 2$.

Now let $F \subseteq E$ be a complemented, separable subspace. We can then find a countable $A \subseteq I$ with $F \subseteq P_A(E)$, so that F is isomorphic to a complemented subspace of l^p, and thus isomorphic to l^p.

Now suppose that $p = 1, 2$ or ∞ (the c_0 case). Then, by [Lindenstrauss, Tzafriri, 1977, Theorem 2.b.10], we know that each such space has exactly one unconditional basis, up to equivalence. For each countably infinite $A \subseteq I$, let $T_A : P_A(E) \to l^p$ be an isomorphism, chosen such that $\|T_A\|\|T_A^{-1}\| \leq 2d(P_A(E), l^p)$, the Banach-Mazur distance. Then it is clear that, if we take an enumeration of A, $A = \{a_n^A : n \in \mathbb{N}\}$, then the sequence $(T_A(e_{a_n^A}))$ is an unconditional basis for l^p, and thus there exists $K_A \geq 1$ such that

$$K_A^{-1}\left(\sum_{n=1}^\infty |b_n|^p\right)^{1/p} \leq \left\|\sum_{n=1}^\infty b_n T_A(e_{a_n^A})\right\| \leq K_A\left(\sum_{n=1}^\infty |b_n|^p\right)^{1/p}$$

for each sequence of scalars (b_n). Then we have that, for a sequence of scalars (b_n),

$$K_A^{-1}\|T_A\|^{-1}\left(\sum_{n=1}^\infty |b_n|^p\right)^{1/p} \leq \left\|\sum_{n=1}^\infty b_n e_{a_n^A}\right\| \leq K_A\|T_A^{-1}\|\left(\sum_{n=1}^\infty |b_n|^p\right)^{1/p}.$$

Thus given an injection $f : \mathbb{N} \to I$, let $B_f \geq 1$ be the minimal constant such that

$$B_f^{-1}\left(\sum_{n=1}^\infty |b_n|^p\right)^{1/p} \leq \left\|\sum_{n=1}^\infty b_n e_{f(n)}\right\| \leq B_f\left(\sum_{n=1}^\infty |b_n|^p\right)^{1/p}$$

holds for all sequences of scalars (b_n). We claim that the family (B_f) is bounded. For if not, let $f_n : \mathbb{N} \to I$ be such that $B_{f_n} \geq n$, for each $n \in \mathbb{N}$. Then let $g : \mathbb{N} \to I$ be an injective function chosen such that $g(\mathbb{N}) = \bigcup_{n\in\mathbb{N}} f_n(\mathbb{N})$. Now pick $N \in \mathbb{N}$, and given a sequence of scalars (b_n), let (c_n) be a sequence of scalars such that

$$c_n = \begin{cases} b_m & : g(n) = f_N(m), \\ 0 & : \text{otherwise.} \end{cases}$$

Then, as F_N and g are injective, and the image of g contains that image of F_N, we see that

$$\left(\sum_{n=1}^{\infty} |b_n|^p\right)^{1/p} = \left(\sum_{n=1}^{\infty} |c_n|^p\right)^{1/p} \leq B_g \left\|\sum_{n=1}^{\infty} c_n e_{g(n)}\right\| = B_g \left\|\sum_{m=1}^{\infty} b_m e_{f_N(m)}\right\|,$$

and similarly

$$\left\|\sum_{m=1}^{\infty} b_m e_{f_N(m)}\right\| = \left\|\sum_{n=1}^{\infty} c_n e_{g(n)}\right\| \leq B_g \left(\sum_{n=1}^{\infty} |c_n|^p\right)^{1/p} = B_g \left(\sum_{n=1}^{\infty} |b_n|^p\right)^{1/p}.$$

As B_{f_N} is minimal, we must have $B_g \geq B_{f_N}$, which contradicts $(B_{f_N})_{N=1}^{\infty}$ being unbounded.

Hence let $M = \sup_f B_f < \infty$. Define $T : E \to l^p(I)$ by $T(e_i) = d_i$, $(d_i)_{i\in I}$ being the standard basis for $l^p(I)$. Then, if $x \in E$, we have $x = \sum_{n=1}^{\infty} a_n e_{f(n)}$ for some injection $f : \mathbb{N} \to I$ and some sequence of scalars (a_n), so that

$$\|T(x)\| = \left(\sum_{n=1}^{\infty} |a_n|^p\right)^{1/p} \leq M \left\|\sum_{n=1}^{\infty} a_n e_{f(n)}\right\| = M\|x\|.$$

Conversely, given $y = \sum_{i\in I} a_i d_i \in l^p(I)$, we can find an injection $f : \mathbb{N} \to I$ such that $f(\mathbb{N}) = \{i \in I : a_i \neq 0\}$. Define $(b_n)_{n=1}^{\infty}$ by

$$b_n = \begin{cases} a_i & : f(n) = i, \\ 0 & : \text{otherwise.} \end{cases}$$

Then we have $y = \sum_{n=1}^{\infty} b_n d_{f(n)} = \sum_{n=1}^{\infty} b_n T(e_{f(n)}) \in l^p(I)$. Let $z = \sum_{n=1}^{\infty} b_n e_{f(n)} \in E$, so that

$$\|z\| = \left\|\sum_{n=1}^{\infty} b_n e_{f(n)}\right\| \leq B_f \left(\sum_{n=1}^{\infty} |b_n|^p\right)^{1/p} \leq M\|y\|.$$

Thus we see that T has an inverse, so that E is isomorphic to $l^p(I)$, as required. □

We note that the case where $1 < p < \infty$, $p \neq 2$, seems to be a good deal harder.

If E is an arbitrary non-separable Banach space, suppose that we only know that every closed ideal J of $\mathcal{B}(E)$ with $\mathcal{K}_{\aleph_1}(E) \subseteq J$ has $J = \mathcal{K}_\kappa(E)$ for some κ. By examining the proof of Proposition 5.7.1 we see that, if F is separable, we gain no information on the ideal structure of $\mathcal{B}(F)$ because then $J_0 \subseteq \mathcal{K}_{\aleph_1}(E)$. So we could ask an easier question: namely, are there more Banach spaces E so that, beyond the operators with separable range, every closed ideal is an ideal of κ-compact operators? However, this is too easy, for consider

$$E = l^1(\mathbb{N}) \oplus l^2(I)$$

for some uncountable I. A moments thought shows that this rather simple example does satisfy our conditions.

We hence conclude that the interesting questions, with regards to compact ideal structure, lie in studying the ideal structure of $\mathcal{B}(E)$ for separable Banach spaces E.

Bibliography

[Arens, 1951] R. ARENS, 'The adjoint of a bilinear operation', *Proc. Amer. Math. Soc.* **2** (1951) 839–848.

[Arveson, 1976] W. ARVESON, *An invitation to C^*-algebras*, (Springer-Verlag, New York-Heidelberg, 1976).

[Barton, Yu, 1996] T. BARTON, XIN-TAI YU, 'A generalized principle of local reflexivity', *Quaestiones Math.* **19** (1996) 353–355.

[Behrends, 1991] E. BEHRENDS, 'On the principle of local reflexivity', *Studia Math.* **100** (1991) 109–128.

[Civin, Yood, 1961] P. CIVIN, B. YOOD, 'The second conjugate space of a Banach algebra as an algebra', *Pacific J. Math.* **11** (1961) 847–870.

[Cowling, Fendler, 1984] M. COWLING, G. FENDLER, 'On representations in Banach spaces', *Math. Ann.* **266** (1984) 307–315.

[Dales, 2000] H. G. DALES, *Banach algebras and automatic continuity*, (Clarendon Press, Oxford, 2000).

[Dales, Lau, 2004] H. G. DALES, A. T.-M. LAU, 'The second dual of Beurling algebras', preprint.

[Davis et al., 1974] W. J. DAVIS, T. FIGIEL, W. B. JOHNSON, A. PEŁCZYŃSKI, 'Factoring weakly compact operators', *J. Functional Analysis* **17** (1974) 311–327.

[Daws, 2004] M. DAWS, 'Arens regularity of the algebra of operators on a Banach space', *Bull. London Math. Soc.* **36** (2004) 493–503.

[Daws, Read, 2004] M. DAWS, C. J. READ, 'Semisimplicity of $\mathcal{B}(E)''$', to appear in *J. Functional Analysis*.

[Daws(2), 2004] M. DAWS, 'Closed ideals in the Banach algebra of operators on classical non-separable spaces', preprint.

[Day, 1941] M. M. DAY, 'Some more uniformly convex spaces', *Bull. Amer. Math. Soc.* **47** (1941) 504–507.

[Defant, Floret, 1993] A. DEFANT, K. FLORET, *Tensor norms and operator ideals*, (North-Holland Publishing Co., Amsterdam, 1993).

[Diestel, 1984] J. DIESTEL, *Sequences and series and Banach spaces*, (Springer-Verlag, New York, 1984).

[Diestel, Uhl, 1977] J. DIESTEL, J. J. UHL, *Vector measures*, (American Mathematical Society, Providence, R.I., 1977).

[Duncan, Hosseiniun, 1979] J. DUNCAN, S. A. R. HOSSEINIUN, 'The second dual of a Banach algebra', *Proc. Roy. Soc. Edinburgh Sect. A* **84** (1979) 309–325.

[Enflo, 1973] P. ENFLO, 'A counterexample to the approximation problem in Banach spaces', *Acta Math.* **130** (1973) 309–317.

[Enflo et al.,1975] P. ENFLO, J. LINDENSTRAUSS, G. PISIER, 'On the "three space problem"', *Math. Scand.* **36** (1975) 199–210.

[Feder, Saphar, 1975] M. FEDER, P. SAPHAR, 'Spaces of compact operators and their dual spaces', *Israel J. Math.* **21** (1975) 38–49.

[Figiel, 1972] T. FIGIEL, 'An example of infinite dimensional reflexive Banach space non-isomorphic to its Cartesian square', *Studia Math.* **42** (1972) 295–306.

[Figiel, Johnson, 1973] T. FIGIEL, W. B. JOHNSON, 'The approximation property does not imply the bounded approximation property', *Proc. Amer. Math. Soc.* **41** (1973) 197–200.

[Figiel et al., 1977] T. FIGIEL, J. LINDENSTRAUSS, V. MILMAN, 'The dimension of almost spherical sections of convex sets', *Acta Math.* **139** (1977) 53–94.

[Godefroy, Iochum, 1988] G. GODEFROY, B. IOCHUM, 'Arens-regularity of Banach algebras and the geometry of Banach spaces', *J. Funct. Anal.* **80** (1988) 47–59.

[Gohberg et al., 1967] I. C. GOHBERG, A. S. MARKUS, I. A. FELDMAN, 'Normally solvable operators and ideals associated with them', *American Math. Soc. Translat.* **61** (1967) 63–84.

[Gramsch, 1967] B. GRAMSCH, ‘Eine Idealstruktur Banachscher Operatoralgebren’, *J. Reine Angew. Math.* **225** (1967) 97–115.

[Grønbæk, Willis, 1993] N. GRØNBÆK, G. A. WILLIS, ‘Approximate identities in Banach algebras of compact operators’, *Canad. Math. Bull.* **36** (1993) 45–53.

[Grosser, 1984] M. GROSSER, ‘Arens semiregular Banach algebras’, *Monatsh. Math.* **98** (1984) 41–52.

[Grosser, 1987] M. GROSSER, ‘Arens semiregularity of the algebra of compact operators’, *Illinois J. Math.* **31** (1987) 544–573.

[Grosser, 1989] M. GROSSER, ‘The trace of certain commutators’, *Rev. Roumaine Math. Pures Appl.* **34** (1989) 413–418.

[Grothendieck, 1953] A. GROTHENDIECK, ‘Résumé de la théorie métrique des produits tensoriels topologiques’, *Bol. Soc. Mat. São Paulo* **8** (1953) 1–79.

[Guerre-Delabriére, 1992] S. GUERRE-DELABRIÉRE, *Classical sequences in Banach spaces*, (Marcel Dekker Inc., New York, 1992).

[Habala et al., 1996] P. HABALA, P. HÁJEK, V. ZIZLER, *Introduction to Banach spaces vol. 2*, (Matfyzpress, 1996).

[Halmos, 1960] P. R. HALMOS, *Naive set theory*, (D. Van Nostrand Company, Princeton, 1960).

[Haydon et al., 1991] R. HAYDON, M. LEVY, Y. RAYNAUD, *Randomly normed spaces*, (Hermann, Paris, 1991).

[Heinrich, 1980] S. HEINRICH, ‘Ultraproducts in Banach space theory’, *J. Reine Angew. Math.* **313** (1980) 72–104.

[Herman, 1968] R.H. HERMAN, ‘On the uniqueness of the ideals of compact and strictly-singular operators’, *Studia Math.* **29** (1968) 161–165.

[Hrbacek, Jech, 1999] K. HRBACEK, T. JECH, *Introduction to set theory, 3rd ed.*, (Marcel Dekker, Inc. New York, 1999).

[Iochum, Loupias, 1989] B. IOCHUM, G. LOUPIAS, ‘Arens regularity and local reflexivity principle for Banach algebras’, *Math. Ann.* **284** (1989) 23–40.

[James, 1951] R. C. JAMES, 'A non-reflexive Banach space isometric with its second conjugate space', *Proc. Nat. Acad. Sci.* **37** (1951) 174–177.

[James, 1972] R. C. JAMES, 'Super-reflexive Banach spaces', *Canad. J. Math.* **24** (1972) 896–904.

[James(2), 1972] R. C. JAMES, 'Reflexivity and the sup of linear functionals', *Israel J. Math.* **13** (1972) 289–300.

[James, 1974] R. C. JAMES, 'A separable somewhat reflexive Banach space with nonseparable dual', *Bull. Amer. Math. Soc.* **80** (1974) 738–743.

[Kadec, Pełczyński, 1962] M. I. KADEC, A. PEŁCZYŃSKI, 'Bases, lacunary sequences and complemented subspaces in the spaces L_p', *Studia Math.* **21** (1962) 161–176.

[Kaijser, 1981] S. KAIJSER, 'On Banach modules. I.', *Math. Proc. Cambridge Philos. Soc.* **90** (1981) 423–444.

[Lau, Ülger, 1996] A. T.-M. LAU, A. ÜLGER, 'Topological centers of certain dual algebras', *Trans. Amer. Math. Soc.* **348** (1996) 1191–1212.

[Laustsen, 2002] N. J. LAUSTSEN, 'Maximal ideals in the algebra of operators on certain Banach spaces', *Proc. Edinb. Math. Soc.* **45** (2002) 523–546.

[Laustsen, Loy, 2003] N. J. LAUSTSEN, R. J. LOY, 'Closed ideals in the Banach algebra of operators on a Banach space', to appear in *proceedings of the conference on topological algebras, their applications, and related results (ed. K. Jarosz), held in Bedlewo, Poland, May 2003*.

[Laustsen et al., 2004] N. J. LAUSTSEN, R. J. LOY, C. J. READ, 'The lattice of closed ideals in the Banach algebra of operators on certain Banach spaces', *J. Funct. Anal.* **214** (2004) 106–131.

[Lindenstrauss, Tzafriri, 1977] J. LINDENSTRAUSS, L. TZAFRIRI, *Classical Banach spaces I*, (Springer-Verlag, Berlin, 1977).

[Lindenstrauss, Tzafriri, 1979] J. LINDENSTRAUSS, L. TZAFRIRI, *Classical Banach spaces II*, (Springer-Verlag, Berlin, 1979).

[Luft, 1968] E. LUFT, 'The two-sided closed ideals of the algebra of bounded linear operators of a Hilbert space', *Czechoslovak Math. J.* **18** (1968) 595–605.

[Megginson, 1998] R. MEGGINSON, *An introduction to Banach space theory*, (Springer-Verlag, New York, 1998).

[Palmer, 1985] T. W. PALMER, 'The bidual of the compact operators', *Trans. Amer. Math. Soc.* **288** (1985) 827–839.

[Palmer, 1994] T. W. PALMER, *Banach algebras and the general theory of *-algebras, Vol 1*, (Cambridge University Press, Cambridge, 1994).

[Pietsch, 1980] A. PIETSCH, *Operator ideals*, (North-Holland, Amsterdam, 1980).

[Pisier, 1983] G. PISIER, 'Counterexamples to a conjecture of Grothendieck', *Acta Math.* **151** (1983) 181–208.

[Robinson, 1966] A. ROBINSON, *Non-standard analysis (reprint)*, (Princeton Landmarks in Mathematics, Princeton University Press, Princeton, 1996).

[Runde, 2002] V. RUNDE, *Lectures on Amenability*, (Springer-Verlag, Berlin, 2002).

[Ryan, 2002] R. A. RYAN, *Introduction to tensor products of Banach spaces*, (Springer-Verlag, London, 2002).

[Schatten, 1950] R. SCHATTEN, *A Theory of Cross-Spaces*, (Annals of Mathematics Studies, no. 26, Princeton University Press, Princeton, 1950).

[Schlumprecht, 2003] TH. SCHLUMPRECHT, 'How many operators exist on a Banach space?', *Trends in Banach spaces and operator theory*, 295–333, Contemp. Math., 321, Amer. Math. Soc., Providence, 2003.

[Singer, 1981] I. SINGER, *Bases in Banach Spaces II*, (Spriner-Verlag, Berlin, 1981).

[Szankowski, 1978] A. SZANKOWSKI, 'Subspaces without the approximation property', *Israel J. Math.* **30** (1978) 123–129.

[Szankowski, 1981] A. SZANKOWSKI, '$B(\mathcal{H})$ does not have the approximation property', *Acta Math.* **147** (1981) 89–108.

[Willis, 1992] G. A. WILLIS, 'The compact approximation property does not imply the approximation property', *Studia Math.* **103** (1992) 99–108.

[Young, 1976] N. J. YOUNG, 'Periodicity of functionals and representations of normed algebras on reflexive spaces.', *Proc. Edinburgh Math. Soc. (2)* **20** (1976/77) 99–120.

[Zippin, 1966] M. ZIPPIN, 'On perfectly homogeneous bases in Banach spaces', *Israel J. Math.* **4** (1966) 265–272.

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