Dual factorization property

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Background

Arens Product

For $m, n \in A^{**}$, $f \in A^*$ and $a, b \in A$

Left Arens Product : $(A^{**}, \Box)$.

$$\begin{aligned} \langle m \Box n, f \rangle &= \langle m, n \Box f \rangle \\ \langle n \Box f, a \rangle &= \langle n, f \Box a \rangle \\ \langle f \Box a, b \rangle &= \langle f, ab \rangle \end{aligned}$$

Right Arens product : $(A^{**}, \triangle)$.

Arens Product

Left topological centre of A^{**} is defined by

$$\begin{aligned} Z_\ell(A^{**}) &= \{m \in A^{**} \mid m\Box n = m\triangle n, \text{ for every } \; n \in A^{**}\} \\ &= \{m \in A^{**} \mid \lambda_m \text{ is } w^* - w^* - \text{continuous}\} \end{aligned}$$

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- **Arens regular** if $Z_\ell(A^{**}) = A^{**}$
- **left strongly Arens irregular** if $Z_\ell(A^{**}) = A$

Multipliers

A **left multiplier on** A is a bounded linear operator $T : A \longrightarrow A$ such that for $a, b \in A$,

$$T(ab) = T(a)b$$

$LM(A)$ and $RM(A)$ denote respectively the left and right multipliers of A .

Strong Topological Centre

Strong Topological Center

Definition

(Neufang, P.) *The strong left topological centre of A^{**} is defined by*

$$SZ_\ell(A^{**}) = \{m \in A^{**} : \lambda_m = T^{**}, \text{ for some } T \in B(A)\}$$

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$$\begin{aligned} \lambda_m : A^{**} &\longrightarrow A^{**} \\ (\lambda_m)_* : A^* &\longrightarrow A^* \\ (\lambda_m)_{**} : A &\longrightarrow A \end{aligned}$$

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Position in A^{**}

$$A \subseteq SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**}) \subseteq A^{**}$$

Strong Topological Centre

Theorem

(Hu, N., R.)

$$SZ_\ell(A^{**}) = Z_\ell(A^{**}) \cap \{m \in A^{**} \mid m \Box A \subseteq A\}$$

Strong Topological Centre

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$$SZ_\ell(A^{**}) = Z_\ell(A^{**}) \cap \{m \in A^{**} \mid m\Box A \subseteq A\}$$

Corollary

1. *if A is an ideal in A^{**} then $SZ_\ell(A^{**}) = Z_\ell$.*
2. *If A is Arens regular then $SZ_\ell(A^{**}) = \{m \in A^{**} \mid m\Box A \subseteq A\}$. In particular for C^*-algebras.*

Difference left / right

Example

(Neufang, P.) Let S be a right zero semigroup i.e.

$$s_1 \cdot s_2 = s_2 \qquad (\forall s_1, s_2 \in S).$$

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$$SZ_\ell((l^1(S))^{**}) = (l^1(S))^{**}$$

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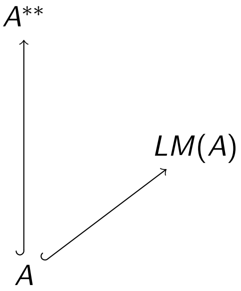
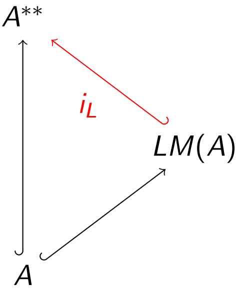
$$\begin{array}{rcl} SZ_\ell((l^1(S))^{**}) & = & (l^1(S))^{**} \\ SZ_r((l^1(S))^{**}) & = & l^1(S) \end{array}$$


A^{**}
$LM(A)$
A


A^{**}
i_L
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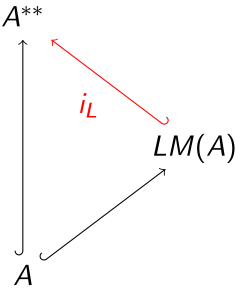
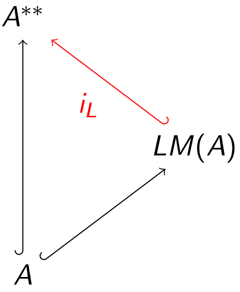


Proposition

(Neufang, P.) *Let A be a Banach algebra with a BAI. Let $\mathcal{E}$ be a fixed mixed unit.*

$$i_L(LM(A)) = \{m \in A^{**} \mid m \Box A \subseteq A \text{ and}$$



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$$i_L(LM(A)) = \{m \in A^{**} \mid m \Box A \subseteq A \text{ and } m \triangle \mathcal{E} = m\}$$

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(Neufang-P.) Let c_0 and c the Banach algebra of all sequences converging to 0 and of all the convergent sequences respectively.

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$$(c_0)^{**} = (c)^{**} = l^\infty \qquad LM(c_0) = l^\infty \qquad LM(c) = c.$$

$$\begin{aligned} SZ_l((c_0)^{**}) &= l^\infty \\ SZ_l((c)^{**}) &= c \end{aligned}$$

Possible cases (Neufang, P.)

$$A \subseteq SZ_\ell(A^{**}) \subseteq Z_\ell(A^{**}) \subseteq A^{**}$$

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- $A = SZ_\ell = Z_\ell$
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- $SZ_\ell = A$ and $Z_\ell = A^{**}$
  - Unital C^*-algebra

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  - $(l^1, \cdot)$

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- $A = SZ_\ell \subsetneq Z_\ell \subsetneq A^{**}$
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- $A \subsetneq SZ_\ell = LM(A) \subsetneq Z_\ell = A^{**}$
  - Non-unital C^*-algebra
- $A \subsetneq SZ_\ell(A^{**}) = Z_\ell \subsetneq A^{**}$
  - The nuclear operators $(N(l^p(G)), *)^{op}$ where G is a locally compact discrete group.

Strong Topological Centre

Corollary

(Hu, N., R.) *Let* $\mathbb{G}$ *be a locally compact co-amenable quantum group then*

$$SZ_\ell(L^1(\mathbb{G})) = L^1(\mathbb{G})$$

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Corollary

(Hu, N., R.) *Let G be a locally compact amenable group, then*

$$SZ_\ell(A(G)^{**}) = A(G)$$

Note that $Z_\ell(A(SU(3))^{**}) \neq A(SU(3))$ but $SU(3)$ is compact

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Corollary

(Hu, N., R.) *Let G be a locally compact amenable group, then*

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Corollary

(Lau) *Let G be a locally compact amenable group. If $A(G)$ is Arens regular, then G is finite.*

Strong Topological Centre

Theorem

(Neufang, P.) *Let A be a Banach algebra with a BAI*

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Theorem

(Neufang,P.) *Let A be a Banach algebra with a BAI*

$$SZ_\ell(A^{**}) = LM(A) \iff A^* = A^*A$$

Dual factorization property

QUESTION

Characterize the Banach algebra A such that $A^* = A^*A$

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(Granirer) For $A(G)$, this property implies that G is **compact**

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3. *left weak dual factorization property* if $A^* = \overline{< A^*A >}$.

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- The space l^2 equipped with pointwise multiplication has the dual factorization property but not the strong dual factorization property.

Dual Factorization Property

Example

- **(Neufang, P.)** A reflexive Banach algebra A has the left and right weak dual factorization property.
- The space l^2 equipped with pointwise multiplication has the dual factorization property but not the strong dual factorization property.
- **(Neufang, P.)** $A(E)$ has the left weak dual factorization property if and only if $I(E^*) = \overline{N(E^*)}^{\|\cdot\|_I}$

With a BAI

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Theorem

(Lau, Ü.) *Let A be a Banach algebra with a BAI, then $A^* = A^*A$ if and only if $(A^{**}, \Box)$ is unital.*

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Theorem

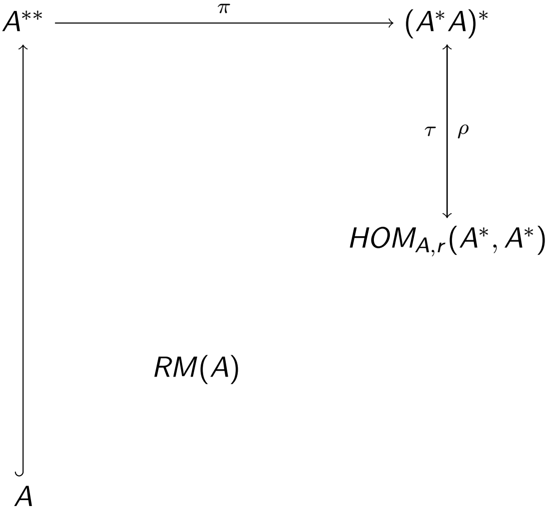
(Lau, Ü.) *Let A be a Banach algebra with a BAI, then $A^* = A^*A$ if and only if*

$$\{m \in A^{**} \mid A \cdot m \subseteq A\} \subseteq Z_\ell(A^{**}).$$

Picture of the situation (with a BAI)

$$
\begin{array}{ccc}
A^{**} & \xrightarrow{\pi} & (A^*A)^* \\
\uparrow & & \tau \updownarrow \rho \\
 & & HOM_{A,r}(A^*, A^*) \\
\Big\uparrow & & \\
A & &
\end{array}
$$

Picture of the situation (with a BAI)



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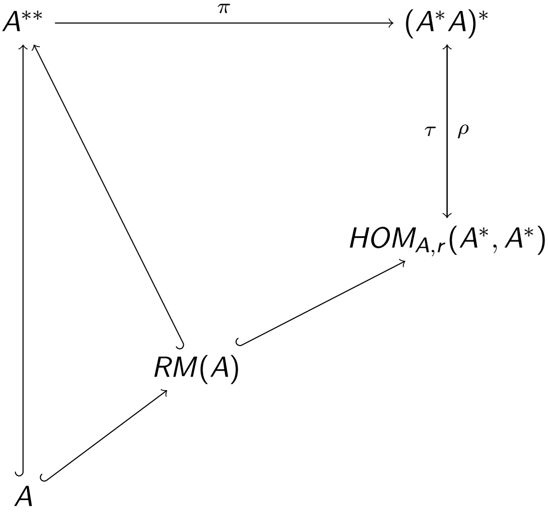

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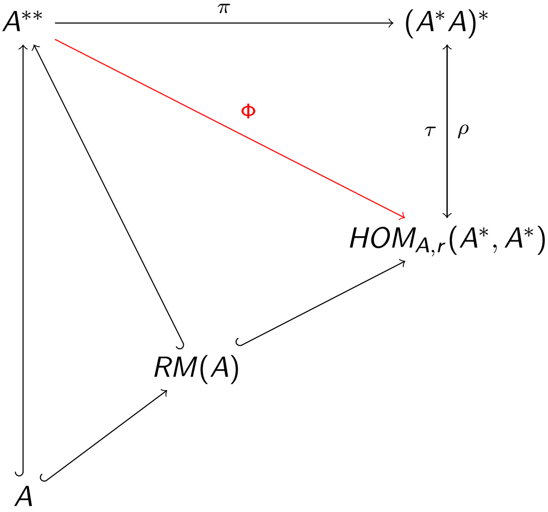

With a BAI

Theorem

(Neufang, P.) *Let A be a Banach algebra with a BAI Then*

1. $A^* = A^*A$ *if and only if*

$$RM(A) = \{m \in A^{**} \mid A \cdot m \subseteq A\}$$

2. $A^* = AA^*$ *if and only if*

$$LM(A) = \{m \in A^{**} \mid m \cdot A \subseteq A\}$$

When A has no BAI

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Definition

Let $(A, \| \|_A)$ be a Banach algebra. A Banach algebra $(B, \| \|_B)$ is a right abstract Segal algebra in A if the following conditions are satisfied :

1. *The algebra B is a dense right ideal of A,*

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3. *There is a constant $M > 0$ such that for each $a \in A$ and $b \in B$,*
$$\|ba\|_B \leq M\|b\|_B\|a\|_A.$$

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3. *There is a constant $M > 0$ such that for each $a \in A$ and $b \in B$,*

$$\|ba\|_B \leq M\|b\|_B\|a\|_A.$$

A Banach algebra $(B, \| \|_B)$ is a symmetric abstract Segal algebra *in A if it's a left and right abstract Segal algebra in A.*

When A has no BAI

Example

- A faithful Banach algebra A is a right abstract Segal algebra in its closure in $RM(A)$.

When A has no BAI

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- A faithful Banach algebra A is a right abstract Segal algebra in its closure in $RM(A)$.
- The Schatten p class $S_p(H)$ on a Hilbert space H is an abstract symmetric Segal algebra in $K(H)$.

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- A faithful Banach algebra A is a right abstract Segal algebra in its closure in $RM(A)$.
- The Schatten p class $S_p(H)$ on a Hilbert space H is an abstract symmetric Segal algebra in $K(H)$.
- The Lebesgue-Fourier algebra is a abstract symmetric Segal algebra in $A(G)$.

When A has no BAI

Lemma

(Mustafayev) *Let* $(A, \| \|_A)$ *be a Banach algebra and* $(B, \| \|_B)$ *an abstract right Segal algebra in* A*. Then there is an injection from* $< B^*B >$ *into* A^**.*

When A has no BAI

Lemma

(Mustafayev) *Let $(A, \| \|_A)$ be a Banach algebra and $(B, \| \|_B)$ an abstract right Segal algebra in A. Then there is an injection from $< B^*B >$ into A^*.*

Corollary

(Mustafayev) *Let $(A, \| \|_A)$ be a Banach algebra with a BAI and $(B, \| \|_B)$ an abstract right Segal algebra in A. Then $< B^*B > \subseteq < A^*A >$.*

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(Neufang, P.) *Let $(A, \| \|_A)$ be a Banach algebra. Then there is no proper abstract right Segal algebra $(B, \| \|_B)$ in A such that $B^* = B^*B$.*

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Theorem

(Neufang, P.) *Let A be a faithful Banach algebra. If A has the left(resp. right) strong dual factorization property then the norm of A is equivalent to the norm of* $RM(A)$ *(resp.* $LM(A)$*).*

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α-Nuclear Operators

Tensor norm

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1. α is reasonable i.e for each $u \in E \otimes F$, $\epsilon(u) \leq \alpha(u) \leq \pi(\mathrm{u})$.
2. α satisfies the property : if for $T_i \in L(E_i, F_i)$, we have that

$$T_1 \otimes T_2 : E_1 \otimes_\alpha E_2 \to F_1 \otimes_\alpha F_2,$$
$$\|T_1 \otimes T_2\| \leq \|T_1\| \|T_2\|.$$

Tensor norm

A tensor norm α on the class of all normed spaces assigns to each pair (E, F) of normed spaces E and F a norm α on the algebraic tensor product $E \otimes F$ such that the following conditions are satisfied :

1. α is reasonable i.e for each $u \in E \otimes F$, $\epsilon(u) \leq \alpha(u) \leq \pi(\mathrm{u})$.
2. α satisfies the property : if for $T_i \in L(E_i, F_i)$, we have that

$$T_1 \otimes T_2 : E_1 \otimes_\alpha E_2 \to F_1 \otimes_\alpha F_2,$$
$$\|T_1 \otimes T_2\| \leq \|T_1\| \|T_2\|.$$

3. for each pair of Banach spaces E and F, and each $u \in E \otimes F$, we have

$$\alpha(u) = \inf\{\alpha(u, M \otimes N) : M \in FIN(E), N \in FIN(F), u \in M \otimes N\}.$$

α-Nuclear Operators

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Let E be a Banach space and α be a tensor norm.

$$J_\alpha : E^* \hat{\otimes}_\alpha E \to B(E)$$

For $e, x \in E$ and $f \in E^*$, we have

$$J_\alpha(f \otimes x)(e) = \langle f, e \rangle x$$

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The Banach of the α-nuclear operators denoted by $N_\alpha(E)$ is the image of J_α equipped with the quotient norm. This algebra is en operator ideal in $B(E)$.

α-Nuclear Operators

$$\begin{array}{ccc} (N_\alpha(E))^{**} & & B(E^{**}) \\ {\scriptstyle \kappa_{N_\alpha}} \Big\uparrow & & \\ N_\alpha(E) & \hookrightarrow & B(E) \end{array}$$

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Define $\phi : E^{**}\hat{\otimes}_\pi E^* \to N_\alpha(E)^*$ by

$$\langle \phi(\Lambda \otimes \mu), T\rangle = \langle T \cdot \Lambda, \mu\rangle$$

with $T \in N_\alpha(E)$, $\mu \in E^*$ and $\Lambda \in E^{**}$.

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Theorem

(Neufang, P.) *Let E be a Banach space. Let α be a tensor norm. Suppose that $N_\alpha(E)$ has the left strong dual factorization property then α is the injective tensor norm.*

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Proof.

$N_\alpha(E)$ has the left strong dual factorization property implies that the norm of $N_\alpha(E)$ and $RM(N_\alpha(E)) = B(E)$ are equivalent. This is true only if α is the injective tensor norm. □

α-Nuclear Operators

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Corollary

Let E be a Banach space such that E^* has the bounded approximation property. Let $A = A(E)$. Then $A^* = A^*A$ if and only if $I(E^*) = N(E^*)$.

What happens for the right ?

Lau and Ülger showed that $A(E)$ does not have the right strong dual factorization property when E^* has the bounded approximation property and $I(E^*) = N(E^*)$ with E not reflexive.

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Proof.

If E reflexive then $A(E)$ Arens regular [Young]. Thus have the right strong dual factorization property.

If $A^* = AA^*$ then $SZ_r(A(E)^{**}) = RM(A(E)) = B(E)$. But $SZ_r(A(E)) = W(E)$ thus we get that $B(E) = W(E)$. So E is reflexive. □

QUESTION

Does the left strong dual factorization property imply the existence of a bounded left approximate identity ?

Lau-Ülger conjecture

Theorem

(Lau, Ü.) *Let A be a weakly sequentially complete Banach algebra with a sequential BAI. The following are equivalent.*

1. $A^* = A^* A$.
2. $A^* = A A^*$.
3. A *is unital.*

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Conjecture

Let A be a weakly sequentially complete Banach algebra with a BAI. If $A^* = A^*A$*, then A is unital.*

Partial answers

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- Banach algebra A such that A^* has the Radon-Nicodym property.
- **(Neufang, P.)** Banach algebra such that $SZ_l(A^{**}) = A$. In particular, left strongly Arens irregular Banach algebra.
- **(Neufang, P.)** Banach algebra which is an ideal in its second dual.