Lecture 5: Extensions to The Quantum Group Setting

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Pontryagin Duality

Let G be a locally compact *abelian* group. There exists a *dual group*

$$\widehat{G} = \{\gamma : G \to \mathbb{T} : \text{continuous homomorphisms}\}.$$

$\widehat{G}$ has a canonical abelian group structure and we have the homeomorphism

$$\widehat{\widehat{G}} \cong G.$$

In particular, we have the Plancherel theorem

$$L_2(\widehat{G}) = L_2(G).$$

Question: How do we generalize this to non-abelian groups ?

Groups Algebras

Let G be a locally compact group and let

$$\lambda : G \to B(L_2(G))$$

denote the left regular representation of G, respectively. We are interested in the following operator algebras.

$$L_\infty(G) \qquad L(G)$$

$$C_0(G) \qquad C^*_\lambda(G)$$

- $L(G)$ denote the left group von Neumann algebra on $L_2(G)$ and
- $C^*_\lambda(G)$ denote the corresponding reduced group C*-algebra.

This represents a nice duality since if G is an abelian group with dual group $\widehat{G}$, we have

$$L_\infty(G) \qquad L(G) = L_\infty(\widehat{G})$$

$$C_0(G) \qquad C^*_\lambda(G) = C_0(\widehat{G})$$

Co-multiplication on $L_\infty(G)$

To recover the group structure from $L_\infty(G)$, we need to consider a co-multiplication

$$\Gamma_a : f \in L_\infty(G) \to \Gamma_a(f) \in L_\infty(G)\bar{\otimes} L_\infty(G)$$

given by

$$\Gamma_a(f)(s,t) = f(st).$$

It is easy to see the Γ_a is a normal injective *-homomorphism such that it satisfies the co-associativity condition

$$(\Gamma_a \otimes \iota)\Gamma_a = (\iota \otimes \Gamma_a)\Gamma_a$$

i.e. we have

$$(\Gamma_a \otimes \iota)\Gamma_a(f)(s,t,u) = f((st)u) = f(s(tu)) = (\iota \otimes \Gamma_a)\Gamma_a(f)(s,t,u).$$

We call $(L_\infty(G), \Gamma_a)$ a commutative Hopf von Neumann algebra.

Co-involution on $L_\infty(G)$

To recover the inverse of G, we can consider a co-involution

$$\kappa_a(h) = \check{h}$$

which is a weak* continuous *-anti-automorphsim on $L_\infty(G)$ such that

$$\Gamma_a \circ \kappa_a = \Sigma(\kappa_a \otimes \kappa_a) \circ \Gamma_a.$$

Moreover, we can use the left Haar measure to obtain a normal faithful weight

$$\varphi_a : h \in L_\infty(\mathbb{G})^+ \to \varphi_a(h) = \int_G h(s)ds \in [0, +\infty]$$

on $L_\infty(G)$ such that $(\iota \otimes \varphi_a)\Gamma(h) = \varphi_a(h)1$ since

$$(\iota \otimes \varphi_a)\Gamma(h)(s) = \int_G h(st)dt = (\int_G h(t)dt) = \varphi_a(h).$$

So we call φ_a a left Haar weight on $L_\infty(G)$.

Commutative Kac Algebras

Then $(L_\infty(G), \Gamma_a, \kappa_a, \varphi_a)$ is a commutative co-involutive Hopf von Neumann algebra with a left Haar weight.

On the other hand, it was shown by Takesaki in 1969 that every commutative co-involutive Hopf von Neumann algebra with a Haar weight Kac algebra co-involutive commutative Hopf von Neumann algebra $(M, \Gamma, \kappa, \varphi)$ must be like this, associated with a locally compact group G

They are just commutative Kac algebras.

Co-involutive Hopf Structure on $L(G)$

There is a natural co-associative co-multiplication

$$\Gamma_G : L(G) \to L(G) \bar{\otimes} L(G)$$

on $L(G)$ given by

$$\Gamma_G(\lambda_s) = \lambda_s \otimes \lambda_s.$$

This is co-commutative in the sense that

$$\Sigma \circ \Gamma_G = \Gamma_G.$$

To recover the inverse, we consider a co-involution κ_G on $L(G)$ given by

$$\kappa_G(\lambda_s) = \lambda_{s^{-1}}.$$

Moreover, we can obtain a normal faithful (Plancherel) weight φ_G on $L(G)$. So $(L(G), \Gamma_G, \kappa_G, \varphi_G)$ is a co-commutative Kac algebra.

Remark If G is a discrete group, then

$$\varphi_G(x) = \psi_G(x) = \langle x\delta_e | \delta_e \rangle.$$

Kac Algebras

G. Kac introduced *Kac algebras* $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$, for unimodular case, in the 60's.

The theory was completed for general (non-unimodular) case in the 70's by two groups: Kac-Vainerman in Ukraine and Enock-Schwartz in France (see Enock-Schwartz's book 1992).

There exists a perfect Pontryagin duality

$$\hat{\hat{\mathbb{K}}} = \mathbb{K}.$$

Examples:

- Commutative Kac algebras $(L_\infty(G), \Gamma_a, \kappa_a, \varphi_a)$

- Co-commutative Kac algebras $(L(G), \Gamma_G, \kappa_G, \varphi_G)$

Non-trivial Example

The first example of non-commutative and non-cocommutative Kac algebra (LCQG) was given by Kac and Paljutkin 1965.

This can be given by considering the same algebra

$$L(G) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C} \oplus M_2$$

but having a twisted (non-co-commutative) co-multiplication. It turns out that this is the only

non-commutative and non-co-commutative

Kac algebra (LCQG) $\mathbb{G}$ of order 8.

Locally Compact Quantum Groups (LCQG)

The notion of *quantum groups* was introduced by Drinfel'd in his 1986 ICM talk. Here, we consider the analysis aspect of quantum groups, i,e we consider the quantization of locally compact groups.

In 1987, Woronowicz discovered $SU_q(2, \mathbb{C})$, a natural quantum deformation of $SU(2, \mathbb{C})$. He showed that $SU_q(2, \mathbb{C})$ does not correspond to any Kac algebra due to the missing of bounded co-involution.

Since then, several different definitions of LCQG have been given by

- Baaj and Skandalis 1993: Regular Multiplicative Unitaries
- Woronowicz 1996: Manageable Multiplicative Unitaries
- Kustermans and Vaes 2000: Quantum Groups, C^*-algebra setting
- Kustermans and Vaes: 2003: Quantum Groups, von Neumann algebra setting.

Kustermans and Vaes' Definition

A *LCQG* is $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ consisting of

(1) a von Neumann algebra M

(2) a *co-multiplication* $\Gamma : M \to M \bar{\otimes} M$, i.e. a unital normal *-homomorphism satisfying the co-associativity condition

$$(id \otimes \Gamma) \circ \Gamma = (\Gamma \otimes id) \circ \Gamma.$$

(3) a left Haar wight φ, i.e. a n.f.s weight φ on M satisfying

$$(\iota \otimes \varphi)\Gamma(x) = \varphi(x)1$$

(4) a right Haar weight ψ, i.e. n.f.s weight ψ on M satisfying

$$(\psi \otimes \iota)\Gamma(x) = \psi(x)1.$$

It is known that for every locally compact quantum group $\mathbb{G} = (M, \Gamma, \varphi, \psi)$, there exists a *dual quantum group* $\widehat{\mathbb{G}} = (\widehat{M}, \widehat{\Gamma}, \widehat{\varphi}, \widehat{\psi})$ such that we may obtain the perfect Pontryagin duality

$$\hat{\hat{\mathbb{G}}} = \mathbb{G}.$$

Every Kac algebra $\mathbb{K} = (M, \Gamma, \kappa, \varphi)$ is a LCQG since we can obtain the right Haar weight

$$\psi = \varphi \circ \kappa$$

via φ and κ !

Indeed, for a general LCQG $\mathbb{G}$, the relation between left and right Haar weights determines a (not necessarily bounded) co-involution called antipode of $\mathbb{G}$, on a dense subspace of M.

Let $\mathbb{G} = (M, \Gamma, \varphi, \psi)$ be a locally compact quantum group. We write

$$L_\infty(\mathbb{G}) = M \text{ and } L_1(\mathbb{G}) = M_*.$$

We also let $C_0(\mathbb{G})$ denote the quantum group C*-subalgebra contained in $L_\infty(\mathbb{G})$.

Commutative LCQGs are exactly $\mathbb{G}_a = (L_\infty(G), \Gamma_a, \varphi_a, \psi_a)$ with $\psi_a = \varphi \circ \kappa_a$, or equivalently, is determined by the right Haar measure of G.

Co-commutative LCQGs are exactly $\widehat{\mathbb{G}}_a = (L(G), \Gamma_G, \varphi_G, \psi_G)$ with the Plancherel weight $\varphi_G = \psi_G$.

- $\mathbb{G}$ is commutative if and only if $\hat{G}$ is co-commutative.

A LCQG $\mathbb{G}$ is **Discrete** if $L_1(\mathbb{G})$ is unital. In this case,

$$L_\infty(\mathbb{G}) = \prod M_{n(\alpha)} \text{ and } C_0(\mathbb{G}) = c_0 - \oplus M_{n(\alpha)}.$$

A LCQG $\mathbb{G}$ is **compact** if $C_0(\mathbb{G})$ is unital.

- $\mathbb{G}$ is discrete if and only if $\widehat{\mathbb{G}}$ is compact.

- $\mathbb{G}$ is finite dim if and only if it is discrete and compact.

In this case, it is a finite dim Kac algebra.

Bicrossed Product

Bicrossed product of finite groups was studied by G. Macky, W.M. Singer 1972, Takeuchi 1981.

Suppose that G_1 and G_2 are two subgroups of a finite group G such that $G_1 \cap G_2 = \{e\}$ and every $g \in G$ can be (uniquely) written as

$$g = g_1 g_2.$$

In this case, we can

- obtain a pair of compatible actions of G_1 and G_2 on each other, and

- a pair of compatible 2-cocycles for these actions so that G_1 and G_2 form a cocycle matched pair.

Then G_1 and G_2 form a matched pair and G can be identified with the bicrossed product of G_1 and G_2.

Bicrossed Product Construction

S. Majid gave the first infinite dimensional non-trivial example of Kac algebras in 1991

More non-trivial examples of LCQGs are given by Baaj-Skandalis 1993, Baaj-Skandalis-Vaes 2003, Vaes-Vainerman 2003, and et al.

Theorem [Baaj-Skadalis 1993]: There exist a locally compact quantum groups $\mathbb{G}$ such that

$$C_0(\mathbb{G}) = K(H) \text{ and } L_\infty(\mathbb{G}) = B(H).$$

Theorem [Baaj-Skadalis-Vaes 2003]: There exist a locally compact quantum groups $\mathbb{G}$ such that

$$L_\infty(\mathbb{G}) = B(H) = L_\infty(\hat{\mathbb{G}}).$$

Theorem [Vainerman and Fima, Fima]: There exists a locally compact quantum groups $\mathbb{G}$ such that both $L_\infty(\mathbb{G})$ and $L_\infty(\hat{\mathbb{G}})$ are type II_1-factors, or type III_λ factors.

In these cases, $\mathbb{G}$ is non-compact and non-discrete.

Banach Algebra Structure on $L_1(\mathbb{G}) = M_*$

The co-multiplication

$$\Gamma : L_\infty(\mathbb{G}) \to L_\infty(\mathbb{G}) \bar{\otimes} L_\infty(\mathbb{G})$$

induces an associative completely contractive multiplication

$$\star = \Gamma_* : f_1 \otimes f_2 \in L_1(\mathbb{G}) \hat{\otimes} L_1(\mathbb{G}) \to f_1 \star f_2 = (f_1 \otimes f_2) \circ \Gamma \in L_1(\mathbb{G})$$

on $L_1(\mathbb{G}) = M_*$ such that $A = (L_1(\mathbb{G}), \star)$ is a faithful completely contractive Banach algebra with

$$\langle L_1(\mathbb{G}) \star L_1(\mathbb{G}) \rangle = L_1(\mathbb{G}).$$

If $\mathbb{G}_a$ is a commutative LCQG, then $\star = \Gamma_*$ is just the convolution on the convolution algebra

$$L_1(\mathbb{G}_a) = L_1(G).$$

If $\hat{\mathbb{G}}_a$ is a co-commutative LCQG, then $\star = \hat{\Gamma}_*$ is just the pointwise multiplication on the Fourier algebra

$$L_1(\hat{\mathbb{G}}_a) = L(G)_* = A(G).$$

Fundamental Unitary Operators

If G is a finite group, we can obtain an operator

$$W = \sum_{g\in G} \delta_g \otimes \lambda_g \in B(\ell_2(G\times G)).$$

This is actually a unitary operator on $\ell_2(G\times G)$ since

$$W\xi(s,t) = \xi(s, s^{-1}t) \text{ for all } \xi \in \ell_2(G\times G).$$

Moreover, we see that for every $f = \sum_{s\in G} f_s\delta_s \in \ell_1(G)$, we have

$$(f\otimes\iota)(W) = \sum_{s\in G} f_s\lambda_s = \lambda(f),$$

for every $\omega \in A(G)$, we have

$$(\iota\otimes\omega)(W) = \sum_{g\in G} \delta_g\omega(g) \in \ell_\infty(G).$$

Now if $\mathbb{G}$ is a LCQG, then there is a left fundamental unitary operator W on $L_2(\mathbb{G}) \otimes L_2(G)$ such that

$$\lambda : f \in L_1 \to \lambda(f) = (f \otimes \iota)(W) \in B(L_2(\mathbb{G}))$$

defines a completely contractive algebra homomorphism. We let

$$C_0(\hat{\mathbb{G}}) = \overline{\{\lambda(f) : f \in L_1(\mathbb{G})\}}^{\|\cdot\|}$$

be the quantum C*-algebra of $\hat{\mathbb{G}}$ and let

$$L_\infty(\hat{\mathbb{G}}) = \overline{\{\lambda(f) : f \in L_1(\mathbb{G})\}}^{s.o.t.}$$

be the quantum von Neumann algebra of $\hat{\mathbb{G}}$.

It turns out that we have

$$C_0(\mathbb{G}) = \overline{\{(\iota \otimes \hat{f})(W) : \hat{f} \in L_1(\hat{\mathbb{G}})\}}^{\|\cdot\|}$$

be the quantum C*-algebra of $\mathbb{G}$ and

$$L_\infty(\mathbb{G}) = \overline{\{(\iota \otimes \hat{f})(W) : \hat{f} \in L_1(\hat{\mathbb{G}})\}}^{s.o.t.}$$

We can also have full quantum C*-algebras $C_0^u(\mathbb{G})$ and $C_0^u(\hat{\mathbb{G}})$.

Quantum Measure Algebra and cb-Centralzer Algebras

We let $M(\mathbb{G}) = C_0(\mathbb{G})^*$ to be the quantum measure algebra of $\mathbb{G}$ and we let $M_u(\mathbb{G}) = C_u(\mathbb{G})^*$ to be the full quantum measure algebra of $\mathbb{G}$.

A cb-map S on $L_1(\mathbb{G})$ is called a cb-left centralizer (resp. a cb-right centralizer) if

$$S(f \star g) = S(f) \star g \text{ (resp. } T(f \star g) = f \star T(g)).$$

We let $LM_{cb}L_1(\mathbb{G})$ (resp. $RM_{cb}(L_1(\mathbb{G}))$) to be the space of completely bounded left (resp. right) centralizers on $L_1(\mathbb{G})$.

We let $M_{cb}(L_1(\mathbb{G}))$ to be the space of completely bounded double centralizers on $L_1(\mathbb{G})$. A completely bounded double centralizer of $L_1(\mathbb{G})$ if S and T are completely bounded maps on $L_1(\mathbb{G})$ such that

$$f \star S(g) = T(f) \star g \qquad \text{for all } f, g \in L_1(\mathbb{G}).$$

Summary

$$C_0(\mathbb{G}) \subseteq L_\infty(G) \qquad C_0(\widehat{\mathbb{G}}) \subseteq L_\infty(\hat{G})$$

$$C_0^u(\mathbb{G}) \qquad C_0^u(\widehat{\mathbb{G}})$$

$$LUC(\mathbb{G}), RUC(\mathbb{G}) \qquad LUC(\widehat{\mathbb{G}}), RUC(\widehat{\mathbb{G}})$$

$$L_1(\mathbb{G}) \subseteq M(\mathbb{G}) \subseteq M_u(\mathbb{G}) \qquad L_1(\widehat{\mathbb{G}}) \subseteq M(\widehat{\mathbb{G}}) \subseteq M_u(\widehat{\mathbb{G}})$$

$$LM_{cb}(L_1(\mathbb{G})), RM_{cb}(L_1(\mathbb{G})) \qquad LM_{cb}(L_1(\widehat{\mathbb{G}})), RM_{cb}(L_1(\widehat{\mathbb{G}}))$$

$$M_{cb}(L_1(\mathbb{G})) \qquad M_{cb}(L_1(\widehat{\mathbb{G}}))$$

$$Q^l_{cb}(\mathbb{G}), Q^r_{cb}(\mathbb{G}) \qquad Q^l_{cb}(\widehat{\mathbb{G}}), Q^r_{cb}(\widehat{\mathbb{G}})$$

$$Q_{cb}(\mathbb{G}) \qquad Q_{cb}(\widehat{\mathbb{G}}).$$

The Dual Structure of $RM_{cb}(L_1(\hat{\mathbb{G}}))$ and $M_{cb}(L_1(\hat{\mathbb{G}}))$

There is a natural complete contraction $Q^r : L_1(\mathbb{G}) \to RM_{cb}(L_1(\hat{\mathbb{G}}))^*$. We let $Q^r_{cb}(L_1(\hat{\mathbb{G}}))$ be the closure of $L_1(\mathbb{G})$ w.r.t. this norm.

Theorem [H-N-R 2009]: There exists a complete isometry

$$RM_{cb}(L_1(\hat{\mathbb{G}})) = Q^r_{cb}(L_1(\hat{\mathbb{G}}))^*.$$

Therefore, $RM_{cb}(L_1(\hat{\mathbb{G}}))$ is a dual operator space. In fact, $RM_{cb}(L_1(\hat{\mathbb{G}}))$ is a dual Banach algebra, i.e. its multiplication is weak* continuous in each component.

Daws considered the cb-double multiplier algebra $M_{cb}(L_1(\mathbb{G}))$ and its predual $Q_{cb}(L_1(\hat{\mathbb{G}}))$.

Theorem [H-N-R 2009, Daws 2010]: There exists a complete isometry

$$M_{cb}(L_1(\hat{\mathbb{G}})) = Q_{cb}(L_1(\hat{\mathbb{G}}))^*,$$

and $M_{cb}(L_1(\hat{\mathbb{G}}))$ is a dual Banach algebra,

Co-amenability of LCQGs

A locally compact quantum group is called co-amenable if $L_1(\mathbb{G})$ has a bounded approximate identity.

Theorem [Bédos-Tuset]: Let G be a LCQG and $\widehat{\mathbb{G}}$ be its dual quantum group. Then TFAE:

1. $\widehat{\mathbb{G}}$ is co-amenable, i.e. $L_1(\widehat{\mathbb{G}})$ has a BAI;
2. $M(\widehat{\mathbb{G}}) = C_0(\widehat{\mathbb{G}})^*$ is unital;
3. $C_0(\widehat{\mathbb{G}}) = C^u(\widehat{\mathbb{G}}) \Leftrightarrow M(\widehat{\mathbb{G}}) = M_u(\widehat{\mathbb{G}})$.

Theorem [K-R 1996 for Kac algebras, H-N-R 2009 for QGs]: Let $\mathbb{G}$ be a locally compact quantum group. TFAE:

- $\widehat{\mathbb{G}}$ is co-amenable;
- $\|\lambda(f)\| = \|f\|$ for all $f \in L_1(\mathbb{G})^+$
- $\lambda_Q : Q_{cb}(L_1(\widehat{\mathbb{G}})) \to C_0(\widehat{\mathbb{G}})$ is an isometric (isomorphic) isomorphism.

Amenable LCQGs

A locally compact quantum group is amenable if there exists a state $m : L_\infty(\mathbb{G}) \to \mathbb{C}$ such that

$$(\iota \otimes m)\Gamma(x) = m(x)1 \text{ for all } x \in L_\infty(\mathbb{G}).$$

Theorem:

1. If $\hat{\mathbb{G}}$ is co-amenable, i.e. $L_1(\hat{\mathbb{G}})$ has a BAI, then $\mathbb{G}$ is amenable, i.e. $L_\infty(\mathbb{G})$ has a left invariant mean.

   The converse is true if $\mathbb{G}$ is a discrete quantum group, i.e. if $L_1(\mathbb{G})$ is unital, or $\mathbb{G} = G$ is a locally compact group.

2. If $\mathbb{G}$ is amenable, then both $C_0(\hat{\mathbb{G}})$ and $C_0^u(\hat{\mathbb{G}})$ are nuclear and $L_\infty(\hat{\mathbb{G}})$ is injective.

   Howerver it is not known whether $C_0(\hat{\mathbb{G}}) = C_0^u(\hat{\mathbb{G}})$, unless $\mathbb{G}$ is discrete, or $\mathbb{G} = G$ is a locally compact group.

Operator Amenability

Since $L_1(\mathbb{G})$ is a c.c. Banach algebra, we can consider the operator amenability of $L_1(\mathbb{G})$.

It is clear that if $L_1(\mathbb{G})$ then $L_1(\mathbb{G})$ has a BAI and $L_\infty(\mathbb{G})$ has a left invariant mean.

Questions:

• It is interesting to know that if $L_1(\mathbb{G})$ is operator amenable, whether $L_1(\hat{\mathbb{G}})$ has a BAI.

• It is also interesting to know that whether $L_1(\mathbb{G})$ is operator amenable if and only if $L_1(\hat{\mathbb{G}})$ is operator amenable.

These are true for locally compact groups , and for discrete or compact Kac algebras (R 1995 and R 1996).