Multi-normed spaces and multi-Banach algebras

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Motivating problem

Let G be a locally compact group, with group algebra $L^1(G)$.

Theorem - B. E. Johnson, 1972 The Banach algebra $L^1(G)$ is amenable if and only if the group G is amenable. $\square$

Theorem - Helemski, Johnson Let A be an amenable Banach algebra. Then E' is injective for each Banach right A-module. $\square$

We do not know if the converse holds. If A is a Banach algebra such that E' is injective for each, or some, Banach right A-module.

The Banach space $L^p(G)$ is a Banach left $L^1(G)$-module in a canonical way.

Some results

Theorem Suppose that G is an amenable locally compact group. Then $L^p(G)$ is an injective Banach left $L^1(G)$-module for each $p \in (1, \infty)$. $\square$

We ask if the converse to this holds.

For partial results, see a paper of D and Polyakov in Proc. London Math. Soc. Attempts on this question led to a theory of multi-norms, which may have a life of its own. See a proto-memoir of 140 pages, and some Bangalore notes.

For a solution in the case where A is $L^1(G)$ and the module is any $L^p(G)$, and more, see the second conference. (Work of Matt Daws, Hung Le Pham, and Paul Ramsden.) Here we give some background on multi-norms; the connections with group algebras and new characterizations of amenability for locally compact groups will come in a talk at the second conference.

A second motivating problem

Again let G be a locally compact group, with measure algebra $(M(G), \star)$. For $\mu \in M(G)$, set

$$T_\mu(f) = \mu \star f \quad (f \in L^p(G))\,.$$

(Here $1 < p < \infty$; usually, $p = 2$, so that $L^2(G)$ is a Hilbert space.)

Then $T_\mu \in \mathcal{B}(L^p(G))$, and the map $\mu \mapsto T_\mu$ is a representation of $M(G)$.

Always $\sigma(T_\mu) \subset \sigma(\mu)$, but maybe $\sigma(T_\mu) \subsetneq \sigma(\mu)$, which can be unfortunate - how can we cure this?

Basic definitions

Let $(E, \|\cdot\|)$ be a normed space.

Definition A **multi-norm** on $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is a norm on E^n, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following hold for all $n \in \mathbb{N}$ and all $x_1, \ldots, x_n \in E$:

(A1) $\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\right\|_n = \|(x_1, \ldots, x_n)\|_n$ for each permutation σ of $\{1, \ldots, n\}$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n$

$$\leq (\max_{i \in \mathbb{N}_n} |\alpha_i|) \|(x_1, \ldots, x_n)\|_n$$

for each $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$;

(A3) $\|(x_1, \ldots, x_n, 0)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$;

(A4) $\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$.

Another representation of multi-norms

Let $\mathbb{M}_{m,n}$ be the algebra of $m \times n$-matrices over $\mathbb{C}$, and give it a norm by identifying it with $\mathcal{B}(\ell_n^\infty, \ell_m^\infty)$.

Let E be a normed space. Then $\mathbb{M}_{m,n}$ acts from E^n to E^m in the obvious way.

Consider a sequence $(\|\cdot\|_n)$ such that each $\|\cdot\|_n$ is a norm on E^n and such that $\|x\|_1 = \|x\|$ for each $x \in E$.

Theorem This sequence of norms is a multi-norm if and only if

$$\|a \cdot x\|_m \leq \|a\| \, \|x\|_n$$

for all $m, n \in \mathbb{N}$, $a \in \mathbb{M}_{m,n}$, and $x \in E^n$. $\square$

Elementary consequences

The following hold for all $x_1, \ldots, x_{n+1} \in E$, etc:

1) $\|(x_1, \ldots, x_n)\|_n \le \left\|(x_1, \ldots, x_n, x_{n+1})\right\|_{n+1}$;

2) $\max \|x_i\| \le \|(x_1, \ldots, x_n)\|_n \le \sum_{i=1}^n \|x_i\|$;

3)

$$\begin{aligned} \|(x_1, \ldots, x_n, y_1, \ldots, x_m)\|_{m+n} \quad &\le \\ &\|(x_1, \ldots, x_n)\|_n + \|(y_1, \ldots, y_m)\|_m \,. \end{aligned}$$

Minimum and maximum multi-norms

Example 1 Set $\|(x_1, \ldots, x_n)\|_n = \max \|x_i\|$. This gives the **minimum** multi-norm.

Example 2 It follows from 2) that there is also a **maximum** multi-norm, say it is $(\|\cdot\|_n^{\max})$.

Note that it is **not** true that $\sum_{i=1}^n \|x_i\|$ gives the maximum multi-norm — because it is not a multi-norm. (It does fit into a more general scenario.)

Another characterization

Let $(E, \|\cdot\|)$ be a normed space. Then a c_0-**norm** on $c_0 \otimes E$ is a norm $\|\cdot\|$ such that $\|a \otimes x\| \leq \|a\| \, \|x\|$ for all $a \in c_0$ and $x \in E$ and such that $T \otimes I_E$ is a bounded linear operator on $(c_0 \otimes E, \|\cdot\|)$ with $\|T \otimes I_E\| = \|T\|$ whenever T is a compact operator on c_0.

Theorem (Daws) Multi-norms on $\{E^n : n \in \mathbb{N}\}$ correspond to c_0-norms on $c_0 \otimes E$. The injective tensor product norm gives the minimum multi-norm, and the projective tensor product norm gives the maximum multi-norm $\square$

[Cf Alexander Helemskii's abstract theory of operator spaces.]

And another characterization of multi-norms

There is a paper 'La structure des sous-espaces de trellis' by J. L. Marcolino Nhani, apparently a student of a student of Pisier, in *Dissertationes Math.*, 2001.

He introduces 'condition (P)': for a normed space E, there is a norm α on $E \otimes c_0$ such that, for each $T \in \mathcal{B}(c_0)$ and each $x \in E \otimes c_0$,

$$\alpha((I_E \otimes T)(x)) \leq \|T\| \, \alpha(x) \, .$$

This condition is equivalent to Daws' condition, and so characterizes multi-norms.

A theorem of Pisier shows that, in this case, E can be identified with a subspace of a certain Banach lattice X. The structure on X gives exactly what I had already called a *Banach lattice multi-norm*; see below.

This relates our theory to that of *operator sequence spaces* of Volker Runde etc.

An associated sequence

Let $(\|\cdot\|_n)$ be a multi-norm on $\{E^n : n \in \mathbb{N}\}$.

Define

$$\varphi_n(E) = \sup\{\|(x_1, \ldots, x_n)\|_n : \|x_i\| \le 1\}.$$

Trivially, $1 \le \varphi_n(E) \le n$ for all $n \in \mathbb{N}$ and

$$\varphi_{m+n}(E) \le \varphi_m(E) + \varphi_n(E)$$

for all $m, n \in \mathbb{N}$. What is the sequence $(\varphi_n(E))$?

In particular $(\varphi_n^{\max}(E))$ is the sequence associated with the maximum multi-norm.

It can be shown quite easily that $\varphi_n^{\max}(E)$ is

$$\sup\left\{\sum_{j=1}^{n}\left\|\lambda_j\right\|\right\},$$

where $\lambda_1, \ldots, \lambda_n \in E'$ and

$$\sum_{j=1}^{n}\left|\langle x,\, \lambda_j\rangle\right| \le 1 \quad (x \in E_{[1]}).$$

Summing norms - I

Let E be a normed space, and take $p \in [1, \infty)$. For $x_1, \ldots, x_n \in E$, set

$$\mu_{p,n}(x_1, \ldots, x_n) = \sup_{\lambda \in E'_{[1]}} \left\{ \left(\sum_{j=1}^{n} \left| \langle x_j, \lambda \rangle \right|^p \right)^{1/p} \right\} .$$

Then

$$\mu_{1,n}(x_1, \ldots, x_n) = \sup \left\{ \left\| \sum_{j=1}^{n} \zeta_j x_j \right\| : \zeta_1, \ldots, \zeta_n \in \mathbb{T} \right\} .$$

For $\lambda_1, \ldots, \lambda_n \in E'$, we have

$$\mu_{1,n}(\lambda_1, \ldots, \lambda_n) = \sup \left\{ \sum_{j=1}^{n} \left| \langle x, \lambda_j \rangle \right| : x \in E_{[1]} \right\} .$$

Let E and F be Banach spaces, and take $T \in \mathcal{B}(E, F)$ and $n \in \mathbb{N}$. Then $\pi_p^{(n)}(T)$ is

$$\sup \left\{ \left(\sum_{j=1}^{n} \left\| T x_j \right\|^p \right)^{1/p} : \mu_{p,n}(x_1, \ldots, x_n) \leq 1 \right\} .$$

Summing norms - II

Definition $\pi_p(T) = \lim_{n\to\infty} \pi_{p,n}^{(n)}(T)$ is the p-*summing norm* of T.

The p-summing operators form an operator ideal.

We write $\pi_p^{(n)}(E)$ for $\pi_p^{(n)}(I_E)$ and $\pi_p(E)$ for $\pi_p(I_E)$.

Theorem Let E be a normed space, and let $n \in \mathbb{N}$. Then

$$\varphi_n^{\max}(E) = \pi_1^{(n)}(E')\,.$$

If $E = F'$, then

$$\varphi_n^{\max}(E) = \pi_1^{(n)}(F)\,.$$

$\square$

Another constant

Let F be a normed space, and let S_F be the unit sphere of F. It is useful to define

$$c_n(F) = \inf\{\mu_1(\lambda_1, \ldots, \lambda_n) : \lambda_1, \ldots, \lambda_n \in S_F\}\,.$$

Easy fact: $\pi_1^{(n)}(F)c_n(F) \geq n$. In fact

$$\overline{\pi}_1^{(n)}(F)c_n(F) = n\,,$$

where $\overline{\pi}_1^{(n)}(F)$ is the version of $\pi_1^{(n)}(F)$ with $\|x_1\| = \cdots = \|x_n\|$ in the definition of $\mu_1(x_1, \ldots, x_n)$.

Guess: I think that there should be a large class of Banach spaces F with the property that there is a constant C_F such that

$$\overline{\pi}_1^{(n)}(F) \geq C_F \pi_1^{(n)}(F)$$

for all $n \in \mathbb{N}$. Is this correct? Is it true for $F = \ell^q$ whenever $q \in [1, 2]$? In the latter case, F is an Orlicz space, and there is a constant C_q such that $c_n(\ell^q) \geq C_q\sqrt{n}$ $(n \in \mathbb{N})$; what is C_q in the complex case?

A lower bound

We shall use the famous theorem of Dvoretzky, sometimes called the theorem on *almost spherical sections*.

Theorem For each $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists $m = m(n, \varepsilon)$ in $\mathbb{N}$ such that, for each normed space E with $\dim E \geq m$, there is an n-dimensional subspace L of E such that $d(L, \ell_n^2) < 1 + \varepsilon$. $\square$

Theorem Let E be an infinite-dimensional normed space. Then $\varphi_n^{\max}(E) \geq \sqrt{n}$ for each $n \in \mathbb{N}$. $\square$

Corollary Let E be an infinite-dimensional normed space. Then $\varphi_n^{\max}(E) \to \infty$, and there is a multi-norm on E not equivalent to the minimum multi-norm. $\square$

Special spaces

Take p with $1 \leq p \leq \infty$, and write q for the conjugate index to p. Take $E = \ell^p$. Thus

$$\varphi_n^{\max}(\ell^p) = \pi_1^{(n)}(\ell^q)\,.$$

We write ℓ_n^p for the n-dimensional space $\mathbb{C}^n$ with the usual ℓ^p-norm. Direct calculations of $\varphi_n^{\max}(\ell^p)$ using Banach–Mazur distance give:

Theorem (i) For each $p \in [1, 2]$, we have

$$\varphi_n^{\max}(\ell_n^p) = \varphi_n^{\max}(\ell^p) = n^{1/p} \quad (n \in \mathbb{N})\,.$$

(ii) For each $p \in [2, \infty]$, there is a constant C_p such that

$$\sqrt{n} \leq \varphi_n^{\max}(\ell_n^p) \leq \varphi_n^{\max}(\ell^p) \leq C_p\sqrt{n} \quad (n \in \mathbb{N})\,.$$

□

In general, I do not know the best constant C_p in the above inequality.

Example 1 : Standard (p, q)-multi-norm

Let Ω be a measure space, and take p, q with $1 \leq p \leq q < \infty$. We consider the Banach space $E = L^p(\Omega)$, with the usual L^p-norm $\|\cdot\|$.

For each family $\mathbf{X} = \{X_1, \ldots, X_n\}$ of pairwise-disjoint measurable subsets of Ω such that $X_1 \cup \cdots \cup X_n = \Omega$, we set

$$r_{\mathbf{X}}((f_1, \ldots, f_n)) =$$

$$\left(\left\|P_{X_1} f_1\right\|^q + \cdots + \left\|P_{X_n} f_n\right\|^q\right)^{1/q}$$

where $P_X : L^p(\Omega) \to L^p(X)$ is the natural projection.

Finally, $\|(f_1, \ldots, f_n)\|_n = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \ldots, f_n))$.

This is the **standard** (p, q)-multi-norm.

Remark Let $q = p$. Then

$$\|(f_1, \ldots, f_n)\|_n = \| \, |f_1| \vee \cdots \vee |f_n| \, \| \, .$$

Example 2 : Measures

Let Ω be a non-empty, locally compact space. Then $M(\Omega)$ is the Banach space of all regular Borel measures on Ω. Take $q \geq 1$.

For each partition $\mathbf{X} = \{X_1, \dots, X_n\}$ of Ω into measurable subsets and each $\mu_1, \dots, \mu_n \in M(\Omega)$, take $r_{\mathbf{X}}((\mu_1, \dots, \mu_n))$ to be

$$(\|\mu_1 \mid X_1\|^q + \cdots + \|\mu_n \mid X_n\|^q)^{1/q} \ ,$$

so that $r_{\mathbf{X}}$ is a seminorm on $M(\Omega)^n$ Then define

$$\|(\mu_1, \dots, \mu_n)\|_n^{(1,q)} = \sup_{\mathbf{X}} r_{\mathbf{X}}((\mu_1, \dots, \mu_n)) \, ,$$

where the supremum is taken over all such families $\mathbf{X}$. Then $\|\cdot\|_n$ is a norm on $M(\Omega)^n$, and it is again easily checked that $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm on $\{M(\Omega)^n : n \in \mathbb{N}\}$. It is the *standard* $(1, q)$*-multi-norm*.

Example 3 - Banach lattice multi-norms

Let $(E, \|\cdot\|)$ be a complex Banach lattice.

[Thus there are lattice operations on $E_{\mathbb{R}}$, the modulus $|x|$ of an element $x \in E_{\mathbb{R}}$ is defined, and the norm is such that $\|x\| \leq \|y\|$ whenever $|x| \leq |y|$ in E. Now $E = E_{\mathbb{R}} \oplus \mathrm{i}E_{\mathbb{R}}$ is a complex Banach lattice.]

Example $L^p(\Omega)$, $L^\infty(\Omega)$, or $C(\Omega)$ with the usual norms and the obvious lattice operations.

Definition Let $(E, \|\cdot\|)$ be a Banach lattice. For $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$, set

$$\|(x_1, \ldots, x_n)\|_n = \| \, |x_1| \vee \cdots \vee |x_n| \, \| \, .$$

Then $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a multi-Banach space. It is the **Banach lattice multi-norm**.

It generalizes the standard (p,p)-multi-norm on $L^p(\Omega)$ and the minimum multi-norms on $L^\infty(\Omega)$ and $C(\Omega)$.

Example 4 - The Schauder multi-norm

Let E be a Banach space with an unconditional Schauder basis (e_n). Set

$$\left|\left|\left|\sum_{n=1}^{\infty}\alpha_n e_n\right|\right|\right| = \sup\left\{\left\|\sum_{n=1}^{\infty}\alpha_n\beta_n e_n\right\| : |\beta_n| \le 1\right\}.$$

This norm is equivalent to the original one.

Let $\mathbf{X} = \{X_1, \dots, X_n\}$ be a partition of $\mathbb{N}$, and define

$$r_{\mathbf{X}}((x_1, \dots, x_n)) = \left|\left|\left|P_{X_1}x_1 + \cdots + P_{X_n}x_n\right|\right|\right|,$$

where P_{X_i} are the obvious projections, and then set

$$|||(x_1, \dots, x_n)|||_n = \sup_{\mathbf{X}} r_{\mathbf{X}}((x_1, \dots, x_n)).$$

We again obtain a multi-norm $(|||\cdot|||_n)$.

Example 5 : The weak (p_1, p_2)–multi-norm

Again E is a Banach space. Recall that the weak p-summing norm is

$$\mu_{p,n}(x_1, \dots, x_n) = \sup_{\lambda \in E'_{[1]}} \left\{ \left(\sum_{j=1}^{n} \left| \langle x_j, \lambda \rangle \right|^p \right)^{1/p} \right\}.$$

Here $x_1, \dots, x_n \in E$.

Now take p_1, p_2 with $1 \le p_1 \le p_2 < \infty$.

Define

$$\|x\|_n^{(p_1,p_2)} = \sup \left\{ \left(\sum_{j=1}^{n} \left| \langle x_j, \lambda_j \rangle \right|^{p_2} \right)^{1/p_2} \right\}$$

taking the sup over all $\lambda_1, \dots, \lambda_n \in E'$ with $\mu_{p_1,n}(\lambda_1, \dots, \lambda_n) \le 1$.

Fact $\{(E^n, \|\cdot\|_n^{(p_1,p_2)}) : n \in \mathbb{N}\}$ is a multi-normed space.

It is the **weak** (p_1, p_2)**–multi-norm** over E.

The weak (p_1, p_2)–multi-norm, continued

Fact The canonical embedding of E into E'' is a multi-isometry (see later) when we consider the weak (p_1, p_2)–multi-norms over E and E''.

Fact We can work out the dual of this multi-norm quite explicitly; there is pleasing duality theory.

Fact There are relations between these. For example, take $1 \leq p_1 \leq p_2 < \infty$ and $1 \leq r_1 \leq r_2 < \infty$. Suppose that $p_2 \geq r_2$ and

$$\frac{1}{p_2} + \frac{1}{r_1} \leq \frac{1}{r_2} + \frac{1}{p_1}.$$

Then $\| \cdot \|_n^{(p_1,p_2)} \leq \| \cdot \|_n^{(r_1,r_2)}$ on E^n.

The weak (p_1, p_2)–multi-norm, connections with $L^1(\Omega)$

Fact For $1 \leq q < \infty$, the weak $(1, q)$–multi-norm on the family $\{L^1(\Omega)^n : n \in \mathbb{N}\}$ is the same as the standard $(1, q)$–multi-norm described above.

Fact There are several other useful identifications using measures and second duals.

Fact Various 'nice' multi-norms that I mentioned (and others) have 'canonical extensions' - and these we now know are suitable weak (p_1, p_2)–multi-norms.

Example 6 :The Hilbert multi-norm

Let $H = \ell^2(S)$ be a Hilbert space. For each family $\mathbf{H} = \{H_1, \ldots, H_n\}$ of closed subspaces of H such that $H = H_1 \perp \cdots \perp H_n$, set

$$r_{\mathbf{H}}((x_1, \ldots, x_n)) = \left(\|P_1 x_1\|^2 + \cdots + \|P_n x_n\|^2\right)^{1/2}$$

where $P_i : H \to H_i$ for $i = 1, \ldots, n$ is the projection, and then set

$$|| \ (x_1, \ldots, x_n)||_n^H = \sup_{\mathbf{H}} r_{\mathbf{H}}((x_1, \ldots, x_n))\,;$$

we obtain multi-norms $||\cdot||_n^H$. We immediately have

$$||(x_1, \ldots, x_n)||_n \leq ||(x_1, \ldots, x_n)||_n^H \ ,$$

where $(\|\cdot\|_n)$ is the standard (2,2)-multi-norm on $\ell^2(S)$.

Maximality of the Hilbert multi-norm

Question Is the Hilbert multi-norm the maximum multi-norm on the family $\{H^n : n \in \mathbb{N}\}$? This seemed to be very likely because I could not think of a bigger one. However it seems to be rather a hard question.

In fact it can be reduced to a question about Hilbert spaces that does not mention multi-norms.

Let H be a Hilbert space. Then the closed unit ball of the dual of $(H^n, \|\cdot\|_n^H)$ is described as follows. Set

$$S := \bigcup \left\{ (\alpha_1 e_1, \ldots, \alpha_n e_n) : \sum_{j=1}^{n} \left|\alpha_j\right|^2 \leq 1 \right\},$$

where the union is taken over all orthonormal subsets $\{e_1, \ldots, e_n\}$ of H. The required unit ball is the weak-$*$-closed convex hull of S, call it K.

On the other hand, the closed unit ball of the dual of $(H^n, \|\cdot\|_n^{\max})$ is

$$\{y = (y_1, \dots, y_n) \in H^n : \mu_{1,n}(y_1, \dots, y_n) \leq 1\}\,;$$

this set, temporarily called L, is equal to the set of $y = (y_1, \dots, y_n) \in H^n$ such that

$$\|\zeta_1 y_1 + \cdots + \zeta_n y_n\| \leq 1$$

for all $\zeta_1, \dots, \zeta_n \in \mathbb{T}$.

Since $\|\cdot\|_n^H \leq \|\cdot\|_n^{\max}$, necessarily $K \subset L$.

To establish the equality of the two multinorms, we need to show that $L \subset K$ for each (implicit) $n \in \mathbb{N}$. In fact, we need

$$\operatorname{ex} L \subset \operatorname{ex} K = S \quad (n \in \mathbb{N})\,,$$

where 'ex' denotes the set of extreme points of a convex set. Is this always the case?

Towards this, I know the following.

Theorem Let H be a Hilbert space of dimension n.

(i) Suppose that $n = 2$. Then $\mathrm{ex}\, L \subset S$.

(ii) Suppose that $n = 3$ and H is a **real** Hilbert space. Then this fails.

(iii) (Pham) Suppose that $n = 3$ and H is complex. Then $\mathrm{ex}\, L \subset S$.

(iv) (Daws) There is a universal constant C with $C\,\|\cdot\|_n^H \geq \|\cdot\|_n^{\max}$, and so the Hilbert multi-norm is equivalent to the maximum multi-norm. [At present C is K_G, Grothendieck's constant. Maybe we have $C = 1$.] $\square$

Multi-topological linear spaces

Let E be a linear space, and let F be a subspace in $E^{\mathbb{N}}$ such that, for each $x \in E$, we have $(x, 0, 0, \dots) \in F$. A subset B of F is **basic** if:

(1) for each permutation σ of $\mathbb{N}$ and $(x_n) \in B$, also $(x_{\sigma(n)}) \in B$;

(2) for each $(x_n) \in B$ and α_n with $|\alpha_n| \leq 1$, also $(\alpha_n x_n) \in B$;

(3) for each $(x_n) \in B$, also $(x_1, x_1, x_2, x_2, x_3, \dots) \in B$;

(4) $(x_n) \in B$ if and only if $(x_1, \dots, x_k, 0, \dots) \in B$ for each $k \in \mathbb{N}$.

Suppose that F has a basis (in the usual sense of topological linear spaces) consisting of basic sets. Then F is a **multi-topological linear space**.

Examples of multi-topological linear spaces

(1) Let E be a multi-normed space. Set

$$F = \{(x_n) : \|(x_1, \dots, x_k)\|_k < \infty \ \ (k \in \mathbb{N})\}\,.$$

Then F is a multi-topological linear space. The basic sets are

$$\{(x_n) \in F : \|(x_1, \dots, x_k)\|_k \le C\}.$$

(2) Let E be a topological linear space, and set $F = E^{\mathbb{N}}$. Set

$$B = U_1 \times U_2 \times \cdots\,,$$

with U_i open in E, a basic set. Then F is a multi-topological linear space. (Box topology.)

Definition Let F be a multi-topological linear space. Then (x_n) is **multi-null** if, for each basic set B, there exists $n_0 \in \mathbb{N}$ such that

$$(x_n, x_{n+1}, \dots) \in B \quad (n \ge n_0)\,.$$

There is a version of Kolmogorov's theorem.

Multi-convergence

Proposition Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let (x_i) be a sequence in E. Then

$$\operatorname*{Lim}_i x_i = 0$$

if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ with

$$\|(x_{n_1}, \ldots, x_{n_k})\|_k < \varepsilon \quad (n_1, \ldots, n_k \geq n_0)\,. \qquad \square$$

These are exactly the **multi-null** sequences.

Theorem Multi-null sequences in a multi-normed space $(E^n, \|\cdot\|_n)$ are the null sequences of some topology on E if and only if the multi-norm is equivalent to the minimum multi-norm. $\square$

Multi-convergence - Examples

1) Let $\{E^n : n \in \mathbb{N}\}$ have the minimum multi-norm. Then $\text{Lim}_i\, x_i = 0$ if and only if $\lim_i x_i = 0$ in $(E, \|\cdot\|)$.

2) Let $E = \ell^p$ with the standard (p, q)-multi-norm, and let

$$x_i = \alpha_i \delta_i \quad (i \in \mathbb{N})\,.$$

Then $\text{Lim}_i\, x_i = 0$ if and only if $\sum |\alpha_i|^q < \infty$. Here δ_i is the sequence $(\delta_{i,j} : j \in \mathbb{N})$.

3) Let $E = L^p(\Omega)$ with the standard (p, p)-multi-norm. A sequence (f_n) is multi-null iff (f_n) is order-bounded and $f_n \to 0$ almost everywhere.

More generally

4) Let $(E, \|\cdot\|)$ be an 'order-continuous' Banach lattice, and consider the Banach lattice multi-norm on $\{E^n : n \in \mathbb{N}\}$. Then a sequence is a multi-null sequence if and only if it converges to 0 'in order'.

Multi-bounded sets and operators

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. A subset B of E is **multi-bounded** if

$$c_B := \sup_{n\in\mathbb{N}} \{\|(x_1, \ldots, x_n)\|_n : x_1, \ldots, x_n \in B\} < \infty\,.$$

Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be multi-Banach spaces. An operator $T \in \mathcal{B}(E, F)$ is **multi-bounded** if $T(B)$ is multi-bounded in F whenever B is multi-bounded in E. The set of these is a linear subspace $\mathcal{M}(E, F)$ of $\mathcal{B}(E, F)$.

For $T \in \mathcal{M}(E, F)$, set

$$\|T\|_{mb} = \sup\{c_{T(B)} : c_B \leq 1\}\,.$$

Theorem Now $((\mathcal{M}(E, F), \|\cdot\|_{mb})$ is a Banach space, and $\mathcal{M}(E)$ is a Banach operator algebra. $\square$

[Recall that these depend on the multi-norm structure, and not just on the Banach space, despite the notation.]

The multi-bounded norm

More generally, for $n \in \mathbb{N}$ and $T_1, \ldots, T_n \in \mathcal{M}(E, F)$, set

$$\|(T_1, \ldots, T_n)\|_{mb,n} = \sup \{c_{T_1(B) \cup \cdots \cup T_n(B)} : c_B \leq 1\}.$$

Theorem Now $((\mathcal{M}(E, F)^n, \|\cdot\|_{mb,n}) : n \in \mathbb{N})$ is a multi-Banach space. $\square$

Examples of $\mathcal{M}(E,F)$ - I

Throughout, $\{(E^n, \|\cdot\|_n) : n \in \mathbb{N}\}$ and $\{(F^n, \|\cdot\|_n) : n \in \mathbb{N}\}$ are multi-normed spaces.

Fact Suppose that E and F are operator sequence spaces (see Lambert, Neufang, and Runde). Then the multi-bounded operators are just the *sequentially bounded operators*. $\square$

Theorem Always

$$\mathcal{N}(E,F) \subset \mathcal{M}(E,F) \subset \mathcal{B}(E,F)\,,$$

where $\mathcal{N}(E,F)$ denotes the space of nuclear operators. $\square$

Theorem Suppose that F has the minimum or E the maximum multi-norm structure. Then $(\mathcal{M}(E,F), \|\cdot\|_{mb}) = (\mathcal{B}(E,F), \|\cdot\|)$. $\square$

Question When exactly do we get the above equality?

Examples of $\mathcal{M}(E, F)$

Theorem We can have $\mathcal{M}(E, F) = \mathcal{B}(E, F)$ and $\mathcal{M}(F, E) = \mathcal{N}(F, E)$. $\square$

Theorem Let $E = \ell^p$ and $F = \ell^q$, where $p, q \geq 1$. Regard them as multi-normed spaces with the standard (p, p) and (q, q) multi-norms, respectively. Then $\mathcal{M}(E, F)$ consists of the *regular* operators. $\square$

[An operator is **regular** if it is the difference of two positive operators.]

Another example

Theorem We can have $\mathcal{K}(E) \not\subset \mathcal{M}(E)$.

Proof Let H be the Hilbert space $\ell^2(\mathbb{N})$, with the standard $(2, 2)$-multi-norm.

Consider the system of vectors

$$(x_r^s : r = 1, \ldots s, s \in \mathbb{N})$$

defined as follows: $x_r^s(k) = 0$ except when

$$k \in \{2^{s-1}, \ldots, 2^s - 1\}\,;$$

at the 2^{s-1} numbers k in this set, $x_r^s(k) = \pm 1/\sqrt{2^{s-1}}$, the values ± 1 being chosen so that $[x_{r_1}^s, x_{r_2}^s] = 0$ when $r_1, r_2 = 1, \ldots, s$ and $r_1 \neq r_2$. Such a choice is clearly possible. Then

$$S := \{x_r^s : r = 1, \ldots, s, s \in \mathbb{N}\}$$

is an orthonormal set in H. Order the set S as (y_n) by using the lexicographic order on the pairs (s, r).

Let $(\alpha_i) \in \ell^\infty$. We define $T \in \mathcal{B}(H)$ by setting

$$Tx_r^s = \alpha_s \delta_n \quad \text{when} \quad x_r^s = y_n\,.$$

It is clear that, in the case where $(\alpha_i) \in c_0$, we have $T \in \mathcal{K}(H)$.

For $k \in \mathbb{N}$, set $N_k = k(k+1)/2$. We see that

$$\left\|(y_1, y_2, \ldots, y_{N_k})\right\|_{N_k}^2 = k\,.$$

However $\left\|(Ty_1, Ty_2, \ldots, Ty_{N_k})\right\|_{N_k}^2 = \sum_{i=1}^k i\,|\alpha_i|^2$.

Now take $\gamma \in (0, 1/2)$, and set $\alpha_i = i^{-\gamma}$. Then

$$\frac{\left\|(Ty_1, Ty_2, \ldots, Ty_{N_k})\right\|_{N_k}}{\left\|(y_1, y_2, \ldots, y_{N_k})\right\|_{N_k}} \geq ck^{(1-2\gamma)/2}$$

for a constant $c > 0$. Since $\gamma < 1/2$, we have $T \notin \mathcal{M}(H)$.

We have shown that $\mathcal{K}(H) \not\subset \mathcal{M}(H)$. $\square$

$\mathcal{M}(E, F)$ for Banach lattices

Let $(E, \|\cdot\|)$ be a Banach lattice. Then E is *monotonically bounded* if each increasing, $\|\cdot\|$-bounded net in E has an (order) upper bound.

Thus each Banach lattice $L^p(\Omega)$ (for $p \in [1, \infty]$) and $C(\Omega)$ (for Ω compact) is monotonically bounded, but c_0 is not monotonically bounded.

Theorem Let $(E, \|\cdot\|)$ and $(F, \|\cdot\|)$ be two monotonically bounded Banach lattices, each with the lattice multi-norms. Then $\mathcal{M}(E, F)$ is the space of all order-bounded operators from E to F. $\square$

Multi-bounded and multi-continuous operators

Let E and F be multi-topological linear spaces. An operator $T : E \to F$ is **multi-continuous** if (Tx_i) is multi-null in F whenever (x_i) is multi-null in E.

Theorem Let $(E^n, \|\cdot\|_n)$ and $(F^n, \|\cdot\|_n)$ be multi-normed spaces, and take $T \in \mathcal{B}(E, F)$. Then T is multi-continuous if and only if T is multi-bounded. $\square$

Banach lattices: examples

Return to the embedding $\mu \mapsto T_\mu$ of $M(G)$ into $\mathcal{B}(L^p(G))$. In fact, it is a mapping into $\mathcal{M}(L^p(G))$, when $L^p(G)$ has the standard (p,p)-multi-norm, and now we do get $\sigma_{\mathcal{M}}(T_\mu) = \sigma(\mu)$ always, where $\sigma_{\mathcal{M}}(T_\mu)$ is the spectrum of T_μ in the Banach algebra $\mathcal{M}(L^p(G))$.

A failure of 'Banach's isomorphism theorem:

Let E be a Banach lattice and consider the Banach lattice multi-norm. Then $\mathcal{M}(E)$ consists of the regular operators; this Banach algebra is $\mathcal{B}_r(E)$. As mentioned, there are examples of $T \in \mathcal{B}_r(E)$ such that T is invertible in $\mathcal{B}(E)$, but the inverse is not in $\mathcal{B}_r(E)$.

The problem of duality

Let E be a Banch space, and let $(\|\cdot\|_n)$ be a multi-norm on $\{E^n : n \in \mathbb{N}\}$.

We might expect that the dual of the multi-normed space is $\mathcal{M}(E, \mathbb{C})$. But this gives just E', and forgets the multi-norm structure.

We could try: $\|\cdot\|'_n$ is the norm on $(E')^n$ which is the dual of the norm $\|\cdot\|_n$ on E^n. We obtain a sequence $\|\cdot\|'_n$ that satisfies (A1), (A2), and (A3), but not (A4). Rather it satisfies:

(B4) $\|(x_1, \ldots, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_{n-1}, 2x_n)\|_n$.

We have characterizations of these 'dual multi-norms' analogous to the above − for example we replace $\mathcal{B}(\ell^\infty_n, \ell^\infty_m)$ by $\mathcal{B}(\ell^1_n, \ell^1_m)$ and $c_0 \otimes E$ by $\ell^1 \otimes E$.

[The second duals of a multi-norm sequence do give a multi-norm.]

So this does not work either.

A consequence of duality

In fact we have a duality theory that involves a long detour through an orthogonality theory for multi-normed spaces that generalizes that of Banach lattices. This gives the concepts of **multi-dual** and **multi-reflexive** spaces.

We have the following, which was the point of the definitions.

Theorem For $1 < p < \infty$, let the families $\{(\ell^p)^n : n \in \mathbb{N}\}$ have the standard (p, p)-multi-norm. Then the multi-dual of the multi- normed space $\{(\ell^p)^n : n \in \mathbb{N}\}$ is $\{(\ell^q)^n : n \in \mathbb{N}\}$ with the standard (q, q)-multi-norm, where q is the conjugate index to p. Hence these multi-normed spaces are multi-reflexive. $\square$

Decompositions - definitions

Definition Let $(E, \|\cdot\|)$ be a normed space. A direct sum decomposition $E = E_1 \oplus \cdots \oplus E_k$ is *valid* if

$$\|\zeta_1 x_1 + \cdots + \zeta_k x_k\| \leq \|x_1 + \cdots + x_k\|$$

for all $\zeta_1, \ldots, \zeta_k \in \overline{\mathbb{D}}$ and $x_1 \in E_1, \ldots, x_k \in E_k$.

Definition Let $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-normed space, and let $E = E_1 \oplus \cdots \oplus E_k$ be a direct sum decomposition of E. Then the decomposition is *small* if

$$\|P_1 x_1 + \cdots + P_k x_k\| \leq \|(x_1, \ldots, x_k)\|_k$$

for all $x_1, \ldots, x_k \in E$.

Fact A small decomposition is valid.

Orthogonality

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. A family $\{E_1, \dots, E_k\}$ of linear subspaces of E is **orthogonal** if, for each partition $\{S_1, \dots, S_j\}$ of $\{1, \dots, k\}$ and each $x_i \in E_i$, we have

$$\|(x_1, \dots, x_k)\|_k = \left\|(y_1, \dots, y_j)\right\|_j ,$$

where $y_i := \sum\{x_r : r \in S_i\}$ $(i = 1, \dots, j)$.

In particular, we require that

$$\|(x_1, \dots, x_k)\|_k = \|x_1 + \cdots + x_k\|$$

whenever $x_i \in E_i$ for $i = 1, \dots, k$.

A set $\{x_1, \dots, x_k\}$ is **orthogonal** if the family $\{\mathbb{C}x_1, \dots, \mathbb{C}x_k\}$ is orthogonal.

Remark It is possible that $\{x_1, x_2, x_3\}$ is not orthogonal, but each of $\{x_1, x_2\}$, $\{x_1, x_3\}$, and $\{x_2, x_3\}$ is orthogonal.

Remark Every orthogonal decomposition is valid.

Orthogonality - Examples

(1) Let H be a Hilbert space with the Hilbert multi-norms Then subspaces are orthogonal if and only if they are orthogonal in the classical sense if and only if they give a valid decomposition.

(2) Let $E = \ell^p(S)$ have the standard (p,p)-multi-norms. Then E has the orthogonal decomposition $E = E_1 \oplus \cdots \oplus E_k$ if and only if there is a partition $\{S_1, \ldots, S_k\}$ of S such that $E_j = \ell^p(S_j)$ $(j = 1, \ldots, k)$.

(3) Let $E = \ell^p(S)$ have the standard (p,q)-multi-norm, with $q \neq p$. Then there are no non-trivial orthogonal decompositions of E.

Continuous functions

Let $E = C(\Omega)$ for a compact space Ω, and let $\{E^n : n \in \mathbb{N}\}$ have the minimum multi-norm. Then the only orthogonal decompositions of E have the form

$$C(\Omega) = C(\Omega_1) \oplus \cdots \oplus C(\Omega_k)$$

for a partition $\{\Omega_1, \ldots, \Omega_k\}$ of Ω into clopen subspaces.

Decompositions - some facts

Fact An orthogonal decomposition is valid.

Fact A small decomposition is not necessarily orthogonal.

Fact An orthogonal decomposition is not necessarily small.

[Maybe the definitions need tweaking to bring 'small' into the definition of 'orthogonality'?]

Orthogonality - Examples

(1) Let H be a Hilbert space with the Hilbert multi-norms. Then subspaces are orthogonal if and only if they are classically orthogonal if and only if they give a valid decomposition.

(2) Let $E = \ell^p(S)$ have the standard (p, p)-multi-norms. Then E has the orthogonal decomposition $E = E_1 \oplus \cdots \oplus E_k$ if and only if there is a partition $\{S_1, \ldots, S_k\}$ of S such that $E_j = \ell^p(S_j)$ $(j = 1, \ldots, k)$.

(3) Let $E = \ell^p(S)$ have the standard (p, q)-multi-norm, with $q \neq p$. Then there are no non-trivial orthogonal decompositions of E.

(4) Let $E = C(\Omega)$ for a compact space Ω, and let $\{E^n : n \in \mathbb{N}\}$ have the minimum multi-norm. Then the only orthogonal decompositions of E have the form

$$C(\Omega) = C(\Omega_1) \oplus \cdots \oplus C(\Omega_k)$$

for a partition $\{\Omega_1, \ldots, \Omega_k\}$ of Ω into clopen subspaces

Decompositions and Banach lattices

Let E be a Banach lattice. Recall that the lattice multi-norms are defined by

$$\|(x_1, \ldots, x_n)\|_n = \|\,|x_1| \vee \cdots \vee |x_n|\,\|$$

for $x_1, \ldots, x_n \in E$.

Recall from that $E = E_1 \perp \cdots \perp E_n$ is a *classically orthogonal decomposition* if $|x_i| \wedge \left|x_j\right| = 0$ whenever $x_i \in E_i$, $x_j \in E_j$, and $i \neq j$.

Easy: a classically orthogonal decomposition is orthogonal for the lattice multi-norm.

Let $E = E_1 \oplus \cdots \oplus E_k$ be orthogonal with respect to the lattice multi-norm. Then

$$\|\,|x_1| \vee \cdots \vee |x_k|\,\| = \|x_1 + \cdots + x_k\|$$

whenever $x_1 \in E_1, \ldots, x_k \in E_k$.

Theorem - Nigel Kalton This already implies that the decomposition is classically orthogonal, and so the new concept of 'orthogonal' coincides with the old one in this case. $\square$

Families of decompositions

Definition Let $(E, \|\cdot\|)$ be a normed space, and consider a family

$$\mathcal{K} = \{(E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}) : \alpha \in A\},$$

where A is an index set, $n_\alpha \in \mathbb{N}$ $(\alpha \in A)$, and

$$E = E_{1,\alpha} \oplus \cdots \oplus E_{n_\alpha,\alpha}$$

is a direct sum decomposition of E for each $\alpha \in A$. The family $\mathcal{K}$ is **closed** provided that the following conditions are satisfied:

(C1) $(E_{\sigma(1),\alpha}, \dots, E_{\sigma(n_\alpha),\alpha}) \in \mathcal{K}$ whenever $(E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}) \in \mathcal{K}$ and $\sigma \in \mathfrak{S}_{n_\alpha}$;

(C2) $(E_{1,\alpha} \oplus E_{2,\alpha}, E_{3,\alpha}, \dots, E_{n_\alpha,\alpha}) \in \mathcal{K}$ whenever $(E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}) \in \mathcal{K}$ and $n_\alpha \geq 2$;

(C3) $\mathcal{K}$ contains all trivial direct sum decompositions.

Orthogonal multi-norms

The families of all direct sum decompositions, of all valid decompositions, of all small decompositions, and of all orthogonal decompositions are closed families of decompositions.

Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. Consider a closed family

$$\mathcal{K} = \{\{E_{1,\alpha}, \ldots, E_{n_\alpha,\alpha}\} : \alpha \in \mathcal{A}\}$$

of orthogonal decompositions of E.

Definition Let $(E^n, \|\cdot\|_n)$ be a multi-normed space. This space is **orthogonal with respect to** $\mathcal{K}$ if

$$\|(x_1, \ldots, x_n)\|_n = \sup_\alpha \left\{ \left\| (P_{1,\alpha}x_1, \ldots, P_{n,\alpha}x_n) \right\|_n \right\},$$

for $x_1, \ldots, x_n \in E$, where the supremum is taken over all $\alpha \in \mathcal{A}$ with $n_\alpha = n$.

Multi-norms from families of decompositions

Let $(E, \|\cdot\|)$ be a normed space, and consider a closed family $\mathcal{K} = \{\{E_{1,\alpha}, \dots, E_{n_\alpha,\alpha}\} : \alpha \in A\}$ of valid decompositions of E. For $n \in \mathbb{N}$ and $x_1, \dots, x_n \in E$, set

$$\|(x_1, \dots, x_n)\|_{n,\mathcal{K}}$$

$$= \sup_{\alpha \in A} \left\{ \left\| P_{1,\alpha} x_1 + \cdots + P_{n_\alpha,\alpha} x_n \right\| : n_\alpha = n \right\} .$$

Then $((E^n, \|\cdot\|_{n,\mathcal{K}}) : n \in \mathbb{N})$ is a multi-normed space, each member of $\mathcal{K}$ is an orthogonal decomposition of E with respect to this multi-norm, and the multi-normed space is orthogonal with respect to $\mathcal{K}$. This is the multi-norm **generated** by $\mathcal{K}$.

Query: what are the conditions on a multi-norm that ensure that it is orthogonal with respect to some closed family of valid decompositions?

Duals of valid decompositions

Fact Let $E = E_1 \oplus \cdots \oplus E_k$ be a valid decomposition of a Banach space E. Then

$$E' = E_1' \oplus \cdots \oplus E_k'$$

is a valid decomposition of the dual space E'.

Consider a closed family

$$\mathcal{K} = \{\{E_{1,\alpha}, \ldots, E_{n_\alpha,\alpha}\} : \alpha \in \mathcal{A}\}$$

of valid decompositions of E. The dual family is

$$\mathcal{K}' = \{\{E'_{1,\alpha}, \ldots, E'_{n_\alpha,\alpha}\} : \alpha \in \mathcal{A}\}\,,$$

and it generates a multi-norm $(\|\cdot\|^{\dagger}_{n,\mathcal{K}} : n \in \mathbb{N})$ on the family $\{(E')^n : n \in \mathbb{N}\}$.

Dual multi-norms

Definition Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of valid decompositions of E. Then the multi-norm on $\{(E')^n : n \in \mathbb{N}\}$ generated by $\mathcal{K}'$ is denoted by

$$(\|\cdot\|^{\dagger}_{n,\mathcal{K}} : n \in \mathbb{N})\,.$$

The multi-normed space

$$(((E')^n, \|\cdot\|^{\dagger}_{n,\mathcal{K}}) : n \in \mathbb{N})$$

is the **multi-dual space** with respect to $\mathcal{K}$.

Orthogonal decompositions for Banach lattices

For example, the family $\mathcal{K}$ of all orthogonal decompositions is closed, and the Banach lattice multi-norm is orthogonal with respect to this family. Moreover, if E is order-continuous, the dual multi-norm is exactly the Banach lattice multi-norm of the dual space E'.

The theorem on duality

Theorem Let $(E, \|\cdot\|)$ be a normed space, and let $\mathcal{K}$ be a closed family of valid decompositions of E. Then

$$(((E')^n, \|\cdot\|^{\dagger}_{n,\mathcal{K}}) : n \in \mathbb{N})$$

is a multi-normed space, each member of $\mathcal{K}'$ is an orthogonal decomposition of E', and this multi-normed space is orthogonal with respect to $\mathcal{K}'$. $\square$

In very many (but not all) cases, the dual multi-norms are independent of the defining family $\mathcal{K}$, as we would wish.

These definitions make the earlier theorems on duality correct (I think!).

Is the above the 'correct' duality theory, or is there a simpler one?

Reduced valid decompositions I

Let $(E, \|\cdot\|)$ be a normed space. A valid decomposition

$$E = E_1 \oplus \cdots \oplus E_k$$

is **reduced** if there is a function $\theta : \mathbb{R}^{+n} \to \mathbb{R}^+$ such that

$$\theta(\|x_1\|, \ldots, \|x_k\|) = \left\| \sum_{i=1}^{k} x_i \right\|$$

whenever $x_i \in E_i$.

A closed family $\mathcal{K}$ of valid decompositions is **reduced** if each member is reduced and the corresponding θ depends only on the value of k.

Reduced valid decompositions II

Suppose that $\mathcal{K}$ is a closed family of valid decompositions which contains non-trivial decompositions of length at least 3.

For $s, t \in \mathbb{R}^+$, set $s \,\square\, t = \theta_2(s, t)$.

Theorem $(\mathbb{R}^+, \square, \cdot\,, \leq)$ is a topological ordered semiring, and hence the only possibilities for the binary operation $\square$ are

$$\begin{aligned} s \square t &= \max\{s, t\} \quad (s, t \in \mathbb{R}^+)\,, \\ s \square t &= (s^p + t^p)^{1/p} \quad (s, t \in \mathbb{R}^+)\,, \end{aligned}$$

for some $p \geq 1$. $\square$

The two possibilities are realised for the normed spaces $C(\Omega)$ and ℓ^p, respectively.

Multi-Banach algebras

Let $(A, \|\cdot\|)$ be a Banach algebra, and let

$$((A^n, \|\cdot\|_n) : n \in \mathbb{N})$$

be a multi-normed space. Then $(A^n, \|\cdot\|_n)$ is a **multi-Banach algebra** if multiplication is a multi-bounded bilinear operator, and so

$$\|(a_1 b_1, \ldots, a_n b_n)\|_n \leq \|(a_1, \ldots, a_n)\|_n \, \|(b_1, \ldots, b_n)\|_n \, .$$

Examples (1) Each Banach algebra is a multi-Banach algebra with respect to both the minimum **and maximum** multi-norms.

(2) Take $1 \leq p \leq q < \infty$. Then $(\ell^p(S), \cdot)$ is a multi-Banach algebra with respect to the standard (p, q)-multi-norm.

(3) Let G be a locally compact group. Then the group algebra $(L^1(G), \star)$ with the standard $(1,1)$-multi-norm is a multi-Banach algebra.

(4) For each multi-Banach space $(E^n, \|\cdot\|_n)$, $(\mathcal{M}(E)^n, \|\cdot\|_{mb,n})$ is a multi-Banach algebra.