Multiplier of locally compact quantum groups and Hilbert C^*-modules

2. Multipliers and Hilbert C^*-modules

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Multipliers of Banach algebras

For a Banach algebra A, we always assume that A is *faithful*: if $a \in A$ and $bac = 0$ for all $b, c \in A$, then $a = 0$.

Recall that the *multiplier algebra* is $M(A)$, consisting of pairs of maps (L, R) with $aL(b) = R(a)b$ for $a, b \in A$. If A is Arens regular and has a bounded approximate identity,

$$M(A) = \big\{x \in A^{**} : xa, ax \in A \ (a \in A)\big\}.$$

Homework: Let $\theta : A \to B(A)$ be the left-regular representation, $\theta(a) : b \mapsto ab$. Show that we can identify $M(A)$ with

$$\big\{T \in B(A) : T\theta(a), \theta(a)T \in \theta(A) \ (a \in A)\big\}.$$

For a locally compact group G, the algebra $L^1(G)$ always has a contractive approximate identity (so is faithful). The algebra $A(G)$ has a bounded approximate identity only when G is amenable, but is always faithful.

Multipliers of $L^1(G)$

We identify $M(G)$ with the dual of $C_0(G)$, and then our coproduct Δ induces a product on $M(G)$:

$$\langle \mu * \lambda, f \rangle = \int_{G\times G} f(st)\ d\mu(s)\ d\lambda(t) \qquad (f \in C_0(G), \mu, \lambda \in M(G)).$$

Theorem

For any locally compact group G, we have an isometric isomorphism between $M(L^1(G))$ and $M(G)$.

Proof.

We embed $L^1(G)$ into $C_0(G)^* = M(G)$ by integration. Then $L^1(G)$ is an ideal in $M(G)$, so we get a contraction $M(G) \to M(L^1(G))$. Conversely, given $(L, R) \in M(L^1(G))$, let (e_α) be a cai for $L^1(G)$, and define $\mu \in M(G)$ to be the weak*-limit of $L(e_\alpha)$. □

Multipliers of $A(G)$

Theorem

Let $A \subseteq C_0(G)$ be a sub-algebra such that:

- *A is a Banach algebra for some norm such that the inclusion $A \to C_0(G)$ is continuous;*
- *for each $s \in G$ there exists $a \in A$ with $a(s) = 1$ and $\|a\| \leq 2$;*
- *for each $s \in G$ there is an open set U containing s, and $a \in A$ with $a|_U \equiv 1$.*

Then we can identify $M(A)$ with $\{f \in C^b(G) : fa \in A \ (a \in A)\}$.

Proof.

As A is commutative, $M(A) = \{T : A \to A : T(ab) = T(a)b \ (a, b \in A)\}$. The rest is homework. □

Observe that $A(G)$ satisfies all these conditions. Observe that $M(C_{00}(G)) = C(G)$, the algebra of all continuous functions.

Completely bounded multipliers of $A(G)$

We see that

$$\begin{aligned} M(A(G)) &= \{T \in B(A(G)) : T(ab) = T(a)b \ (a, b \in A(G))\} \\ &= \{f \in C^b(G) : fa \in A(G) \ (a \in A(G))\}. \end{aligned}$$

Given $T \in M(A(G))$, we see that $T^* \in B(VN(G))$. Hence

$$I \otimes T^* : M_n \otimes VN(G) = M_n(VN(G)) \to M_n(VN(G)),$$

where $M_n(VN(G))$ is a von Neumann algebra acting on $\ell^2_n \otimes L^2(G)$.

We say that T is *completely bounded* if $I \otimes T^*$ is bounded, uniformly in $n \in \mathbb{N}$. Write $M_{cb}A(G)$ for the algebra of such multipliers.

So, formally, $M_{cb}A(G)$ is a subalgebra of $C^b(G)$. Can we find a characterisation which doesn't involve maps on $VN(G)$?

Gilbert's Theorem

Theorem

Let $f \in C^b(G)$. The following are equivalent:

1. $f \in M_{cb}A(G)$;
2. *there exists a Hilbert space K, and bounded continuous maps $\alpha, \beta : G \to K$, such that $f(st^{-1}) = (\alpha(s)|\beta(t))$ for $s, t \in G$.*

Proof.

See Jolissaint, 1992. (History: Jolissaint points us to Cowling and Haagerup, Inventiones, 1989. They point us to Bozejko and Fendler, 1984, who attribute the result to Gilbert, unpublished, late 70s). □

Completely bounded maps

Remember that a map T from a C*-algebra A to $B(H)$ is completely bounded if and only if we can find a $*$-representation $\pi : A \to B(K)$ and bounded maps $P, Q : H \to K$ with

$$T(x) = Q^* \pi(x) P \qquad (x \in A).$$

So, suppose we have $\alpha, \beta : G \to K$ bounded and continuous with

$$f(st^{-1}) = \big(\alpha(s) \big| \beta(t)\big) \qquad (s, t \in G).$$

Then define $\tilde{\alpha} : L^2(G) \to L^2(G, K) = L^2(G) \otimes K$ by $\tilde{\alpha}(\xi) = (\xi(r)\alpha(r))_{r \in G}$. Notice then that

$$(\lambda(s) \otimes 1)\tilde{\alpha}(\xi) = \big(\xi(s^{-1}r)\alpha(s^{-1}r)\big)_{r \in G} \qquad (s \in G).$$

If we form $\tilde{\beta}$ similarly, then we can define a completely bounded map $T : VN(G) \to B(L^2(G))$ by

$$T(x) = \tilde{\beta}^*(x \otimes 1)\tilde{\alpha} \qquad (x \in VN(G)).$$

Notice that T is clearly normal.

Easy direction continued

Then for $\xi, \eta \in L^2(G)$ and $s \in G$,

$$\begin{aligned}
\big(T(\lambda(s^{-1}))\xi\big|\eta\big) &= \big(\tilde{\beta}^*(\lambda(s^{-1}) \otimes 1)\tilde{\alpha}\xi\big|\eta\big) \\
&= \int_G \big(\xi(sr)\alpha(sr)\big|\eta(r)\beta(r)\big) \ dr \\
&= \int_G \xi(sr)\overline{\eta(r)}(\alpha(sr)|\beta(r)) \ dr \\
&= f(srr^{-1}) \int_G \xi(sr)\overline{\eta(r)} \ dr = f(s)\big(\lambda(s^{-1})\xi\big|\eta\big).
\end{aligned}$$

Thus $T(\lambda(s^{-1})) = f(s)\lambda(s^{-1})$. As T is normal, it follows that T maps into $VN(G)$.

Remember that $A(G) \to C_0(G)$ is the map $\omega_{\xi,\eta} \mapsto \big((\lambda(s^{-1})\xi|\eta)\big)_{s \in G}$. It follows that T is the adjoint of the multiplier induced by f. Thus f is completely bounded.

The converse

Now suppose that f is completely bounded, say inducing $T : VN(G) \to VN(G)$ which has the form

$$T(x) = Q^*\pi(x)P \qquad (x \in VN(G)),$$

where $\pi : VN(G) \to B(K)$ is a $*$-representation. As T is normal, we may suppose that π is too. Notice that we may assume that $\pi(1) = 1$. Then the map $\sigma : G \to B(K); s \mapsto \pi(\lambda(s))$ is a continuous unitary representation. Now notice that for $s \in G$ and $\omega \in A(G)$,

$$f(s)\langle\lambda(s^{-1}),\omega\rangle = \langle\lambda(s^{-1}), f\omega\rangle = \langle T(\lambda(s^{-1})),\omega\rangle = \langle Q^*\pi(\lambda(s^{-1}))P,\omega\rangle,$$

and so $Q^*\sigma(s)P = f(s^{-1})\lambda(s)$. Pick $\xi_0 \in L^2(G)$ a unit vector, and define

$$\alpha(s) = \sigma(s^{-1})P\lambda(s)\xi_0, \quad \beta(s) = \sigma(s^{-1})Q\lambda(s)\xi_0 \qquad (s \in G).$$

Thus, for $s, t \in G$,

$$\begin{aligned}\big(\alpha(s)\big|\beta(t)\big) &= \big(Q^*\sigma(t^{-1})^*\sigma(s^{-1})P\lambda(s)\xi_0\big|\lambda(t)\xi_0\big) \\ &= f(st^{-1})\big(\lambda(ts^{-1})\lambda(s)\xi_0\big|\lambda(t)\xi_0\big) = f(st^{-1}).\end{aligned}$$

Hilbert C*-modules

We have seen that the space $C^b(G,K)$ is very useful. How can we think about this abstractly?

Let A be a C*-algebra. A *pre-Hilbert C*-module* over A is a *right* A-module E where E admits an A-valued sesquilinear map satisfying:

1. $(x|\cdot)$ is a *linear* map $E \to A$ and $(x|y)^* = (y|x)$;
2. $(x|x) \geq 0$ in the C*-algebra sense, and $(x|x) = 0 \implies x = 0$;
3. $(x|y \cdot a) = (x|y)a$.

Henceforth all inner-products will be linear on the right.

We can define a norm on E by $\|x\| = \|(x|x)\|_A^{1/2}$; this is a norm, which follows by first showing a Cauchy-Schwarz inequality for $(\cdot|\cdot)$. When E is complete, we say that E is *Hilbert C*-module*.

Notice that
$\|x \cdot a\|^2 = \|(x \cdot a|x)a\| = \|(x|x \cdot a)^* a\| = \|a^*(x|x)a\| \leq \|(x|x)\| \|a^* a\| = \|x\|^2 \|a\|^2$.

Examples

- If $A = \mathbb{C}$, then we just recover the notion of a Hilbert space.
- Given A, we can turn A into a Hilbert C*-module over itself by defining $(a|b) = a^* b$.
- Let K be a Hilbert space, and form the algebraic tensor product $A \odot K$. This becomes a pre-Hilbert C*-module for

$$\big(a \otimes \xi \big| b \otimes \eta\big) = a^* b(\xi|\eta),$$

with the module action $(a \otimes \xi) \cdot b = ab \otimes \xi$.

To show that this is positive definite, notice that we can write any tensor in $A \odot K$ as $\sum a_k \otimes \xi_k$ with the (ξ_k) being orthonormal. Then the inner-product is $\sum_k a_k^* a_k \geq 0$.

Let $A \otimes K$ be the completion of $A \odot K$.

Indeed, if (e_i) is an orthonormal basis for K, then $A \otimes K$ consists of those families (a_i) in A such that $\sum_i a_i^* a_i$ converges in A (notice that this is *weaker* than $\sum_i \|a_i\|^2 < \infty$).

Morphisms

Let E and F be Hilbert C*-modules over A. A map $T : E \to F$ is *adjointable* if there exists a map $T^* : F \to E$ with

$$\big(T(x)\big|y\big) = \big(x|T^*(y)\big) \qquad (x \in E, y \in F).$$

Homework: Show that T and T^* are linear, and using the Closed Graph Theorem, that T and T^* are bounded. Show furthermore that T and T^* are A-module maps.

Unlike for Hilbert spaces, if T is bounded and linear (and an A-module map) then we cannot always find T^*.

Clearly $T^{**} = T$, and it's easy to see that $\|T^*T\| = \|T\|^2$. We write $\mathcal{L}(E,F)$ for the collection of adjointable maps $E \to F$. Then $\mathcal{L}(E)$ is a C*-algebra.

Consider the "finite-rank" map

$$\theta_{x,y} : E \to F;\ z \mapsto x \cdot (y|z) \qquad (z \in E),$$

where $x \in F$ and $y \in E$. This is adjointable, because $\theta_{x,y}^* = \theta_{y,x}$. Let $\mathcal{K}(E)$ be the closure of the linear span of such maps in $\mathcal{L}(E,F)$.

Links with multipliers

Theorem

Consider A as a module over itself. Then $\mathcal{K}(A) \cong A$ and $\mathcal{L}(A) \cong M(A)$.

Proof.

Let $a, b \in A$, so that $\theta_{a,b}(c) = a(b|c) = ab^*c$, and hence $\theta_{a,b}$ is left multiplication by ab^*. As A has an approximate identity, it's not hard to see that $\mathcal{K}(A) \cong A$. Given $T \in \mathcal{L}(A)$, define $(L, R) \in M(A)$ by

$$L(a) = T(a), \quad R(a) = T^*(a^*)^* \qquad (a \in A).$$

Then $a^*L(b) = (a|T(b)) = (T^*(a)|b) = T^*(a)^*b = R(a^*)b$. So we have a map $\mathcal{L}(A) \to M(A)$. This is onto, as given $(L, R) \in M(A)$, the map L is adjointable, with $L^*(a) = R(a^*)^*$. □

For general E, we have that $\mathcal{K}(E)$ is an ideal in $\mathcal{L}(E)$, and, considering $\mathcal{K}(E)$ as a C*-algebra, we have that $M(\mathcal{K}(E)) \cong \mathcal{L}(E)$.

Links with continuous functions

Theorem

We can identify $C_0(G) \otimes K$ with $C_0(G, K)$.

Proof.

Algebraically, we identify $f \otimes \xi$ with $(f(s)\xi)_{s \in G}$. Then

$$\Big\| \sum_k f_k \otimes \xi_k \Big\|^2 = \Big\| \sum_{j,k} f_j^* f_k (\xi_j | \xi_k) \Big\|_{C_0(G)} = \sup_{s \in G} \Big| \sum_{j,k} \overline{f_j(s)} f_k(s) (\xi_j | \xi_k) \Big|$$

$$= \sup_{s \in G} \Big| \sum_{j,k} \big(f_j(s)\xi_j \big| f_k(s)\xi_k \big) \Big| = \sup_{s \in G} \Big\| \sum_k f_k(s)\xi_k \Big\|^2.$$

Finally, a partition of unity argument shows that the image of $C_0(G) \odot K$ is dense in $C_0(G, K)$. □

For morphisms

Theorem

We can identify $\mathcal{L}(C_0(G), C_0(G) \otimes K)$ *with* $C^b(G, K)$.

Proof.

Given $F \in C^b(G, K)$, we define $T : C_0(G) \to C_0(G, K)$ by $T(f) = \big(f(s)F(s)\big)_{s\in G}$. This is adjointable, with $T^*(g) = \big((F(s)|g(s))\big)_{s\in G}$ for $g \in C_0(G, K)$.
Conversely, given $T : C_0(G) \to C_0(G, K)$ adjointable, notice that

$$T^*(g)(s)f(s) = (T^*(g)|f)(s) = (g|T(f))(s) = (g(s)|T(f)(s)) \qquad (s \in G).$$

It follows that if $f_1(s) = f_2(s) = 1$, then $(g(s)|T(f_1)(s)) = (g(s)|T(f_2)(s))$ for all g; thus $T(f_1)(s) = T(f_2)(s)$. Let this be $F(s)$, so $T(f) = fF$. Similar arguments show that F is continuous, and bounded. □

More links between multipliers and adjointable maps

Theorem

We have that $\mathcal{K}(A \otimes K) \cong A \otimes_{\min} B_0(K)$ as C^*-algebras, and thus $\mathcal{L}(A \otimes K) \cong M(A \otimes_{\min} B_0(K))$.

Proof.

Algebraically, we define

$$\Theta : \theta_{a\otimes\xi, b\otimes\eta} \mapsto ab^* \otimes \theta_{\xi,\eta},$$

where $\theta_{\xi,\eta} : K \to K; \gamma \mapsto \xi(\eta|\gamma)$, so $\theta_{\xi,\eta} \in B_0(K)$. It's easy to see that this defines a homomorphism between a dense subalgebra of $\mathcal{K}(A \otimes K)$ and a dense subalgebra of $A \otimes B_0(K)$.
So it remains to show that Θ is an isometry (or at least bounded) and so extends by continuity: this I believe is tricky! (See [Lance]). □

Back to groups

Remember that we have $W \in M\big(C_0(G) \otimes B_0(H)\big)$ for $H = L^2(G)$,

$$W\xi(s,t) = \xi(s, s^{-1}t) \qquad (s,t \in G).$$

Thus W induces some

$$\mathcal{W} \in \mathcal{L}(C_0(G) \otimes H) = \mathcal{L}\big(C_0(G,H)\big).$$

Homework: follow the isomorphisms through to show that

$$\mathcal{W} : f \mapsto \big(\lambda(s)f(s)\big)_{s\in G} \qquad (f \in C_0(G,H)).$$

(Indeed, if H is any Hilbert space, and σ is a unitary representation of G on H, then we can define $\mathcal{V} \in \mathcal{L}\big(C_0(G,H)\big)$ in the same way $f \mapsto (\sigma(s)f(s))_{s\in G}$. This induces a unitary $V \in M(C_0(G) \otimes B_0(H))$: this is the notion of a *corepresentation*.)

Slicing

Given $\xi \in K$, we can define

$$\iota \otimes \xi : A \otimes K \to A; \quad a \otimes \eta \mapsto a(\xi|\eta).$$

This is adjointable with

$$(\iota \otimes \xi)^* : A \to A \otimes K; \quad a \mapsto a \otimes \xi.$$

(This requires a small amount of work: homework!)

Fix a unit vector $\xi_0 \in K$. Using these maps, we can view $\mathcal{L}(A, A \otimes K)$ as a complemented submodule of $\mathcal{L}(A \otimes K)$:

$$\mathcal{L}(A, A \otimes K) \to \mathcal{L}(A \otimes K); \quad \alpha \mapsto \alpha(\iota \otimes \xi_0),$$
$$\mathcal{L}(A \otimes K) \to \mathcal{L}(A, A \otimes K); \quad \mathcal{T} \mapsto \mathcal{T}(\iota \otimes \xi_0)^*,$$

which follows, as $(\iota \otimes \eta)(\iota \otimes \xi)^* = (\eta|\xi)\,\mathrm{id}_A$.

Applications

Let $\alpha \in \mathcal{L}(A, A \otimes K)$ and $\mathcal{T} \in \mathcal{L}(A \otimes K)$ be related by $\alpha = \mathcal{T}(\iota \otimes \xi_0)^*$. Let $A \subseteq B(H)$. As $\mathcal{L}(A \otimes K) \cong M(A \otimes B_0(K))$, let $\mathcal{T}$ be related to $T \in M(A \otimes B_0(K)) \subseteq B(H \otimes K)$.

- We can define an operator $\tilde{\alpha} : H \to H \otimes K$ by $\tilde{\alpha}(\xi) = T(\xi \otimes \xi_0)$. Then:
  - $\tilde{\alpha}$ only depends upon α (and not $\mathcal{T}$).
  - Infact, $\tilde{\alpha}^* \tilde{\alpha} = \alpha^* \alpha \in \mathcal{L}(A) \cong M(A) \subseteq B(H)$.
  - This *is* the generalisation of the way we moved from $C^b(G, K)$ to $\mathcal{B}(L^2(G), L^2(G, K))$.
- Given a non-degenerate $*$-homomorphism $\phi : A \to B$, we can extend $\phi \otimes \iota$ to multiplier algebras, and hence define $S = (\phi \otimes \iota)T \in M(B \otimes B_0(K))$. Form $\mathcal{S}$ using S, and let $\phi * \alpha = \mathcal{S}(\iota \otimes \xi_0)^* \in \mathcal{L}(B, B \otimes K)$. Then:
  - $\phi * \alpha$ only depends upon α, not $\mathcal{T}$.
  - indeed, for any $\xi \in K$, we have that $(\iota \otimes \xi)(\phi * \alpha) = \phi\big((\iota \otimes \xi)\alpha\big)$ in $\mathcal{L}(B)$.

Example

For example, we can apply the second construction to Δ to give a map

$$\mathcal{L}(C_0(G), C_0(G) \otimes K) \to \mathcal{L}(C_0(G \times G), C_0(G \times G) \otimes K); \quad \alpha \mapsto \Delta * \alpha.$$

This is just the map

$$C^b(G, K) \to C^b(G \times G, K); \quad \alpha \mapsto \big(\alpha(st)\big)_{(s,t)\in G\times G}.$$

Gilbert's theorem asks that

$$f(st^{-1}) = \big(\beta(t)\big|\alpha(s)\big) \qquad (s, t \in G).$$

(Remember: inner-products are linear on the right now!) This is equivalent to

$$f(r) = \big(\beta(s)\big|\alpha(rs)\big) \qquad (s, r \in G).$$

In our abstract language, we get

$$(1 \otimes \beta)^*(\Delta * \alpha) = f \otimes 1 \in M(C_0(G) \otimes C_0(G)) = C^b(G \times G).$$