Lecture 1: Introduction to Operator Spaces

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Operator Spaces

A Natural Quantization of Banach Spaces

Banach Spaces

A Banach space is a complete normed space $(V/\mathbb{C}, \|\cdot\|)$.

In Banach spaces, we consider

Norms and Bounded Linear Maps.

Classical Examples:

$$C_0(\Omega), \;\; M(\Omega) = C_0(\Omega)^*, \;\; \ell_p(I), \;\; L_p(X, \mu), \quad 1 \leq p \leq \infty.$$

Hahn-Banach Theorem: Let $V \subseteq W$ be Banach spaces. We have

$$\begin{array}{ccc} W & & \\ \uparrow & \searrow \tilde{\varphi} & \\ V & \xrightarrow{\varphi} & \mathbb{C} \end{array}$$

with $\|\tilde{\varphi}\| = \|\varphi\|$.

It follows from the Hahn-Banach theorem that for every Banach space $(V, \|\cdot\|)$ we can obtain an isometric inclusion

$$(V, \|\cdot\|) \hookrightarrow (\ell_\infty(I), \|\cdot\|_\infty)$$

where we may choose $I = V_1^*$ to be the closed unit ball of V^*.

So we can regard $\ell_\infty(I)$ as the home space of Banach spaces.

Classical Theory	**Noncommutative Theory**
$\ell_\infty(I)$	$B(H)$
Banach Spaces $(V, \|\cdot\|) \hookrightarrow \ell_\infty(I)$	Operator Spaces $(V, ??) \hookrightarrow B(H)$
	norm closed subspaces of $B(H)$?

Matrix Norm and Concrete Operator Spaces [Arveson 1969]

Let $B(H)$ denote the space of all bounded linear operators on H. For each $n \in \mathbb{N}$,

$$H^n = H \oplus \cdots \oplus H = \{[\xi_j] : \xi_j \in H\}$$

is again a Hilbert space. We may identify

$$M_n(B(H)) \cong B(H \oplus \ldots \oplus H)$$

by letting

$$\left[T_{ij}\right] \left[\xi_j\right] = \left[\sum_j T_{i,j}\xi_j\right],$$

and thus obtain an operator norm $\|\cdot\|_n$ on $M_n(B(H))$.

A concrete operator space is norm closed subspace V of $B(H)$ together with the canonical operator matrix norm $\|\cdot\|_n$ on each matrix space $M_n(V)$.

Examples of Operator Spaces

- Every C*-algebra A, i.e. norm closed *-subalgebra of some $B(H)$, is an operator space.

- $A = C_0(\Omega)$ or $A = C_b(\Omega)$ for locally compact space.

- Every operator algebra, i.e. norm closed subalgebra of some $B(H)$, is an operator space.

- Every von Neumann algebra M, i.e. a strong operator topology (resp. w,o.t , weak* topology) closed *-subalgebra of $B(H)$.

- $L_\infty(X, \mu)$ for some measure space (X, μ).

- Weak* closed operator algebras of some $B(H)$.

Completely Bounded Maps

Let $\varphi : V \to W$ be a bounded linear map. For each $n \in \mathbb{N}$, we can define a linear map

$$\varphi_n : M_n(V) \to M_n(W)$$

by letting

$$\varphi_n([v_{ij}]) = [\varphi(v_{ij})].$$

The map φ is called completely bounded if

$$\|\varphi\|_{cb} = \sup\{\|\varphi_n\| : n \in \mathbb{N}\} < \infty.$$

We let $CB(V,W)$ denote the space of all completely bounded maps from V into W.

In general $\|\varphi\|_{cb} \neq \|\varphi\|$. Let t be the transpose map on $M_n(\mathbb{C})$. Then

$$\|t\|_{cb} = n, \text{ but } \|t\| = 1.$$

Theorem: If $\varphi : V \to W = C_b(\Omega)$ is a bounded linear map, then φ is completely bounded with

$$\|\varphi\|_{cb} = \|\varphi\|.$$

Proof: Given any contractive $[v_{ij}] \in M_n(V)$, $[\varphi(v_{ij})]$ is an element in

$$M_n(C_b(\Omega)) = C_b(\Omega, M_n) = \{[f_{ij}] : x \in \Omega \to [f_{ij}(x)] \in M_n\}.$$

Then we have

$$\begin{aligned}
\|[\varphi(v_{ij})]\|_{C_b(\Omega,M_n)} &= \sup\{\|[\varphi(v_{ij})(x)]\|_{M_n} : x \in \Omega\} \\
&= \sup\{|\sum_{i,j=1}^{n} \alpha_i \varphi(v_{ij})(x)\beta_j| : x \in \Omega, \|\alpha\|_2 = \|\beta\|_2 = 1\} \\
&= \sup\{|\varphi(\sum_{i,j=1}^{n} \alpha_i v_{ij}\beta_j)(x)| : x \in \Omega, \|\alpha\|_2 = \|\beta\|_2 = 1\} \\
&\le \|\varphi\| \sup\{\|[\alpha_i][v_{ij}][\beta_j]\| : \|\alpha\|_2 = \|\beta\|_2 = 1\} \\
&\le \|\varphi\|\|[v_{ij}]\| \le \|\varphi\|.
\end{aligned}$$

This shows that $\|\varphi_n\| \le \|\varphi\|$ for all $n = 1, 2, \cdots$. Therefore, we have

$$\|\varphi\| = \|\varphi_2\| = \cdots = \|\varphi_n\| = \cdots = \|\varphi\|_{cb}.$$

Arveson-Wittstock-Hahn-Banach Theorem

Let $V \subseteq W \subseteq B(H)$ be operator spaces.

$$\begin{array}{ccc} W & & \\ \uparrow & \searrow \tilde{\varphi} & \\ V & \xrightarrow{\varphi} & B(H) \end{array}$$

with $\|\tilde{\varphi}\|_{cb} = \|\varphi\|_{cb}$.

In particular, if $B(H) = \mathbb{C}$, we have $\|\varphi\|_{cb} = \|\varphi\|$. This, indeed, is a generalization of the classical Hahn-Banach theorem.

Operator Space Structure on Banach Spaces

Let V be a Banach space. Then there are many different operator space structures on V.

Min(V): We may obtain a minimal operator space structure on V given by

$$x \in V \hookrightarrow \hat{x} \in \ell_\infty(I) = \prod_{\varphi \in I} \mathbb{C}_\varphi.$$

Max(V): We may obtain a maximal operator space structure on V given by

$$x \in V \hookrightarrow \tilde{x} \in \ell_\infty(\tilde{I}, B(\ell_2(\mathbb{N}))) = \prod_{\varphi \in B(V,\tilde{I})} B(\ell_2(\mathbb{N}))_\varphi,$$

where $\tilde{I} = B(\ell_2(\mathbb{N}))_1$ and for each $\varphi \in \tilde{I}$, we get

$$\tilde{x} : \varphi \in \tilde{I} \to \varphi(x) \in B(\ell_2(\mathbb{N})).$$

Column and Row Hilbert Spaces

Let $H = C^m$ be an m-dimensional Hilbert space .

H_c: There is a natural column operator space structure on H given by

$$H_c = M_{m,1}(C) \subseteq M_m(C).$$

H_r: Similarly, there is a row operator space structure given by

$$H_r = M_{1,m}(C) \subseteq M_m(C).$$

Moreover, Pisier introduced an OH structure on H by considering the complex interperlation over the matrix spaces

$$M_n(OH) = (M_n(H_c), M_n(H_r))_{\frac{1}{2}} = (M_n(MAX(H)), M_n(MIN(H)))_{\frac{1}{2}}.$$

All these matrix norm structures are distinct from $MIN(H)$ and $MAX(H)$.

Abstract Operator Spaces

Theorem [R 1988]: Let V be a Banach space with a norm $\|\cdot\|_n$ on each matrix space $M_n(V)$. Then V is completely isometrically isometric to a concrete operator space if and only it satisfies

M1. $\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{n+m} = \max\{\|x\|_n, \|y\|_m\}$

M2. $\|\alpha x \beta\|_n \leq \|\alpha\| \|x\|_n \|\beta\|$

for all $x \in M_n(V), y \in M_m(V)$ and $\alpha, \beta \in M_n(C)$.

Dual Operator Spaces

Let V and W be operator spaces. Then the space $CB(V, W)$ of all completely bounded maps from V into W is an operator space with a canonical operator space matrix norm given by

$$M_n(CB(V, W)) = CB(V, M_n(W)).$$

In particular, if we let $W = \mathbb{C}$, then the dual space $V^* = CB(V, \mathbb{C})$ has a natural operator space matrix norm given by

$$M_n(V^*) = CB(V, M_n(\mathbb{C})).$$

We call V^* the operator dual of V.

More Examples

- $T(\ell_2(\mathbb{N})) = K(\ell_2(\mathbb{N}))^* = B(\ell_2(\mathbb{N}))_*$;

- $M(\Omega) = C_0(\Omega)^*$, operator dual of C*-algebras A^*;

Operator Preduals

Let V be a dual space with a predual V_* and let V be an operator space. Then V^* is an operator space (with the natural dual operator space structure).

Due to the Hahn-Banach theorem, we have the isometric inclusion

$$V_* \hookrightarrow V^*.$$

This defines an operator space structure on V_*, called the dual operator space structure on V_*.

Question: Do we the complete isometry $(V_*)^* = V$? More precisely, can we guarantee that we have the isometric isomorphism

$$M_n((V_*)^*) = M_n(V) \text{ for each } \; n \in \mathbb{N}?$$

Answer: No. Excercise.

Theorem [E-R 1990]: Let M be a von Neumann algebra and M_* the unique predual of M. With the dual operator space strucutre $M_* \hookrightarrow M^*$ on M_*, we have the complete isometry

$$(M_*)^* = M.$$

Therefore, we can say that M_* is the operator predual of M.

Qurstion: What can we say if $M = V$ is not a von Neumann algebra ?

Quotient Operator Spaces

Let $V \hookrightarrow W$ be operator spaces. Then there exists a natural quotient operator space structure on W/V given by the isometric identification

$$M_n(W/V) = M_n(W)/M_n(V) = \{x + M_n(V) : x = [x_{ij}] \in M_n(W)\}.$$

We call W/V the quotient operator space.

Now let $M \subseteq B(H)$ be a von Neumann algebra. Then M is a weak* closed subsapce of $B(H)$. Its predual M_* can be isometrically identified with the quotient space $T(H)/M_\perp$. Then we can also obtain a quotient operator space strucutre on M_*

Theorem: Let $M \subseteq B(H)$ be a von Neumann algebra. We have the complete isometry

$$T(H)/M_\perp = M_*.$$

Proof: Since the restriction map $\omega \in T(H) \to f = \omega_{|M} \in M_*$ is a complete contraction with kernel $M_\perp$, it induces a complete contraction

$$\pi : T(H)/M_\perp \to M_*$$

On the other hand, let us assume that $\Phi = [f_{ij}] \in M_n(M_*) = CB^\sigma(M, M_n)$. It is known from von Neumann algebra theory that every normal cb map has a norm preserving mornal cb extension $\tilde{\Phi} \in CB^\sigma(B(H), M_n) = M_n(T(H))$.

Therefore, $\pi : T(H)/M_\perp \to M_*$ must be a completely isometric isomorphism

Theorem [P-B 1990]: If V has MIN (respectively, MAX) operator space structure, then V^* has MAX (respectively, MIN) operator space structure, , i.e. we have the complete isometries

$$MIN(V)^* = MAX(V^*) \text{ and } MAX(V)^* = MIN(V^*).$$

If G is a locally compact group, then

- $C_0(G)$ and $L_\infty(G)$ have the MIN operator space structure, and

- $M(G)$ and $L_1(G)$ have the natural MAX operator space structure.

Banach Algebras Associated with Locally Compact Groups

Let G be a locally compact group with a left Haar measure ds.

Then we have commutative C*-algebras and von Neumann algebras

$$C_0(G) \subseteq C_b(G) \subseteq L_\infty(G)$$

with pointwise multiplication.

Moreover, we have a natural Banach algebra structure on the convolution algebra $L_1(G) = L_\infty(G)_*$ and the measure algebra $M(G) = C_0(G)^*$ given by

$$f \star g(t) = \int_G f(s)g(s^{-1}t)ds$$

and

$$\langle \mu \star \nu, h \rangle = \int_G h(st)d\mu(s)d\nu(t)$$

for all $h \in L_\infty(G)$.

Group C*-algebras and Group von Neumann Algebras

For each $s \in G$, there exists a unitary λ_s on $L_2(G)$ given by

$$\lambda_s \xi(t) = \xi(s^{-1}t)$$

Then λ induces a contractive *-representation $\lambda : L_1(G) \to B(L_2(G))$ given by

$$\lambda(f) = \int_G f(s) \lambda_s ds.$$

We let $C^*_\lambda(G) = \overline{\lambda(L_1(G))}^{\|\cdot\|}$ denote the reduced group C*-algebra of G.

We let $L(G) = \overline{\lambda(L_1(G))}^{s.o.t} = \{\lambda_s : s \in G\}'' \subseteq B(L_2(G))$ be the left group von Neumann algebra of G.

If G is an abelian group, then $L_1(G)$ is commutative. Therefore, $C_\lambda^*(G)$ and $L(G)$ are commutative and we have

$$C_\lambda^*(G) = C_0(\widehat{G}) \text{ and } L(G) = L_\infty(\widehat{G}),$$

where $\widehat{G} = \{\chi : G \to \mathbb{T} : \text{continuous homo}\}$ is the dual group of G.

Example: Let $G = \mathbb{Z}$. Then $\ell_1(\mathbb{Z})$ is unital commutative. In this case, we have

$$C_\lambda^*(\mathbb{Z}) = C(\mathbb{T}) \text{ and } L(\mathbb{Z}) = L_\infty(\mathbb{T}).$$

Therefore, for a general group G, we can regard $C_\lambda^*(G)$ and $L(G)$ as the dual object of $C_0(G)$ and $L_\infty(G)$, respectively.

Fall Group C*-algebra

Let $\pi_u : G \to B(H_u)$ be the universal representation of G. Then π_u induces a contractive *-representation $\pi_u : L_1(G) \to B(H_u)$ given by

$$\pi_u(f) = \int_G f(s)\pi_u(s)ds.$$

We let $C^*(G) = \overline{\pi_u(L_1)}^{\|\cdot\|}$ denote the full group C*-algebra of G.

It is known that we have a canonical C*-algebra quotient

$$\pi_\lambda : C^*(G) \to C^*_\lambda(G).$$

Fourier Algebra $A(G)$

Let

$$A(G) = \{f : G \to \mathbb{C} : f(s) = \langle \lambda_s \xi | \eta \rangle\}$$

be the space of all coeffient of regular representation λ. It was shown by Eymard in 1964 that $A(G)$ with the norm

$$\|f\|_{A(G)} = \inf\{\|\xi\|\|\eta\| : f(s) = \langle \lambda_s \xi | \eta \rangle\}$$

and pointwise publication is a commutative Banach algebra, i.e. we have

$$\|fg\|_{A(G)} \le \|f\|_{A(G)} \|g\|_{A(G)}.$$

We call $A(G)$ the Fourier algebra of G.

We note that $A(G)$ with the above norm is isometrically isomorphic to the predual $L(G)_*$. More over, if G is an abelian group, we have

$$A(G) = L_1(\widehat{G}).$$

Therefore, we can regard $A(G)$ as the natural dual of $L_1(G)$.

Operator Space Structure on $A(G)$

It is known that we can isometrically identify $A(G)$ with the predual $L(G)_*$ of the group von Neumann algebra. Then we can obtain a natural operator space structure on $A(G)$ given by the canonical inclusion

$$A(G) \hookrightarrow A(G)^{**} = L(G)^*.$$

With this operator space structure, we have the complete isometry

$$A(G)^* = L(G).$$

We also have canonical operator space structures on

$$B_\lambda(G) = C^*_\lambda(G)^* \text{ and } B(G) = C^*(G)^*.$$

We have the completely isometric inclusion

$$A(G) \hookrightarrow B_\lambda(G) \hookrightarrow B(G).$$

A continuous function $\varphi : G \to \mathbb{C}$ is called a multiplier of $A(G)$ if the multiplication map m_φ defines a map on $A(G)$, i.e. we have

$$m_\varphi : \psi \in A(G) \to \varphi\psi \in A(G).$$

We let $MA(G)$ denote the space of all multipliers of $A(G)$.

We let $M_{cb}A(G)$ denote the space of all completely bounded multipliers of $A(G)$, i.e. $\|m_\varphi\|_{cb} < \infty$.

There exists a natural operator space structure on

$$M_{cb}A(G) \subseteq CB(A(G), A(G)).$$

Classical Case	Noncommutative Case
$L_\infty(G)$	$L(G)$
$C_0(G)$	$C^*_\lambda(G)$
$L_1(G)$	$A(G)$
$M(G)$	$B_\lambda(G) \subseteq B(G) \subseteq M_{cb}A(G) \subseteq MA(G).$

If G is amenable, we have

$$B_\lambda(G) = B(G) = M_{cb}A(G) = MA(G).$$

Amenability of G

A locally compact group G is called amenable if there exists a left invariant mean on $L_\infty(G)$, i.e. there exists a positive linear functional

$$m : L_\infty(G) \to \mathbb{C}$$

such that $m(1) = 1$ and $m({}_s h) = m(h)$ for all $s \in G$ and $h \in L_\infty(G)$, where we define ${}_s h(t) = h(st)$.

Theorem: The following are equivalent:

1. G is amenable;

2. G satisfies the Følner condition: for every $\varepsilon > 0$ and compact subset $C \subseteq G$, there exists a compact subset $K \subseteq G$ such that
$$\frac{|K \Delta sK|}{\mu(K)} < \varepsilon \qquad \text{for all } s \in C;$$

3. $A(G)$ has a bounded (or contractive) approximate identity.

Applications to Related Areas

- C*-algebras and von Neumann algebras
- Non-self-adjoint operator algebras
- Abstract harmonic analysis/locally compact quantum groups
- Non-commutative L_p-spaces
- Non-commutative probablilty/non-commutative matingale theory
- Non-commutative harmonic analysis
- Quantum information theory

Related Books

- E.G.Effros and Z-J.Ruan, *Operator spaces*, London Math. Soc. Monographs, New Series 23, Oxford University Press, New York, 2000

- Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics, 78. Cambridge University Press, Cambridge, 2002.

- Pisier, *An introduction to the theory of operator spaces*, London Mathematical Society Lecture Note Series 294, Cambridge University Press, Cambridge, 2003.

- Blecher and Le Merdy, *Operator algebras and their modules-an operator space approach*, London Mathematical Society Monographs. New Series, 30. Oxford University Press, New York 2004.