Dual Banach algebras: an overview

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Banach $\mathfrak{A}$-bimodules and derivations

Definition

Let $\mathfrak{A}$ be a Banach algebra, and let E be a Banach $\mathfrak{A}$-bimodule. A bounded linear map $D : \mathfrak{A} \to E$ is called a derivation if

$$D(ab) := a \cdot Db + (Da) \cdot b \qquad (a, b \in \mathfrak{A}).$$

If there is $x \in E$ such that

$$Da = a \cdot x - x \cdot a \qquad (a \in \mathfrak{A}),$$

we call D an inner derivation.

Amenable Banach algebras

Remark

If E is a Banach $\mathfrak{A}$-bimodule, then so is E^*:

$$\langle x, a \cdot \phi \rangle := \langle x \cdot a, \phi \rangle \qquad (a \in \mathfrak{A}, \phi \in E^*, x \in E)$$

and

$$\langle x, \phi \cdot a \rangle := \langle a \cdot x, \phi \rangle \qquad (a \in \mathfrak{A}, \phi \in E^*, x \in E).$$

We call E^* a dual Banach $\mathfrak{A}$-bimodule.

Definition (B. E. Johnson, 1972)

$\mathfrak{A}$ is called amenable if, for every dual Banach $\mathfrak{A}$-bimodule E, every derivation $D : \mathfrak{A} \to E$, is inner.

Amenability for groups and Banach algebras

Theorem (B. E. Johnson, 1972)

The following are equivalent for a locally compact group G:

1. $L^1(G)$ *is an amenable Banach algebra;*
2. *the group G is amenable.*

Theorem (H. G. Dales, F. Ghahramani, & A. Ya. Helemskiĭ, 2002)

The following are equivalent:

1. $M(G)$ *is amenable;*
2. G *is amenable and discrete.*

Virtual digaonals

Definition (B. E. Johnson, 1972)

An element $\mathbf{D} \in (\mathfrak{A}\hat{\otimes}\mathfrak{A})^{**}$ is called a virtual diagonal for $\mathfrak{A}$ if

$$a \cdot \mathbf{D} = \mathbf{D} \cdot a \qquad (a \in \mathfrak{A})$$

and

$$a\Delta^{**}\mathbf{D} = a \qquad (a \in \mathfrak{A}),$$

where $\Delta : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}$ denotes multiplication.

Theorem (B. E. Johnson, 1972)

$\mathfrak{A}$ is amenable if and only if $\mathfrak{A}$ has a virtual diagonal.

Amenable C^*-algebras

Theorem (A. Connes, U. Haagerup, et al.)

The following are equivalent for a C^*-algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ *is nuclear;*
2. $\mathfrak{A}$ *is amenable.*

Theorem (S. Wasserman, 1976)

The following are equivalent for a von Neumann algebra $\mathfrak{M}$:

1. $\mathfrak{M}$ *is nuclear;*
2. $\mathfrak{M}$ *is subhomogeneous, i.e.,*

$$\mathfrak{M} \cong M_{n_1}(\mathfrak{M}_1) \oplus \cdots \oplus M_{n_k}(\mathfrak{M}_k)$$

with $n_1, \ldots, n_k \in \mathbb{N}$ and $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ abelian.

Normality

Theorem (S. Sakai, 1956)

A C^*-algebra can be faithfully represented on a Hilbert space as a von Neumann algebra if and only if it is the dual space of some Banach space. The predual space is unique.

Definition (R. Kadison, BEJ, & J. Ringrose, 1972)

Let $\mathfrak{M}$ be a von Neumann algebra, and let E be a dual Banach $\mathfrak{M}$-bimodule. Then E is called normal if the module actions

$$\mathfrak{M} \times E \to E, \quad (a, x) \mapsto \begin{cases} a \cdot x \\ x \cdot a \end{cases}$$

are separately weak*-weak* continuous.

Connes-amenability of von Neumann algebras

Definition

Let $\mathfrak{M}$ a von Neumann algebra, and let E be a normal Banach $\mathfrak{M}$-bimodule. We call a derivation $D : \mathfrak{M} \to E$ normal if it is weak*-weak* continuous.

Theorem (R. Kadison, BEJ, & J. Ringrose, 1972)

Suppose that $\mathfrak{M}$ is a von Neumann algebra containing a weak* dense amenable C^*-subalgebra. Then, for every normal Banach $\mathfrak{M}$-bimodule E, every normal derivation $D : \mathfrak{M} \to E$ is inner.

Definition (A. Connes, 1976; A. Ya. Helemskiĭ, 1991)

$\mathfrak{M}$ is Connes-amenable if for every normal Banach $\mathfrak{M}$-bimodule every normal derivation $D : \mathfrak{M} \to E$ is inner.

Injectivity, semidiscreteness, and hyperfiniteness

Definition

A von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathfrak{H})$ is called

1. injective if there is a norm one projection $\mathcal{E} : \mathcal{B}(\mathfrak{H}) \to \mathfrak{M}'$ (this property is independent of the representation of $\mathfrak{M}$ on $\mathfrak{H}$);
2. semidiscrete if there is a net $(S_\lambda)_\lambda$ of unital, weak*-weak* continuous, completely positive finite rank maps such that

$$S_\lambda a \stackrel{\text{weak}^*}{\longrightarrow} a \qquad (a \in \mathfrak{M});$$

3. hyperfinite if there is a directed family $(\mathfrak{M}_\lambda)_\lambda$ of finite-dimensional *-subalgebras of $\mathfrak{M}$ such that $\bigcup_\lambda \mathfrak{M}_\lambda$ is weak* dense in $\mathfrak{M}$.

Connes-amenability, and injectivity, etc.

Theorem (A. Connes, et al.)

The following are equivalent:

1. $\mathfrak{M}$ *is Connes-amenable;*
2. $\mathfrak{M}$ *is injective;*
3. $\mathfrak{M}$ *is semidiscrete;*
4. $\mathfrak{M}$ *is hyperfinite.*

Corollary

A C^**-algebra* $\mathfrak{A}$ *is amenable if and only if* $\mathfrak{A}^{**}$ *is Connes-amenable.*

Normal virtual digaonals, I

Notation

Let $\mathcal{B}^2_\sigma(\mathfrak{M}, \mathbb{C})$ denote the separately weak* continuous bilinear functionals on $\mathfrak{M}$.

Observations

1. $\mathcal{B}^2_\sigma(\mathfrak{M}, \mathbb{C})$ is a closed submodule of $(\mathfrak{M}\hat{\otimes}\mathfrak{M})^*$.
2. $\Delta^*\mathfrak{M}_* \subset \mathcal{B}^2_\sigma(\mathfrak{M}, \mathbb{C})$, so that Δ^{**} drops to a bimodule homomorphism $\Delta_\sigma : \mathcal{B}^2_\sigma(\mathfrak{M}, \mathbb{C})^* \to \mathfrak{M}$.

Normal virtual digaonals, II

Definition (E. G. Effros, 1988)

An element $\mathbf{D} \in \mathcal{B}^2_\sigma(\mathfrak{M}, \mathbb{C})^*$ is called a normal virtual diagonal for $\mathfrak{M}$ if

$$a \cdot \mathbf{D} = \mathbf{D} \cdot a \qquad (a \in \mathfrak{M})$$

and

$$a \Delta_\sigma \mathbf{D} = a \qquad (a \in \mathfrak{M}).$$

Theorem (E. G. Effros, 1988)

$\mathfrak{M}$ is Connes-amenable if and only if $\mathfrak{M}$ has a normal virtual diagonal.

Dual Banach algebras: the definition

Definition

A Banach algebra $\mathfrak{A}$ is called dual if there is a Banach space $\mathfrak{A}_*$ with $(\mathfrak{A}_*)^* = \mathfrak{A}$ such that multiplication in $\mathfrak{A}$ is separately weak* continuous.

Remarks

1. There is no reason for $\mathfrak{A}_*$ to be unique. The same Banach algebra $\mathfrak{A}$ can therefore carry different dual Banach algebra structures. (Often, the predual is clear from the context.)
2. The notions of Connes-amenability and normal virtual diagonals carry over to dual Banach algebras without modifications.

Some examples

Examples

1. Every von Neumann algebra.
2. $M(G)$ for every locally compact group G ($M(G)_* = \mathcal{C}_0(G)$).
3. $B(G)$ for every locally compact group G ($B(G)_* = C^*(G)$).
4. $\mathcal{B}(E)$ for every reflexive Banach space E ($\mathcal{B}(E)_* = E \hat{\otimes} E^*$).
5. Let $\mathfrak{A}$ be a Banach algebra and let $\mathfrak{A}^{**}$ be equipped with either Arens product. Then $\mathfrak{A}^{**}$ is a dual Banach algebra if and only if $\mathfrak{A}$ is Arens regular.
6. All weak* closed subalgebras of a dual Banach algebra are again dual Banach algebras.

Amenability and Connes-amenability, I

Proposition

Let $\mathfrak{A}$ be a dual Banach algebra, and let $\mathfrak{B}$ be a norm closed, amenable subalgebra of $\mathfrak{A}$ that is weak* dense in $\mathfrak{A}$. Then $\mathfrak{A}$ is Connes-amenable.

Corollary

If $\mathfrak{A}$ is amenable and Arens regular. Then $\mathfrak{A}^{**}$ is Connes-amenable.

Theorem (VR, 2001)

Suppose that $\mathfrak{A}$ is Arens regular and an ideal in $\mathfrak{A}^{**}$. Then the following are equivalent:

1. *$\mathfrak{A}$ is amenable;*
2. *$\mathfrak{A}^{**}$ is Connes-amenable.*

Amenability and Connes-amenability, II

Corollary

Let E be reflexive and have the approximation property. Then the following are equivalent:

1. $\mathcal{K}(E)$ *is amenable;*
2. $\mathcal{B}(E)$ *is Connes-amenable.*

Example (N. Grønbæk, BEJ, & G. A. Willis, 1994)

Let $p, q \in (1, \infty) \setminus \{2\}$ such that $p \neq q$. Then $\mathcal{K}(\ell^p \oplus \ell^q)$ is not amenable.
Hence, $\mathcal{B}(\ell^p \oplus \ell^q)$ is not Connes-amenable.

Normal virtual diagonals, III

Proposition

Suppose that $\mathfrak{A}$ has a normal virtual diagonal. Then $\mathfrak{A}$ is Connes-amenable.

Theorem (VR, 2003)

The following are equivalent for a locally compact group G:

1. *G is amenable;*
2. *$M(G)$ is Connes-amenable;*
3. *$M(G)$ has a normal virtual diagonal.*

Corollary

$\ell^1(G)$ is amenable if and only if it is Connes-amenable.

Weakly almost periodic functions

Definition

A bounded continuous function $f : G \to \mathbb{C}$ is called weakly almost periodic if $\{L_x f : x \in G\}$ is relatively weakly compact in $\mathcal{C}_b(G)$. We set

$$\mathrm{WAP}(G) := \{f \in \mathcal{C}_b(G) : f \text{ is weakly almost periodic}\}.$$

Remark

$\mathrm{WAP}(G)$ is a commutative C^*-algebra.
Its character space wG is a compact semigroup with separately continuous multiplication containing G as a dense subsemigroup.
This turns $\mathrm{WAP}(G)^* \cong M(wG)$ into a dual Banach algebra.

Normal virtual diagonals, IV

Proposition

The following are equivalent:

1. *G is amenable;*
2. $\mathrm{WAP}(G)^*$ *is Connes-amenable.*

Theorem (VR, 2006)

Suppose that G has small invariant neighborhoods, e.g, is compact, discrete, or abelian. Then the following are equivalent:

1. $\mathrm{WAP}(G)^*$ *has a normal virtual diagonal;*
2. *G is compact.*

Minimally weakly almost periodic groups

Definition

A bounded continuous function $f : G \to \mathbb{C}$ is called almost periodic if $\{L_x f : x \in G\}$ is relatively compact in $\mathcal{C}_b(G)$. We set

$$\mathrm{AP}(G) := \{f \in \mathcal{C}_b(G) : f \text{ is almost periodic}\}.$$

We call G minimally weakly almost periodic (m.w.a.p.) if

$$\mathrm{WAP}(G) = \mathrm{AP}(G) + \mathcal{C}_0(G).$$

Normal virtual diagonals, V

Proposition

Suppose that G is amenable and m.w.a.p. Then $\mathrm{WAP}(G)$ *has a normal virtual diagonal.*

Examples

1. All compact groups are m.w.a.p..
2. $\mathrm{SL}(2, \mathbb{R})$ is m.w.a.p., but not amenable.
3. The motion group $\mathbb{R}^N \rtimes \mathrm{SO}(N)$ is m.w.a.p. for $N \geq 2$ and amenable.

Question

Does $\mathrm{WAP}(G)^*$ have a normal virtual diagonal if and only if G is amenable and m.w.a.p.?

Daws' representation theorem

Recall...

$\mathcal{B}(E)$ is a dual Banach algebra for reflexive E, as is each weak* closed subalgebra of it.

Theorem (M. Daws, 2007)

Let $\mathfrak{A}$ be a dual Banach algebra. Then there are a reflexive Banach space E and an isometric, weak*-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$.

In short...

Every dual Banach algebra "is" a subalgebra of $\mathcal{B}(E)$ for some reflexive E.

Injectivity, I

Recall...

A von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathfrak{H})$ is called injective if there is a norm one projection $\mathcal{E} : \mathcal{B}(\mathfrak{H}) \to \mathfrak{M}'$.

Question

How does the notion of injectivity extend to dual Banach algebras? And how does this extended notion relate to Connes-amenability?

Quasi-expectations

Theorem (J. Tomiyama, 1970)

Let $\mathfrak{A}$ be a C^*-algebra, let $\mathfrak{B}$ be a C^*-subalgebra of $\mathfrak{A}$, and let $\mathcal{E} : \mathfrak{A} \to \mathfrak{B}$ be a norm one projection, an expectation. Then

$$\mathcal{E}(abc) = a(\mathcal{E}b)c \qquad (a, c \in \mathfrak{B},\ b \in \mathfrak{A}).$$

Definition

Let $\mathfrak{A}$ be a Banach algebra, and let $\mathfrak{B}$ be a closed subalgebra. A bounded projection $\mathcal{Q} : \mathfrak{A} \to \mathfrak{B}$ is called a quasi-expectation if

$$\mathcal{Q}(abc) = a(\mathcal{Q}b)c \qquad (a, c \in \mathfrak{B},\ b \in \mathfrak{A}).$$

Injectivity, II

"Definition"

We call $\mathfrak{A}$ "injective" if there are a reflexive Banach space E, an isometric, weak*-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$, and a quasi-expectation $\mathcal{Q} : \mathcal{B}(E) \to \pi(\mathfrak{A})'$.

Easy

Connes-amenability implies "injectivity", but...

Example

For $p, q \in (1, \infty) \setminus \{2\}$ with $p \neq q$, $\mathcal{B}(\ell^p \oplus \ell^q)$ is not Connes-amenable, but trivially "injective".

Injectivity, III

Definition (M. Daws, 2007)

A dual Banach algebra $\mathfrak{A}$ is called injective if, for each reflexive Banach space E and for each weak*-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$, there is a quasi-expectation $\mathcal{Q} : \mathcal{B}(E) \to \pi(\mathfrak{A})'$.

Theorem (M. Daws, 2007)

The following are equivalent for a dual Banach algebra $\mathfrak{A}$:

1. $\mathfrak{A}$ *is injective;*
2. $\mathfrak{A}$ *is Connes-amenable.*

A bicommutant theorem

Question

Does von Neumann's bicommutant theorem extend to general dual Banach algebras?

Example

Let

$$\mathfrak{A} := \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{C} \right\}.$$

Then $\mathfrak{A} \subset \mathcal{B}(\mathbb{C}^2)$ is a dual Banach algebra, but $\mathfrak{A}'' = \mathcal{B}(\mathbb{C}^2)$.

Theorem (M. Daws, 2010)

Let $\mathfrak{A}$ be a unital dual Banach algebra. Then there are a reflexive Banach space E and a unital, isometric, weak*-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$ such that $\pi(\mathfrak{A}) = \pi(\mathfrak{A})''$.

Uniqueness of the predual, I

Question

Let $\mathfrak{A}$ be a dual Banach algebra with predual $\mathfrak{A}_*$. To what extent is $\mathfrak{A}_*$ unique?

Example

Let E be a Banach space with two different preduals, e.g., $E = \ell^1$.
Equip E with the zero product.
Then E is dual Banach algebra, necessarily with two different preduals.

Uniqueness of the predual, II

Theorem (G. Godefroy & P. D. Saphar, 1988)

If E is reflexive, then $E\hat{\otimes}E^*$ is the unique isometric predual of $\mathcal{B}(E)$.

Theorem (M. Daws, 2007)

If E is reflexive and has the approximation property, then $E\hat{\otimes}E^*$ is the unique isomorphic predual of $\mathcal{B}(E)$ turning it into a dual Banach algebra.

Amenability and Connes-amenability

Question

Does every Connes-amenable dual Banach algebra contain a closed, weak* dense, amenable subalgebra?

Question

Let $\mathfrak{A}$ be Arens regular such that $\mathfrak{A}^{**}$ is Connes-amenable. Is $\mathfrak{A}$ amenable?

Theorem (VR, 2001)

Suppose that every bounded linear map from $\mathfrak{A}$ to $\mathfrak{A}^*$ is weakly compact and that $\mathfrak{A}^{**}$ has a normal virtual diagonal. Then $\mathfrak{A}$ is amenable.

Representation theory

Recall...

A von Neumann algebra $\mathfrak{M} \subset \mathcal{B}(\mathfrak{H})$ is injective if and only if there is an expectation $\mathcal{E} : \mathcal{B}(\mathfrak{H}) \to \mathfrak{M}$.

Question

Consider the following property of a unital dual Banach algebra $\mathfrak{A}$:

> *For each reflexive Banach space E, and for each unital, weak*-weak* continuous algebra homomorphism $\pi : \mathfrak{A} \to \mathcal{B}(E)$, there is a quasi-expectation $\mathcal{Q} : \mathcal{B}(E) \to \overline{\pi(\mathfrak{A})}$.*

How does this property relate to injectivity?

Fourier–Stieltjes algebras

Question

For which G is $B(G)$ Connes-amenable?

Conjecture

$B(G)$ is Connes-amenable if and only if G has an abelian subgroup of finite index.

Theorem (F. Uygul, 2007)

The following are equivalent for discrete G:

1. *$B(G)$ is Connes-amenable;*
2. *G has an abelian subgroup of finite index.*