Multipliers of locally compact quantum groups and Hilbert C*-modules
1. Locally compact groups, duality, and multiplier algebras

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Locally compact groups

Let G be a locally compact group, and consider $C_0(G)$, $C^b(G)$ and $L^\infty(G)$. These are two C*-algebras and a von Neumann algebra: they depend only on the topological and measure space properties of G.

We turn $L^1(G)$ into a Banach algebra for the convolution product:

$$(f * g)(s) = \int_G f(t)g(t^{-1}s)\, dt.$$

This *does* remember the structure of G, in the following sense: if $L^1(G)$ and $L^1(H)$ are *isometrically* isomorphic as Banach algebras, then G is, as a topological group, isomorphic to H.

At the Operator algebra level

Define a map $\Delta : L^\infty(G) \to L^\infty(G \times G)$ by

$$\Delta(F)(s,t) = F(st) \qquad (F \in L^\infty(G), s,t \in G).$$

This is a unital, injective, $*$-homomorphism which is normal (weak*-continuous). The pre-adjoint is a map $L^1(G \times G) \to L^1(G)$. As $L^1(G) \otimes L^1(G)$ embeds into $L^1(G \times G)$, we get a bilinear map on $L^1(G)$. This is actually the convolution product, as

$$\begin{aligned}\langle F, \Delta_*(f \otimes g)\rangle = \langle \Delta(F), f \otimes g\rangle &= \int_{G\times G} F(st)f(s)g(t)\, ds\, dt \\ &= \int_G F(t) \int_G f(s)g(s^{-1}t)\, ds\, dt = \langle F, f * g\rangle.\end{aligned}$$

For the C*-algebra

Notice that we can also interpret Δ as a $*$-homomorphism $C_0(G) \to C^b(G \times G)$,

$$\Delta(F)(s,t) = F(st) \qquad (F \in C_0(G), s,t \in G).$$

but not as a map into $C_0(G \times G)$.

The map $\Delta : C_0(G) \to C^b(G \times G)$ "almost" maps into $C_0(G \times G)$. Indeed, for $f, g \in C_0(G)$,

$$\big((f \otimes 1)\Delta(g)\big)(s,t) = f(s)g(st) \to 0 \text{ as } (s,t) \to \infty.$$

So $(f \otimes 1)\Delta(g) \in C_0(G \times G)$, and similarly $(1 \otimes f)\Delta(g) \in C_0(G \times G)$.

Notice also that the linear span of elements of the form $(f \otimes 1)\Delta(g)$ is dense in $C_0(G \times G)$.

Multiplier algebras

For a C*-algebra A, we can regard A as being a self-adjoint closed subalgebra of $\mathcal{B}(H)$; or as A being a subalgebra of its bidual A^{**}. If A acts non-degenerately on H (so $\operatorname{lin}\{a(\xi) : a \in A, \xi \in H\}$ is dense in H) then the *multiplier algebra* of A is

$$\begin{aligned} M(A) &= \{T \in \mathcal{B}(H) : Ta, aT \in A \ (a \in A)\} \\ &= \{x \in A^{**} : xa, ax \in A \ (a \in A)\}. \end{aligned}$$

These are seen to be closed self-adjoint algebras containing A as an ideal.

An abstract way to think of $M(A)$ is as the pairs of maps (L, R) from A to A with $aL(b) = R(a)b$. A little closed graph argument shows that L and R are bounded, and that

$$L(ab) = L(a)b, \quad R(ab) = aR(b) \qquad (a, b \in A).$$

The involution in this picture is $(L, R)^* = (R^*, L^*)$ where $R^*(a) = R(a^*)^*$, $L^*(a) = L(a^*)^*$. You can move between these pictures by a bounded approximate identity argument.

$M(A)$ is the largest C*-algebra containing A as an *essential* ideal: if $x \in M(A)$ and $axb = 0$ for all $a, b \in A$, then $x = 0$.

Back to groups

We have that $M(C_0(G)) = C^b(G) = C(\beta G)$ (homework!)

Notice then that $\Delta : C_0(G) \to M(C_0(G \times G))$ (actually stronger, as $(f \otimes 1)\Delta(g) \in C_0(G \times G)$.)

A useful topology to put on $M(A)$ is the *strict* topology:

$$x_i \to x \quad \Leftrightarrow \quad x_i a \to xa, x_i^* a \to x^* a \quad (a \in A).$$

Theorem

Let $\theta : A \to M(B)$ be a $*$-homomorphism. Then the following are equivalent:

1. *θ is non-degenerate: $\operatorname{lin}\{\theta(a)b : a \in A, b \in B\}$ is dense in B;*
2. *$\theta(e_i) \to 1$ strictly for some (or all) bai's (e_i) in A;*
3. *there is an extension $\tilde\theta : M(A) \to M(B)$ which is unital, and strictly continuous on bounded sets.*

Notice that Δ is non-degenerate.

Group C*-algebras

We let G act on $L^2(G)$ by the left-regular representation:

$$(\lambda(s)f)(t) = f(s^{-1}t) \qquad (f \in L^2(G), s, t \in G).$$

The s^{-1} arises to make $G \mapsto B(H); s \mapsto \lambda(s)$ a group homomorphism.

We can integrate this to get a contractive homomorphism $\tilde\lambda : L^1(G) \to B(L^2(G))$. The action of $L^1(G)$ on $L^2(G)$ is just convolution.

Let the norm closure of $L^1(G)$ in $B(L^2(G))$ be $C_r^*(G)$, the (reduced) group C*-algebra. The weak-operator closure is $VN(G)$, the group von Neumann algebra. Equivalently, $VN(G)$ is $\{\lambda(s) : s \in G\}''$.

We claim that there is a normal, unital injective $*$-homomorphism $\Delta : VN(G) \to VN(G \times G)$ satisfying

$$\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s) = \lambda(s, s).$$

Here we identify $VN(G)\overline{\otimes}VN(G)$ with $VN(G \times G)$. *If* Δ exists, then it's uniquely defined by this property.

Construction of Δ

Define $\hat W : L^2(G \times G) \to L^2(G \times G)$ by

$$\hat W \xi(s, t) = \xi(ts, t) \qquad (\xi \in L^2(G \times G), \xi, \eta \in G).$$

Then $\hat W$ is unitary, and

$$\begin{aligned} (\hat W^*(1 \otimes \lambda(r))\hat W \xi)(s, t) &= ((1 \otimes \lambda(r))\hat W \xi)(t^{-1}s, t) \\ &= (\hat W \xi)(t^{-1}s, r^{-1}t) = \xi(r^{-1}s, r^{-1}t) \\ &= (\lambda(r) \otimes \lambda(r))\xi(s, t). \end{aligned}$$

So we could *define* Δ by

$$\Delta(x) = \hat W^*(1 \otimes x)\hat W \qquad (x \in VN(G)).$$

Then obviously Δ is an injective, unital, normal $*$-homomorphism, and $\Delta(\lambda(s)) = \lambda(s) \otimes \lambda(s)$, so by normality, Δ must map into $VN(G \times G)$.

At the C^*-algebra level

We expect that Δ should restrict to give a non-degenerate map $C_r^*(G) \to M(C_r^*(G \times G))$. This is indeed so:

- Notice that $\lambda(s) f * \xi = (s \cdot f) * \xi$ for $s \in G, f \in L^1(G), \xi \in L^2(G)$), where $(s \cdot f)(t) = f(s^{-1}t)$. So $\lambda(s)\tilde{\lambda}(L^1(G)) \subseteq \tilde{\lambda}(L^1(G))$.
- By density, $\lambda(s) \in M(C_r^*(G))$ for all s.
- So also $\lambda(s,s) \in M(C_r^*(G \times G))$, and we can integrate the map $G \to M(C_r^*(G \times G)); s \mapsto \lambda(s,s)$ to get a homomorphism $L^1(G) \to M(C_r^*(G \times G))$.
- This "is" the map Δ, so by density, we're done.

Checking that Δ is non-degenerate is a touch more work:

- We can find "nice" bai's in $L^1(G)$, and then Δ takes these to a strict bai in $M(C_r^*(G \times G))$;
- It's not too hard to check that for $f \in L^1(G)$ and $h \in L^1(G \times G)$, we have that $\Delta(\tilde{\lambda}_G(f))\tilde{\lambda}_{G\times G}(h) = \tilde{\lambda}_{G\times G}(g)$ for some $g \in L^1(G \times G)$. Then Δ is non-degenerate by density.

Back to $C_0(G)$

There is a unitary W associated to $C_0(G)$ and $L^\infty(G)$, given by $W\xi(s,t) = \xi(s, s^{-1}t)$, and which satisfies

$$\Delta(x) = W^*(1 \otimes x)W \qquad (x \in L^\infty(G) \text{ or } x \in C_0(G)).$$

The map $\tilde{\lambda} : L^1(G) \to C_r^*(G)$ is actually

$$\tilde{\lambda}(f) = (f \otimes \iota)W \qquad (f \in L^1(G)).$$

This needs some explanation!

- Given $\xi, \eta \in L^2(G)$, we define $(\omega_{\xi,\eta} \otimes \iota)W \in B(L^2(G))$ by

$$\big((\omega_{\xi,\eta} \otimes \iota)W\gamma \big| \delta\big) = \big(W(\xi \otimes \gamma)\big|\eta \otimes \delta\big) \qquad (\gamma, \delta \in L^2(G)).$$

- We let $f = \xi\overline{\eta} \in L^1(G)$ (pointwise product). Then part of the claim is that $(\omega_{\xi,\eta} \otimes \iota)W$ depends only on f.

Do the calculation!

$$\begin{aligned}\big((\omega_{\xi,\eta} \otimes \iota)W\gamma \big| \delta\big) &= \big(W(\xi \otimes \gamma)\big|\eta \otimes \delta\big) \\ &= \int_{G\times G} \xi(s)\gamma(s^{-1}t)\overline{\eta(s)\delta(t)}\; ds\; dt = (f * \gamma|\delta),\end{aligned}$$

Thus indeed $(\omega_{\xi,\eta} \otimes \iota)W = \tilde{\lambda}(f)$ where $f = \xi\overline{\eta}$.

Similarly, we calculate $(\iota \otimes \omega_{\xi,\eta})W$:

$$\begin{aligned}\big((\iota \otimes \omega_{\xi,\eta})W\gamma \big| \delta\big) &= \big(W(\gamma \otimes \xi)\big|\delta \otimes \eta\big) \\ &= \int_{G\times G} \gamma(s)\xi(s^{-1}t)\overline{\delta(s)\eta(t)}\; ds\; dt = (f\gamma|\delta).\end{aligned}$$

Here $f \in C_0(G)$ is the map $f(s) = \int_G \xi(s^{-1}t)\overline{\eta(t)}\; dt$. Such f are linearly dense in $C_0(G)$.

So W determines Δ, the algebra $C_0(G)$, and the map $\tilde{\lambda}$. In this sense, W completely determines G.

What happens for $VN(G)$?

Using the coproduct Δ, we can turn the predual of $VN(G)$ into a Banach algebra. This is the Fourier algebra $A(G)$: for the moment, we just view this abstract as the predual of $VN(G)$.

Given $\xi, \eta \in L^2(G)$, let $\omega_{\xi,\eta} \in A(G)$ be the normal functional

$$VN(G) \to \mathbb{C}; \quad x \mapsto \big(x(\xi)\big|\eta\big).$$

As $VN(G)$ is in *standard position* (big von Neumann algebra machinery) on $L^2(G)$, it follows that actually every member of $A(G)$ takes this form.

Let's try to define $\hat{\lambda} : A(G) \to C_0(G)$ by

$$\omega_{\xi,\eta} \mapsto (\omega_{\xi,\eta} \otimes \iota)\hat{W}.$$

The Fourier algebra $A(G)$

$$((\omega_{\xi,\eta} \otimes \iota)\hat{W}\gamma|\delta) = \int_{G\times G} \xi(ts)\gamma(t)\overline{\eta(s)\delta(t)}\ ds\ dt = (f\gamma|\delta),$$

where $f \in C_0(G)$ is the map $f(t) = \int_G \xi(ts)\overline{\eta(s)}\ ds = (\lambda(t^{-1})\xi|\eta)$. As λ is weak-operator continuous, it follows immediately that f is continuous, and it's easy to see that actually $f \in C_0(G)$. So

$$\hat{\lambda}(\omega_{\xi,\eta}) = (\omega_{\xi,\eta} \otimes \iota)\hat{W} = f, \quad f(t) = \langle \lambda(t^{-1}), \omega_{\xi,\eta} \rangle.$$

As $\{\lambda(t^{-1}) : t \in G\}$ generates $VN(G)$, we see that $\hat{\lambda}$ is injective.

For $\omega_1, \omega_2 \in A(G)$, we have that

$$\begin{aligned}\hat{\lambda}(\omega_1\omega_2)(t) &= \langle \lambda(t^{-1}), \omega_1\omega_2 \rangle = \langle \Delta(\lambda(t^{-1})), \omega_1 \otimes \omega_2 \rangle \\ &= \langle \lambda(t^{-1}) \otimes \lambda(t^{-1}), \omega_1 \otimes \omega_2 \rangle = \hat{\lambda}(\omega_1)(t)\hat{\lambda}(\omega_2)(t).\end{aligned}$$

Thus $\hat{\lambda}$ is a homomorphism. It is usual to identify $A(G)$ with it's image in $C_0(G)$; so $A(G)$ is a commutative Banach algebra, dense in $C_0(G)$ (and actually with spectrum G).

Finishing the duality picture

We perform a similar calculation:

$$\begin{aligned}((\iota \otimes \omega_{\xi,\eta})\hat{W}\gamma|\delta) &= (\hat{W}(\gamma \otimes \xi)|\delta \otimes \eta) = \int_{G\times G} \gamma(ts)\xi(t)\overline{\delta(s)\eta(t)}\ ds\ dt \\ &= \int_{G\times G} \gamma(t^{-1}s)\xi(t^{-1})\nabla(t^{-1})\overline{\delta(s)\eta(t^{-1})}\ ds\ dt \\ &= \int_G \int_G \xi(t^{-1})\nabla(t^{-1})\overline{\eta(t^{-1})}\gamma(t^{-1}s)\ dt\ \overline{\delta(s)}\ ds \\ &= (f * \gamma|\delta),\end{aligned}$$

where $f \in L^1(G)$ is the function $f(t) = \xi(t^{-1})\nabla(t^{-1})\overline{\eta(t^{-1})}$. Here ∇ is the group modular function.

So we have that operators of the form $(\iota \otimes \omega_{\xi,\eta})\hat{W}$ are linearly dense in $C_r^*(G)$.

Again, $\hat{W}$ allows us to build the algebra $C_r^*(G)$, the coproduct Δ and the map $\hat{\lambda}$.

In fact, $\hat{W} = \sigma W^*\sigma$, where $\sigma \in B(L^2(G \times G))$ is the swap map $\sigma\xi(s,t) = \xi(t,s)$.

Where does W live?

We have been considering W as a unitary in $B(L^2(G \times G))$; however, right slices of W land in $C_0(G)$, and left slices in $C_r^*(G)$.

Set $H = L^2(G)$. The C*-algebra $C_0(G) \otimes B_0(H)$ is the closure of the algebraic tensor product $C_0(G) \odot B_0(H)$ acting on $L^2(G \times G)$; here $B_0(H)$ is the compact operators on H. We can thus identify $M\big(C_0(G) \otimes B_0(H)\big)$ with a subalgebra of $B(L^2(G \times G))$.

Theorem

We can identify $C_0(G) \otimes B_0(H)$ with $C_0(G, B_0(H))$, the (norm) continuous functions $f : G \to B_0(H)$ which vanish at infinity. This identifies $a \otimes T$ with f where $f(s) = a(s)T$.

Proof.

Consider $L^2(G \times G)$ as the vector-valued $L^2(G, H)$. Then $C_0(G, B_0(H))$ acts on $L^2(G, H)$ pointwise: $(f\xi)(s) = f(s)\xi(s)$. It follows that $C_0(G, B_0(H))$ is isometrically represented on $L^2(G \times G)$, in a way compatible with the action of $C_0(G) \odot B_0(H)$. It remains to show that $C_0(G) \odot B_0(H)$ is dense in $C_0(G, B_0(H))$: this follows by a partition of unity argument. □

Where does W live? (continued)

Recall that we identified $M(C_0(G))$ with $C^b(G)$. The multiplier algebra of $B_0(H)$ is simply $B(H)$.

Similarly, we can identify $M\big(C_0(G, B_0(H))\big)$ with $C^b_{str}(G, B(H))$, the bounded functions $F : G \to B(H)$ which are strictly continuous. Such a function F acts on $L^2(G, H)$ by $(F\xi)(s) = F(s)\xi(s)$.

Theorem

The operator W is a member of $M\big(C_0(G) \otimes B_0(H)\big) = C^b_{str}(G, B(H))$.

Proof.

Under the identification $L^2(G \times G) = L^2(G, H)$, W acts as $(W\xi)(s) = \lambda(s)\xi(s)$. Thus $W \in C^b_{str}(G, B(H))$ is the map $s \mapsto \lambda(s)$; we simply check that this is strictly continuous. □

Where does $\hat{W}$ live?

Theorem

The operator $\hat{W}$ is a member of $M\big(C_r^*(G) \otimes B_0(H)\big)$.

Proof.

Homework! (Actually, I know of no particularly nice proof). □

Theorem

The operator W is a member of $M\big(C_0(G) \otimes C_r^*(G)\big)$.

Proof.

Again, $C_0(G) \otimes C_r^*(G) = C_0(G, C_r^*(G))$ and the multiplier algebra is $C^b_{str}(G, M(C_r^*(G)))$. Then check that $s \mapsto \lambda(s)$ does map into $M(C_r^*(G))$, and is strictly continuous. □