Around Compact Quantum Groups

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What is a compact group?

Well, it's a compact topological space G with the structure of a group such that the group action is jointly continuous, and the inverse is continuous.

It's a unital commutative C*-algebra A with a unital $*$-homomorphism $\Delta : A \to A \otimes_{\min} A$ which is:

- Co-associative, $(\mathrm{id} \otimes \Delta)\Delta = (\Delta \otimes \mathrm{id})\Delta$
- "Cancellative", that is, the sets

$$\{(a \otimes 1)\Delta(b) : a, b \in A\}, \quad \{(1 \otimes a)\Delta(b) : a, b \in A\},$$

  have dense linear span in $A \otimes_{\min} A$.

Equivalence, easy direction

If G is a compact group, set

$$A = C(G) = \{\text{continuous functions } G \to \mathbb{C}\},$$

identify $A \otimes_{\min} A = C(G \times G)$, define

$$\Delta(f) \in C(G \times G), \quad \Delta(f) : (s,t) \mapsto f(st) \qquad (f \in C(G), s,t \in G).$$

Finally observe that

$$(a \otimes 1)\Delta(b) : (s,t) \mapsto a(s)b(st),$$

will separate the points of $G \times G$ (by varying a and b) so by Stone-Weierstrass,

$$\mathrm{lin}\{(a \otimes 1)\Delta(b) : a, b \in A\}$$

is a dense subalgebra of $C(G \times G)$.

Equivalence, hard direction

Gelfand-Naimark tells us that a unital commutative C*-algebra A has the form $C(X)$ for some compact space X. So again $A \otimes_{\min} A = C(X \times X)$. Then $\Delta : C(X) \to C(X \times X)$ a unital $*$-homomorphism induces a continuous map $\theta : X \times X \to X$ such that

$$f(\theta(s,t)) = \Delta(f)(s,t) \qquad (s,t \in X, f \in C(X)).$$

The category of unital commutative C*-algebras and unital $*$-homomorphisms is dual to the category of compact spaces and continuous maps.

Δ co-associative implies that θ is associative, so X is a compact semigroup.
The cancellation rules for Δ imply that X is cancellative, that is

$$st = rt \implies s = r, \quad ts = tr \implies s = r.$$

Exercise: A compact semigroup with cancellation is a compact group.

Compact quantum groups

Simply remove the word “commutative”!
For example, let Γ be a discrete group, and let Γ act on $\ell^2(\Gamma)$ by left translation:

$$\lambda(s)f : t \mapsto f(s^{-1}t) \qquad (s, t \in \Gamma, f \in \ell^2(\Gamma)).$$

Let $C_r^*(\Gamma)$ be the (reduced) group C*-algebra: that is, the norm closed algebra, acting on $\ell^2(\Gamma)$, generated by $\lambda(\Gamma)$. So $C_r^*(\Gamma)$ is commutative if and only if Γ is.
There is a $*$-homomorphism

$$\Delta : C_r^*(\Gamma) \to C_r^*(\Gamma) \otimes_{\min} C_r^*(\Gamma) = C_r^*(\Gamma \times \Gamma),$$
$$\Delta : \lambda(s) \mapsto \lambda(s) \otimes \lambda(s) = \lambda(s, s) \qquad (s \in \Gamma).$$

Compact or Discrete?

Hang on: we're saying that for *discrete* Γ, we have that $C_r^*(\Gamma)$ is a *compact* quantum group?
If Γ were abelian, then the fourier transform tells us that

$$C_r^*(\Gamma) \cong C(\hat{\Gamma}),$$

where $\hat{\Gamma}$ is the Pontryagin dual of Γ. As Γ is discrete, $\hat{\Gamma}$ is compact.
As $C(G)$ is our "commutative" base algebra, this weird terminology is forced upon us.

Twisted $SU(2)$

From Woronowicz in the C*-setting, but independently discovered by Soibelman and Vaksman

$C(SU(2))$ is the commutative C*-algebra generated by a, b with

$$a^*a + b^*b = 1.$$
$$aa^* + bb^* = 1, \quad b^*b = bb^*, \quad ab = ba, \quad ab^* = b^*a.$$

We introduce a real parameter $\mu \in [-1, 1] \setminus \{0\}$, and let $C(SU_\mu(2))$ be the (non-commutative) C*-algebra generated by a, b with

$$a^*a + b^*b = 1, \quad aa^* + \mu^2 bb^* = 1,$$
$$b^*b = bb^*, \quad ab = \mu ba, \quad ab^* = \mu b^*a.$$

There exists a coproduct Δ with

$$\Delta(a) = a \otimes a - \mu b^* \otimes b, \quad \Delta(b) = b \otimes a + a^* \otimes b.$$

Why?

(Topological) Quantum groups grew out of:

- How do we extend the notion of the Fourier transform, or more specifically, the Pontryagin Duality, to non-abelian groups? Ideally, we'd like a self-dual category into which all proper groups fit (so Tannaka-Krein duality doesn't quite fit the bill). This lead to *Kac algebras*.
- But $SU_\mu(2)$ does not fit into this framework! Indeed, we have rather few examples of Kac algebras.

We now have a simple set of axioms for objects which are called "locally compact quantum groups", and which encompass all known examples.

Haar state

A compact group admits a unique Haar measure: a probability measure which is invariant under the group action. In our language, this corresponds to a *state* φ on $C(G)$ with

$$(\varphi \otimes \mathrm{id})\Delta(a) = (\mathrm{id} \otimes \varphi)\Delta(a) = \varphi(a) \qquad (a \in C(G)).$$

Woronowicz and van Daele showed that such a state always exists on a compact quantum group (A, Δ)
Applying the GNS construction gives a Hilbert space H and a $*$-representation of A on H. This is the analogue of $C(G)$ acting on $L^2(G)$ by pointwise multiplication.

Corepresentation theory

A (finite-dimensional) corepresentation of (A, Δ) is a matrix $u \in \mathbb{M}_n(A)$ with

$$\Delta(u_{ij}) = \sum_{k=1}^{n} u_{ik} \otimes u_{kj}.$$

- All irreducible corepresentations of (A, Δ) are finite-dimensional.
- Using the Haar state, it's possible to show that any finite-dimensional corepresentation u is *equivalent* to a unitary corepresentation: $u^* u = uu^* = I_n$.
- There is an (infinite-dimensional) corepresentation of (A, Δ) on H; all finite-dimensional corepresentations are sub-corepresentations of this. So we have a generalised Peter-Weil theory.
- There is a character theory.

Links with algebra

Let $\mathcal{A}$ be the collection of matrix entries of all irreducible corepresentations of (A, Δ).

- Then $\mathcal{A}$ is a $*$-algebra.
- $\mathcal{A}$ is norm-dense in A.
- Δ restricts to give a map $\mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ (algebraic tensor product).
- We can turn $(\mathcal{A}, \Delta)$ into a Hopf $*$-algebra: there exists an antipode and counit.
- But, in general, these are *unbounded*, and so don't make sense on A.
- $(\mathcal{A}, \Delta)$ is unique.

More algebra

We can go in reverse! Dijkhuizen and Koornwinder showed that if $(\mathcal{A}, \Delta)$ is a Hopf $*$-algebra which is spanned by the matrix entries of its finite-dimensional unitary corepresentations, then there is a compact quantum group (A, Δ) such that $\mathcal{A}$ is given by A.

Drinfeld's approach to quantum groups starts with a Lie group G, the Lie algebra $\mathfrak{g}$ and the enveloping algebra $U(\mathfrak{g})$ which is naturally a Hopf algebra. It is this Hopf algebra which is deformed to get a "quantum group". If we start with $SU(2)$, and we now take the $*$-representations of this deformed enveloping algebra, we naturally get a Hopf $*$-algebra which is isomorphic to the Hopf $*$-algebra associated to $C(SU_\mu(G))$.

So we are really studying the "dual" world to what is often understood by the term "quantum group".

Back to analysis

- For a compact group G, we start with $C(G)$, and find a Haar measure to form $L^2(G)$.
- Then $C(G)$ is a C*-algebra acting on $C(G)$.
- Consider the strong operator topology closure: this gives us $L^\infty(G)$, a von Neumann algebra.
- Δ extends to $L^\infty(G)$: it has the same formula.
- $L^\infty(G)$ has a predual: $L^1(G)$.
- Then Δ induces a Banach algebra product on $L^1(G)$, which is the usual convolution product.

Interlude: KMS condition

- The Haar state on $C(G)$ or $C_r^*(\Gamma)$ is a *trace*:

$$\varphi(ab) = \varphi(ba) \qquad (a, b \in A).$$

- For a compact quantum group, this is true if and only if we really have a Kac algebra.
- So not so for $C(SU_\mu(2))$ say.
- But φ is KMS.
- Loosely speaking, this means that there is an automorphism σ of $\mathcal{A}$ such that

$$\varphi(ab) = \varphi(b\sigma(a)) \qquad (a, b \in \mathcal{A}).$$

For compact quantum groups

Can do the same thing:

$$\begin{aligned}(A, \Delta) &\implies \text{Haar state } \varphi \implies \text{Hilbert space } H \\ &\implies \text{von Neumann algebra } M \\ &\implies \text{predual } M_* \text{ has structure of a Banach algebra.}\end{aligned}$$

If we start with discrete Γ then we get $A(\Gamma)$, the *Fourier algebra*: this encodes information about positive definite functions on Γ, and so forth.
If we start with $C(SU_\mu(2))$, we get: Who knows?

Projectivity

If $\mathfrak{A}$ is a Banach algebra, then in the category of left $\mathfrak{A}$-modules, P is *projective* if:

$$\begin{tikzcd} & X \arrow[d, two heads] \\ P \arrow[ur, dotted, "?"] \arrow[r] & Y \end{tikzcd}$$

Complication: it might be impossible to solve this diagram problem for purely toplogical reasons. So we insist that the map $X \to Y$ is *admissible*, that is, there is a bounded linear (but not necessarily $\mathfrak{A}$-linear) right inverse.

Then $\mathfrak{A}$ is (left) projective if, as a module over itself, it is projective.

For group algebras

- For a locally compact group G, we can still form $L^1(G)$ with convolution.
- Then $L^1(G)$ is projective if and only if G is compact.

- Similarly, the Fourier algebra $A(\Gamma)$ can be defined in general.
- If we work in the right category, then $A(\Gamma)$ is projective if and only if Γ is discrete.

The "right category" is the category of *completely bounded maps*.

For compact quantum groups

Let M_* be the predual convolution algebra associated to a (locally) compact quantum group (A, Δ). Again, we work with completely bounded maps.

- If M_* is projective, then A is compact.
- The converse is tricky!
- Abstract nonsense implies that M_* is projective if and only if

$$\Delta_* : M_* \widehat{\otimes} M_* \to M_*$$

  the product map ($\widehat{\otimes}$ is the topological tensor product which linearises bilinear, completely bounded maps) has a completely bounded right inverse, which is an M_* module map.
- (Daws) If this inverse map is contractive, then A is a Kac algebra.
- The proof ends up showing that the modular automorphism of φ must be trivial.