Topological centres and SIN quantum groups

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Recall: Topological centres

Let A be a Banach algebra with a faithful multiplication. Left and right Arens products on A^{**} extend the multiplication on A.

- The left and right topological centres of A^{**} are

$$\mathfrak{Z}_t(A^{**},\Box) = \{m \in A^{**} : n \longmapsto m\Box n \text{ is } w^*\text{-}w^* \text{ cont.}\},$$

$$\mathfrak{Z}_t(A^{**},\Diamond) = \{m \in A^{**} : n \longmapsto n\Diamond m \text{ is } w^*\text{-}w^* \text{ cont.}\}.$$

- The canonical quotient map $q : A^{**} \longrightarrow \langle A^*A\rangle^*$ yields

$$(\langle A^*A\rangle^*,\Box) \cong (A^{**},\Box)/\langle A^*A\rangle^{\perp}.$$

- The topological centre of $\langle A^*A\rangle^*$ is

$$\mathfrak{Z}_t(\langle A^*A\rangle^*) = \{m \in \langle A^*A\rangle^* : n \longmapsto m\Box n \text{ is } w^*\text{-}w^* \text{ cont.}\}.$$

Algebraic descriptions of topological centres

- We have

$$\mathfrak{Z}_t(A^{**},\Box) = \{m \in A^{**} : m\Box n = m\Diamond n \ \forall \ n \in A^{**}\},$$

$$\mathfrak{Z}_t(A^{**},\Diamond) = \{m \in A^{**} : n\Box m = n\Diamond m \ \forall \ n \in A^{**}\}.$$

- If $\langle A^* A \rangle$ is two-sided introverted in A^*, then $\Diamond$ is also defined on $\langle A^* A \rangle^*$. In this case,

$$\mathfrak{Z}_t(\langle A^* A \rangle^*) = \{m \in \langle A^* A \rangle^* : m\Box n = m\Diamond n \ \forall \ n \in \langle A^* A \rangle^*\}.$$

- **Question**: In general, can $\mathfrak{Z}_t(\langle A^* A \rangle^*)$ also be described in terms of TWO products?

Right-left subalgebras and quotient algebras

We define $A^{**}_R \;:=\; \{m \in A^{**} : \langle A^*A\rangle \Diamond m \subseteq \langle A^*A\rangle\}$.

- A^{**}_R is a subalgebra of $(A^{**}, \Diamond)$.
- Let $\langle A^*A\rangle^*_R := q(A^{**}_R) = \{m \in \langle A^*A\rangle^* : \langle A^*A\rangle \Diamond m \subseteq \langle A^*A\rangle\}$.

  Then

  $$(\langle A^*A\rangle^*_R, \Diamond) \;\cong\; (A^{**}_R, \Diamond)/\langle A^*A\rangle^{\perp}\,.$$

  $\langle A^*A\rangle^*_R = \langle A^*A\rangle^*$ iff $\langle A^*A\rangle$ is two-sided introverted in A^*.
- Both A^{**}_R and $\langle A^*A\rangle^*_R$ are left topological semigroups.

  We can also consider $\mathfrak{Z}_t(A^{**}_R)$ and $\mathfrak{Z}_t(\langle A^*A\rangle^*_R)$.
- More general, for any left introverted subspace X of A^*,

  the algebra X^*_R can be defined.

An algebraic description of $\mathfrak{Z}_t(\langle A^*A\rangle^*)$

Proposition. (H.-N.-R.) Let A be a Banach algebra. Then

$$\mathfrak{Z}_t(\langle A^*A\rangle^*) \;=\; \{m \in \langle A^*A\rangle^*_R : \; m\Box n \;=\; m\Diamond n \quad \forall\, n \in \langle A^*A\rangle^*\}.$$

Corollary. If $m \in \langle A^*A\rangle^*$, then

$$m \in \mathfrak{Z}_t(\langle A^*A\rangle^*) \quad\Longleftrightarrow\quad A\cdot m \subseteq \mathfrak{Z}_t(A^{**},\Box)\,.$$

Corollary. If $\langle A^2\rangle = A$ (e.g., $A = L_1(\mathbb{G})$), then

$$A\cdot \mathfrak{Z}_t(A^{**},\Box) \subseteq A \quad\Longleftrightarrow\quad A\cdot \mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq A\,.$$

Strong identity of $\langle A^*A\rangle^*$

- Recall: If $\langle A^2\rangle = A$, then A has a BRAI iff $\langle A^*A\rangle^*$ is unital (Grosser-Losert 84).

So, a LCQG $\mathbb{G}$ is co-amenable iff $(LUC(\mathbb{G})^*, \Box)$ is unital, where $LUC(\mathbb{G}) = \langle L_\infty(\mathbb{G}) \star L_1(\mathbb{G})\rangle$.

- If e is an identity of $(\langle A^*A\rangle^*, \Box)$, then e is a left identity of $(\langle A^*A\rangle^*_R, \Diamond)$.

- $e \in \langle A^*A\rangle^*$ is called a **strong identity** if e is an identity of $(\langle A^*A\rangle^*, \Box)$ and an identity of $(\langle A^*A\rangle^*_R, \Diamond)$.

When does $\langle A^*A\rangle^*$ have a strong identity?

Proposition. (H.-N.-R.) Suppose that $\langle A^2\rangle = A$. T.F.A.E.

(i) $\langle A^*A\rangle^*$ has a strong identity;

(ii) $\langle A^*A\rangle^*_R$ is right unital;

(iii) A has a BRAI and $\langle A^*A\rangle = \langle AA^*A\rangle$;

(iv) $id \in \mathfrak{Z}_t(\langle A^*A\rangle^*_R)$,

where $\langle A^*A\rangle^* \subseteq B(A^*)$ canonically.

SIN quantum groups

- Recall: A LCG G is SIN if e_G has a basis of compact sets invariant under inner automorphisms.

  It is known that G is SIN iff $LUC(G) = RUC(G)$ (Milnes 90).

- A LCQG $\mathbb{G}$ is called **SIN** if $LUC(\mathbb{G}) = RUC(\mathbb{G})$.

  This class includes: discrete, compact, co-commutative $\mathbb{G}$, and $\mathbb{G}$ with $L_1(\mathbb{G})$ having a central approximate identity.

Corollary. T.F.A.E.

(i) $\mathbb{G}$ is a co-amenable SIN quantum group;

(ii) $LUC(\mathbb{G})^*_R$ is right unital;

(iii) $LUC(\mathbb{G})^*$ has a strong identity;

(iv) $id \in \mathfrak{Z}_t(LUC(\mathbb{G})^*_R)$.

The commutative quantum group case

Let G be a locally compact group.

- Recall: For $m \in LUC(G)^*$ and $f \in LUC(G)$,

$$m_r(f)(s) := \langle m, f_s \rangle \quad (s \in G).$$

$$Z_U(G) := \{m \in LUC(G)^* : m_r(f) \in LUC(G) \ \forall f \in LUC(G)\}.$$

- For $f \in LUC(G)$, $m \in Z_U(G)$, and $n \in LUC(G)^*$, let

$$\langle f, m * n \rangle := \langle m_r(f), n \rangle .$$

Then $(Z_U(G), *)$ is a Banach algebra.

The commutative quantum group case

- $\mathfrak{Z}_t(LUC(G)^*) = \{m \in Z_U(G) : m \Box n = m * n \; \forall n \in LUC(G)^*\}$ (Lau 86).

By our algebraic description of $\mathfrak{Z}_t(\langle A^* A \rangle^*)$, we obtained

$$\mathfrak{Z}_t(LUC(G)^*) = \{m \in LUC(G)^*_R : m \Box n = m \Diamond n \; \forall n \in LUC(G)^*\}.$$

- **Question:** Do we have $(LUC(G)^*_R, \Diamond) = (Z_U(G), *)$?
- **Answer:** They are equal iff G is SIN.

The commutative quantum group case

- Note that for any Banach algebra A and any left introverted subspace X of A^*, the algebra X_R^* can be defined.

  We shall see that $Z_U(G)$ has the form X_R^*.

- $LUC_{\ell_\infty}(G) \ := \ LUC(G)$ as a subspace of $\ell_\infty(G)$.

- $LUC_{\ell_\infty}(G)$ is left introverted in $\ell_\infty(G) = \ell_1(G)^*$.

  Then $(LUC_{\ell_\infty}(G)^*, \Box_{\ell_1})$ and $(LUC_{\ell_\infty}(G)^*_R, \Diamond_{\ell_1})$ are defined.

So, there are five Banach algebras associated with $LUC(G) \cdots$

The five Banach algebras associated with $LUC(G)$

In general, we have $(LUC_{\ell_\infty}(G)^*, \Box_{\ell_1}) \;=\; (LUC(G)^*, \Box)\,;$

$$(Z_U(G), *) \;=\; (LUC_{\ell_\infty}(G)^*_R, \Diamond_{\ell_1}) \;\neq\; (LUC(G)^*_R, \Diamond)\,.$$

- So, $(Z_U(G), *)$ has the form $(X^*_R, \Diamond)$.

It can be seen that T.F.A.E.

(i) $LUC(G)^* \;=\; LUC(G)^*_R\,;$

(ii) G is SIN;

(iii) $LUC_{\ell_\infty}(G)^* \;=\; LUC_{\ell_\infty}(G)^*_R\,.$

Note that the equalities in (i) and (iii) are equalities of SPACES.

Some algebraic characterizations of SIN groups

Theorem. (H.-N.-R.) Let G be a locally compact group. T.F.A.E.

(i) G is SIN;

(ii) $(LUC(G)^*_R, \Diamond) = (Z_U(G), *)$;

(iii) $LUC(G)^*_R$ is a subalgebra of $Z_U(G)$;

(iv) $\delta_e \in \mathfrak{Z}_t(LUC(G)^*_R)$;

(v) $(LUC(G)^*_R, \Diamond)$ is unital;

(vi) $LUC(G)^*$ has a strong identity.

- In (iv), (v), $LUC(G)^*_R$ cannot be replaced by $Z_U(G)$, since δ_e is always an identity of $(Z_U(G), *)$.

Compact and discrete groups

- In general, the three algebras $(LUC(G)^*, \Box)$, $(LUC(G)^*_R, \Diamond)$, and $(Z_U(G), *)$ are different.
- G is compact $\iff$ $(LUC(G)^*, \Box) = (LUC(G)^*_R, \Diamond)$.

  In this case, $(LUC(G)^*, \Box) = (LUC(G)^*_R, \Diamond) = (Z_U(G), *)$.
- G is discrete $\iff$ $(UC(\widehat{G})^*, \Box) = (UC(\widehat{G})^*_R, \Diamond)$.
- The equivalence holds for some general quantum groups.

An auxiliary topological centre of $\langle A^* A \rangle^*$ – motivation

- **Some asymmetry phenomena** (Lau-Ülger 96; H.-N.-R.):

$$\mathfrak{Z}_t(\langle A^* A \rangle^*) = RM(A) \quad \Longleftrightarrow \quad A \cdot \mathfrak{Z}_t(A^{**}, \Box) \subseteq A\,;$$

$$\mathfrak{Z}_t(A^{**}, \Box) = A \quad \Longleftrightarrow \quad \mathfrak{Z}_t(A^{**}, \Box) \cdot A \subseteq A\,.$$

- **Interrelationship between topological centre problems:**

$$m \in \mathfrak{Z}_t(\langle A^* A \rangle^*) \quad \Longleftrightarrow \quad A \cdot m \subseteq \mathfrak{Z}_t(A^{**}, \Box)\,;$$

$$m \in \;? \quad \Longleftrightarrow \quad A \cdot m \subseteq \mathfrak{Z}_t(A^{**}, \Diamond)\,.$$

- **Automatic normality problem for certain right A-module maps on A^*.**

One subspace of $\langle A^*A\rangle^*$ can help for all of these problems.

Definition. (H.-N.-R.) For a Banach algebra A, the **auxiliary topological centre** of $\langle A^*A\rangle^*$ is defined by

$$\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond = \{m \in \langle A^*A\rangle^* : n\Diamond m = n\Box m \text{ in } A^{**}\ \forall n \in \langle A^{**}A\rangle\}.$$

Similarly, $\mathfrak{Z}_t(\langle AA^*\rangle^*)_\Box$ can be defined.

- $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond = \mathfrak{Z}_t(\langle A^*A\rangle^*)$ if $\mathfrak{Z}_t(A^{**},\Box) = \mathfrak{Z}_t(A^{**},\Diamond)$.
- Under the canonical quotient map $q : A^{**} \longrightarrow \langle A^*A\rangle^*$,

  $$\mathfrak{Z}_t(A^{**},\Box) \longrightarrow \mathfrak{Z}_t(\langle A^*A\rangle^*), \quad \mathfrak{Z}_t(A^{**},\Diamond) \longrightarrow \mathfrak{Z}_t(\langle A^*A\rangle^*)_\Diamond .$$

$\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamondsuit$ – some applications

- For $m \in \langle A^*A\rangle^*$, we have

$$m \in \mathfrak{Z}_t(\langle A^*A\rangle^*) \quad \Longleftrightarrow \quad A\cdot m \subseteq \mathfrak{Z}_t(A^{**}, \square)\,;$$

$$m \in \mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamondsuit \quad \Longleftrightarrow \quad A\cdot m \subseteq \mathfrak{Z}_t(A^{**}, \diamondsuit)\,.$$

- If $\langle A^2\rangle = A$ (e.g., $A = L_1(\mathbb{G})$), then

$$A\cdot \mathfrak{Z}_t(A^{**}, \square) \subseteq A \quad \Longleftrightarrow \quad A\cdot \mathfrak{Z}_t(\langle A^*A\rangle^*) \subseteq A\,;$$

$$A\cdot \mathfrak{Z}_t(A^{**}, \diamondsuit) \subseteq A \quad \Longleftrightarrow \quad A\cdot \mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamondsuit \subseteq A\,.$$

Proposition. (H.-N.-R.) If A is of type (M), then

$$\mathfrak{Z}_t(A^{**},\Box) = A \quad\Longleftrightarrow\quad \mathfrak{Z}_t(\langle AA^*\rangle^*)_\Box = LM(A)\,;$$

$$\mathfrak{Z}_t(A^{**},\diamondsuit) = A \quad\Longleftrightarrow\quad \mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamondsuit = RM(A)\,.$$

- Surprisingly, **LSAI** and **RSAI** of A are not related to the usual topo centres $\mathfrak{Z}_t(\langle A^*A\rangle^*)$ and $\mathfrak{Z}_t(\langle AA^*\rangle^*)$, but related to **auxiliary topo centres** $\mathfrak{Z}_t(\langle AA^*\rangle^*)_\Box$ and $\mathfrak{Z}_t(\langle A^*A\rangle^*)_\diamondsuit$.

Corollary. If A is of type (M) with $\mathfrak{Z}_t(A^{**},\Box) = \mathfrak{Z}_t(A^{**},\Diamond)$ (e.g., A is commutative), then

$$A \text{ is SAI} \quad \Longleftrightarrow \quad \mathfrak{Z}_t(\langle A^*A\rangle^*) \; = \; RM(A)\,.$$

- "$\Longleftarrow$" was shown by Lau-Losert (93) for $A(G)$ with G amenable.
- There exist unital WSC Banach algebras A such that $\mathfrak{Z}_t(A^{**},\Box) \; = \; A \; \subsetneqq \; \mathfrak{Z}_t(A^{**},\Diamond)$. In this case, the above equivalence does not hold.

Module homomorphisms on A^*

$$B_A(A^*) := \text{bounded right } A\text{-module maps on } A^*.$$

$$B_A^\sigma(A^*) := \text{normal bounded right } A\text{-module maps on } A^*.$$

$$B_{A^{**}}(A^*) := \text{bounded right } (A^{**}, \diamondsuit)\text{-module maps on } A^*.$$

- $RM(A) \cong B_A^\sigma(A^*) \subseteq B_{A^{**}}(A^*) \subseteq B_A(A^*)$.

  In fact, we have

$$B_{A^{**}}(A^*) = \{T \in B_A(A^*) : T^*(A) \subseteq \mathfrak{Z}_t(A^{**}, \diamondsuit)\}.$$

The canonical representation of $\langle A^*A\rangle^*$ on A^*

- Let $\Phi : \langle A^*A\rangle^* \longrightarrow B_A(A^*)$ be the contractive and injective algebra homo $m \longmapsto m_L$, where $m_L(f) = m\Box f$.

  Then Φ is surjective if A has a BRAI.

- Let A be a completely contractive Banach algebra. Then

$$\Phi : \langle A^*A\rangle^* \longrightarrow CB_A(A^*)$$

  is a c.c. algebra homomorphism. If A has a BRAI, then

$$\Phi(\langle A^*A\rangle^*) \subseteq CB_A(A^*) \subseteq B_A(A^*) = \Phi(\langle A^*A\rangle^*);$$

  in this case, we have

$$B_A(A^*) = CB_A(A^*) \quad \text{and} \quad RM(A) = RM_{cb}(A).$$

Topological centres and automatic normality

Using the canonical repn $\Phi : \langle A^* A \rangle^* \longrightarrow B_A(A^*)$, we can study Arens irregularity properties of A through module maps on A^*.

For example, we have the following generalization of a result by Neufang (00) on $L_1(G)$.

Proposition. (H.-N.-R.) If A is of type (M). T.F.A.E.

(i) $\mathfrak{Z}_t(A^{**}, \Diamond) = A$;

(ii) $B_{A^{**}}(A^*) = B_A^\sigma(A^*)$.

Commutation relations

Consider the two sequences:

$$B^{\sigma}_{A}(A^*) \quad \subseteq \quad B_{A^{**}}(A^*) \quad \subseteq \quad B_A(A^*)\,;$$

$$_{A}B(A^*)^c \quad \subseteq \quad {}_{A^{**}}B(A^*)^c \quad \subseteq \quad {}_{A}B^{\sigma}(A^*)^c\,,$$

where "c" denotes commutant in $B(A^*)$.

- If $\langle A^2 \rangle = A$, then

$$B^{\sigma}_{A}(A^*) \subseteq {}_{A}B(A^*)^c \subseteq B_{A^{**}}(A^*) \subseteq B_A(A^*)$$
$$= {}_{A^{**}}B(A^*)^c = {}_{A}B^{\sigma}(A^*)^c\,.$$

- If A has a BLAI, then

$$B^{\sigma}_{A}(A^*) \subseteq {}_{A}B(A^*)^c = B_{A^{**}}(A^*) \subseteq B_A(A^*)$$
$$= {}_{A^{**}}B(A^*)^c = {}_{A}B^{\sigma}(A^*)^c\,.$$

SAI and bicommutant theorem

In the following, $LM(A),\ RM(A)\ \subseteq\ B(A^*)$.

Proposition. (H.-N.-R.) Let A be a Banach algebra of type (M).

(i) A is LSAI $\iff$ $LM(A)^{cc} = LM(A)$;

(ii) A is RSAI $\iff$ $RM(A)^{cc} = RM(A)$.

- There is even a unital WSC A which is LSAI but not RSAI.

  So, the above bicommutation relations are not equivalent.

Corollary. Let A be a unital WSC involutive Banach algebra (e.g., $A = L_1(\mathbb{G})$ of discrete $\mathbb{G}$). Then

$$A \text{ is SAI} \iff A^{cc} = A .$$

The convolution quantum group algebra case

Let $\mathbb{G}$ be a LCQG. In the following, "c" is taken in $B(L_\infty(\mathbb{G}))$.

Corollary. If $L_1(\mathbb{G})$ separable, T.F.A.E.

(i) $M(\mathbb{G})^{cc} = M(\mathbb{G})$;

(ii) $\mathbb{G}$ is co-amenable and $L_1(\mathbb{G})$ is SAI .

Proposition. (H.-N.-R.)

$$\mathbb{G} \text{ is compact} \iff RM(L_1(\mathbb{G}))^c = LM(L_1(\mathbb{G})) .$$

The Fourier algebra case

Corollary. Let G be a locally compact group.

(i) $B(G)^{cc} = B(G) \iff G$ is amenable and $A(G)$ is SAI.

(ii) $A(G)^{cc} = A(G) \iff G$ is compact and $A(G)$ is SAI.

(1) $B(G)^{c} = B(G) \iff G$ is amenable and discrete.

(2) $A(G)^{c} = A(G) \iff G$ is finite.

- The above $B(G)$ can also be replaced by $B_\lambda(G)$.