Multi-normed spaces and amenability conditions for locally compact groups

Paul Ramsden

June 2010

Multi-normed spaces

Definition

A multi-normed space is a Banach space E equipped with a sequence of norms $\{\|\cdot\|_n : n \in \mathbb{N}\}$ on the linear spaces $\{E^n : n \in \mathbb{N}\}$ satisfying:

(A1) $\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\right\|_n = \|(x_1, \ldots, x_n)\|_n$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n \leq (\max_{i \in \mathbb{N}_n} |\alpha_i|) \|(x_1, \ldots, x_n)\|_n$;

(A3) $\|(x_1, \ldots, x_{n-1}, 0)\|_n = \|(x_1, \ldots, x_{n-1})\|_{n-1}$;

(A4) $\|(x_1, \ldots, x_{n-2}, x, x)\|_n = \|(x_1, \ldots, x_{n-2}, x)\|_{n-1}$.

Example

$\|(x_1, \ldots, x_n)\|_n = \max \{\|x_i\| : i \in \mathbb{N}_n\}$

Multi-bounded sets

Definition

Let E be a multi-normed space. A subset $B \subset E$ is multi-bounded if

$$mb(B) := \sup \{\|(x_1, \ldots, x_n)\|_n : x_1, \ldots, x_n \in B,\ n \in \mathbb{N}\} < \infty\,.$$

Tensor norms

Definition

A norm α on the linear space $c_0 \otimes E$ is a c_0-norm if:

(i) $\alpha(x \otimes y) = \|x\|\,\|y\|$ for every $x \in c_0$ and $y \in E$, and

(ii) $T \otimes I_E$ is bounded on $(c_0 \otimes E,\ \alpha)$ with $\|T \otimes I_E\| \leq \|T\|$ for each $T \in \mathcal{B}(c_0)$.

- $\varepsilon(z) \leq \alpha(z) \leq \pi(z)$

Proposition (Daws)

The study of multi-norms over E is equivalent to the study of c_0-norms on $c_0 \otimes E$. □

The multi-norm $\leftrightarrow$ c_0-norm correspondence

- E is a multi-normed space.
- $\mathcal{M}(\ell^1, E) = \{T : \ell^1 \to E : \|T\| = mb[\{T(\delta_k) : k \in \mathbb{N}\}] < \infty\}$
  - $\|T\| = mb[T(\ell^1_{[1]})]$
  - $\mathcal{M}(\ell^1, E) = \mathcal{M}(\min(\ell^1), E)$
- $c_0 \otimes E \subset \mathcal{M}(\ell^1, E)$ defines a c_0-norm on $c_0 \otimes E$
- Conversely, given a c_0-norm α, define

$$\|(x_1, \ldots, x_n)\|_n = \alpha\left(\sum \delta_i \otimes x_i\right)$$

- Results about E often go via the space $\mathcal{M}(\ell^1, E)$

The maximum multi-norm

The maximum multi-norm over E is defined by

$$\|(x_1, \ldots, x_n)\|_n^{\max} = \sup \|(x_1, \ldots, x_n)\|_n^{\alpha} \quad (n \in \mathbb{N},\, x_1, \ldots, x_n \in E),$$

where the supremum is taken over all multi-norms $\|\cdot\|_n^{\alpha}$ on E.

- $\|(x_1, \ldots, x_n)\|_n^{\max} = \pi\left(\sum_{i=1}^n \delta_i \otimes x_i\right)$

Proposition

For each $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_n) \in E^n$, we have

$$\|x\|_n^{\max} = \sup\left\{\left|\sum_{i=1}^{n} \langle x_i, \lambda_i\rangle\right| : \lambda = (\lambda_1, \ldots, \lambda_n) \in (E')^n,\ \mu_{1,n}(\lambda) \le 1\right\}.$$

□

The (p,q)-multi-norm

Proposition

Let $1 \le p \le q < \infty$. For each $n \in \mathbb{N}$ we define a norm on E^n by

$$\|x\|_n^{(p,q)} = \sup \left\{ \left(\sum_{i=1}^{n} |\langle x_i, \lambda_i \rangle|^q \right)^{1/q} : \lambda \in (E')^n,\ \mu_{p,n}(\lambda) \le 1 \right\},$$

where $x = (x_1, \dots, x_n) \in E^n$. Then the family $\{\| \cdot \|_n^{(p,q)} : n \in \mathbb{N}\}$ is a multi-norm over E, called the (p,q)-multi-norm. □

- $\| \cdot \|_n^{(1,1)} = \| \cdot \|_n^{\max}$
- Obvious: $\| \cdot \|_n^{(1,q)} \le \| \cdot \|_n^{(p,q)} \le \| \cdot \|_n^{(q,q)}$
- Less obvious: $\| \cdot \|_n^{(q,q)} \le \| \cdot \|_n^{(p,p)} \le \| \cdot \|_n^{(1,1)}$

The $(1,q)$-multi-norm on $L^1(\Omega)$

Theorem

Let Ω be a measure space, and let $1 \le q < \infty$. Then

$$\|(f_1, \dots, f_n)\|_n^{(1,q)} = \sup_{\mathbf{X}} \left(\sum_{i=1}^{n} \|\chi_{X_i} f_i\|^q \right)^{1/q} \qquad (f_1, \dots, f_n \in L^1(\Omega)).$$

where the supremum is taken over all measurable partitions $\mathbf{X} = (X_1, \dots, X_n)$ of Ω. □

- It follows that

$$\|(f_1, \dots, f_n)\|_n^{\max} = \|(f_1, \dots, f_n)\|_n^{(1,1)} = \| |f_1| \vee \dots \vee |f_n| \| .$$

~~(p,q)-amenability~~ (p,q)-invariant means

Definition

Let G be a locally compact group, and let $1 \leq p \leq q$. A mean $\Lambda \in L^1(G)''$ is (p,q)-invariant if the set $\{s \cdot \Lambda : s \in G\}$ is multi-bounded in the (p,q)-multi-norm.

Proposition

Let G be a locally compact group, then G is amenable if and only if there exists a mean $\Lambda \in L^1(G)''$ such that the set $\{s \cdot \Lambda : s \in G\}$ is relatively weakly-compact in $L^1(G)''$. □

Injective Banach modules

- Let A be a Banach algebra, and let $E \in A$-$\mathbf{mod}$ be faithful.
- Then $\mathcal{B}(A,E) \in A$-$\mathbf{mod}$ with the multiplication

$$(a \cdot T)(b) = T(ba) \quad (a, b \in A,\ T \in \mathcal{B}(A,E)) \,.$$

- We define the *canonical embedding* $\Pi : E \to \mathcal{B}(A,E)$ by the formula

$$\Pi(x)(a) = a \cdot x \quad (a \in A,\ x \in E) \,.$$

Definition

The module E is injective if there exists a left A-module morphism $\rho : \mathcal{B}(A,E) \to E$ with $\rho \circ \Pi = I_E$.

The $L^1(G)$ module $L^p(G)$

- For each $1 < p < \infty$, $L^p(G) \in L^1(G)$-**mod** with the multiplication

$$(a \cdot f)(t) = \int_G a(s) f(s^{-1}t)\, \mathrm{d}m(s) \quad (a \in L^1(G), f \in L^p(G))\,.$$

- G amenable $\Longrightarrow L^p(G)$ injective.

Theorem

Let G be a locally compact group, and let $1 < p < \infty$. Then:

$$L^p(G) \text{ injective } \Longrightarrow L^\infty(G) \text{ has a } (p,p)\text{-invariant mean}\,. \quad \square$$

Multi-bounded vs weakly-compact

Proposition

Suppose that $\mathcal{M}(\ell^1, E) \subset \mathcal{W}(\ell^1, E)$, the space of weakly-compact operators. Then every multi-bounded subset of E is relatively weakly-compact. $\square$

Proposition

Let E be equipped with the (p,q)-multi-norm. Then

$$T \in \mathcal{M}(\ell^1, E) \Longleftrightarrow T' \in \Pi_{q,p}(E', \ell^\infty)\,. \quad \square$$

Multi-bounded vs weakly-compact

- $\Pi_{p,p}(X,Y) \subset \mathcal{W}(X,Y)$.
- $T \in \mathcal{W}(X,Y) \iff T' \in \mathcal{W}(Y',X')$

Corollary

Every (p,p)-multi-bounded subset of E is relatively weakly-compact. □

Theorem

Let G be a locally compact group, and let $1 < p < \infty$. Then:

$$L^p(G) \text{ injective } \iff G \text{ is amenable}\,.$$ □

Multi-bounded vs weakly-compact

Example

- $1 \leq p < q$, and set $E = c_0$
- $(\sum_{i=1}^{n} \delta_i : n \in \mathbb{N}) \subset E$ is (p,q)-multi-bounded but not relatively weakly-compact.

- $1 \leq p < q < r$, and set $E = L^1(\Omega)$
- $\Pi_{q,p}(E',\ell^\infty) = \Pi_{q,1}(E',\ell^\infty) \subset \Pi_{r,r}(E',\ell^\infty)$

Corollary

Let $1 \leq p < q < r$, and let Ω be a locally compact space. Then

$$D\,\|\cdot\|_n^{(r,r)} \leq \|\cdot\|_n^{(1,q)} \leq \|\cdot\|_n^{(p,q)} \leq C\,\|\cdot\|_n^{(1,q)}$$

further, every (p,q)-multi-bounded subset of $L^1(\Omega)$ is relatively weakly-compact. □

Følner type conditions

(WFC) There exists $\varepsilon_0 \in (0,2)$ such that for every finite subset $F \subset G$, there exists a compact set $C \subset G$ such that

$$\frac{m(tC\Delta C)}{m(C)} < \varepsilon_0 \quad (t \in F)\,.$$

(FC) For every $\varepsilon > 0$ and every finite set $F \subset G$, there exists a compact set $C \subset G$ such that

$$\frac{m(tC\Delta C)}{m(C)} < \varepsilon \quad (t \in F)\,.$$

(SFC) For every $\varepsilon > 0$ and every compact set $K \subset G$, there exists a compact set $C \subset G$ such that

$$\frac{m(KC\Delta C)}{m(C)} < \varepsilon\,.$$

Følner type conditions

(PA) For every $\varepsilon > 0$, there exists $n_\varepsilon \in \mathbb{N}$ such that for every finite set $F \subset G$ with $|F| \geq n_\varepsilon$, there exists a compact subset $C \subset G$ such that

$$\frac{m(FC)}{m(C)\,|F|} < \varepsilon\,.$$

Theorem (Dales & Polyakov (2003))

Let G be a discrete group, and let $1 < p < \infty$. Then:

$$\ell^p(G) \text{ injective} \implies G \text{ -pseudo-amenable} \implies \mathbb{F}_2 \not\subset G. \quad \square$$

- $\exists$ an invariant mean $\Lambda \in L^1(G)'' \implies$ (FC)
- $\exists$ an (p,q)-invariant mean $\Lambda \in L^1(G)'' \implies$ (PA)

Theorem (H. L. Pham)

The the following are equivalent:

- G *is amenable.*
- *For every* $\varepsilon > 0$*, there exists* $n_\varepsilon \in \mathbb{N}$ *such that for every finite subset* $F \subset G$ *with* $|F| \geq n_\varepsilon$*, there exists a compact set* $C \subset G$ *such that*

$$\frac{m(EC)}{m(C)\,|E|} < \varepsilon \quad (E \subset F,\ |E| \geq n_\varepsilon)\,.$$

□