Amenability of operator algebras on Banach spaces, I

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Philosophical musings

A quote

Big things are fucking rarely amenable!

N.N., Istanbul, 2004.

Thus...

$$\text{AMENABLE} \approx \text{SMALL}$$

Mean things

Definition

Let G be a locally compact group. A mean on $L^\infty(G)$ is a functional $M \in L^\infty(G)^*$ such that $\langle 1, M \rangle = \|M\| = 1$.

Definition (J. von Neumann 1929; M. M. Day, 1949)

G is amenable if there is a mean on $L^\infty(G)$ which is left invariant, i.e.,

$$\langle L_x \phi, M \rangle = \langle \phi, M \rangle \qquad (x \in G,\, \phi \in L^\infty(G)),$$

where

$$(L_x \phi)(y) := \phi(xy) \qquad (y \in G).$$

Some amenable groups. . .

Examples

1. Compact groups amenable: M = Haar measure.
2. Abelian groups are amenable: use Markov–Kakutani to get M.

Really nice hereditary properties!

1. If G is amenable and $H < G$, then H is amenable.
2. If G is is amenable and $N \lhd G$, then G/N is amenable.
3. If G and $N \lhd G$ are such that N and G/N are amenable, then G is amenable.
4. If $(H_\alpha)_\alpha$ is a directed union of closed, amenable subgroups of G such that $G = \overline{\bigcup_\alpha H_\alpha}$, then G is amenable.

... and a non-amenable one, I

Example

Let $\mathbb{F}_2$ be the free group in two generators.
Assume that there is a left invariant mean M on $\ell^\infty(\mathbb{F}_2)$.
Define

$$\mu : \mathfrak{P}(\mathbb{F}_2) \to [0,1], \quad E \mapsto \langle \chi_E, M \rangle.$$

Then

- μ is finitely additive,
- $\mu(\mathbb{F}_2) = 1$, and
- $\mu(xE) = \mu(E) \qquad (x \in \mathbb{F}_2,\ E \subset \mathbb{F}_2)$.

. . . and a non-amenable one, II

Example (continued. . .)

For $x \in \{a, b, a^{-1}, b^{-1}\}$ set

$$W(x) := \{w \in \mathbb{F}_2 : w \text{ starts with } x\}.$$

Let $w \in \mathbb{F}_2 \setminus W(a)$. Then $a^{-1}w \in W(a^{-1})$, therefore

$$w \in aW(a^{-1}),$$

and thus

$$\mathbb{F}_2 = W(a) \cup aW(a^{-1}).$$

Similarly,

$$\mathbb{F}_2 = W(b) \cup bW(b^{-1})$$

holds.

. . . and a non-amenable one, III

Example (continued even further)

Since

$$\mathbb{F}_2 = \{\epsilon\} \,\dot{\cup}\, W(a) \,\dot{\cup}\, W(a^{-1}) \,\dot{\cup}\, W(b) \,\dot{\cup}\, W(b^{-1}),$$

we have

$$\begin{aligned} 1 &= \mu(\mathbb{F}_2) \\ &\geq \mu(W(a)) + \mu(aW(a^{-1})) + \mu(W(b)) + \mu(bW(b^{-1})) \\ &\geq \mu(W(a) \cup aW(a^{-1})) + \mu(W(b) \cup bW(b^{-1})) \\ &= \mu(\mathbb{F}_2) + \mu(\mathbb{F}_2) \\ &= 2, \end{aligned}$$

which is nonsense.

Consequences

Hence, G is amenable if...

1. G is solvable or
2. G is locally finite.

But G is not amenable if...

G contains $\mathbb{F}_2$ as closed subgroup, e.g., if

- $G = \mathrm{SL}(N, \mathbb{R})$ with $N \geq 2$,
- $G = \mathrm{GL}(N, \mathbb{R})$ with $N \geq 2$, or
- $G = \mathrm{SO}(N, \mathbb{R})$ with $N \geq 3$ equipped with the discrete topology.

Amenable Banach algebras via approximate diagonals

Definition (B. E. Johnson, 1972)

A Banach algebra $\mathfrak{A}$ is said to be amenable if it possesses an approximate diagonal, i.e., a bounded net $(\mathbf{d}_\alpha)_\alpha$ in the projective tensor product $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ such that

$$a \cdot \mathbf{d}_\alpha - \mathbf{d}_\alpha \cdot a \to 0 \qquad (a \in \mathfrak{A})$$

and

$$a\Delta\mathbf{d}_\alpha \to a \qquad (a \in \mathfrak{A})$$

with $\Delta : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}$ denoting multiplication.

Amenable Banach algebras and amenable groups

Theorem (B. E. Johnson, 1972)

The following are equivalent for a locally compact group G:

1. *G is amenable;*
2. *$L^1(G)$ is amenable.*

Grand theme

Let $\mathcal{C}$ be a class of Banach algebras. Characterize the amenable members of $\mathcal{C}$!

More from abstract harmonic analysis

Theorem (H. G. Dales, F. Ghahramani, & A. Ya. Helemskiĭ, 2002)

The following are equivalent:

1. $M(G)$ *is amenable;*
2. G *is amenable and discrete.*

Theorem (B. E. Forrest & VR, 2005)

The following are equivalent:

1. $A(G)$ *is amenable;*
2. G *has an abelian subgroup of finite index.*

From old to new...

Hereditary properties

1. If $\mathfrak{A}$ is amenable and $\theta : \mathfrak{A} \to \mathfrak{B}$ is a bounded homomorphism with dense range, then $\mathfrak{B}$ is amenable. In particular, if $I \lhd \mathfrak{A}$, then $\mathfrak{A}/I$ is amenable.
2. If $I \lhd \mathfrak{A}$ is such that both I and $\mathfrak{A}/I$ are amenable, then $\mathfrak{A}$ is amenable.
3. If $\mathfrak{A}$ is amenable and $I \lhd \mathfrak{A}$, then the following are equivalent:
   1. I is amenable;
   2. I has a bounded approximate identity;
   3. I is weakly complemented in $\mathfrak{A}$, i.e., $I^\perp$ is complemented in $\mathfrak{A}^*$.

Grand theme, reprise!

The "meaning" of amenability

What does it mean for a member of a class $\mathcal{C}$ of Banach algebras to be amenable for the following classes $\mathcal{C}$?

1. all C^*-algebras;
2. all von Neumann algebras;
3. all norm closed, but not necessarily self-adjoint subalgebras of $\mathcal{B}(\mathfrak{H})$;
4. all algebras $\mathcal{K}(E)$;
5. all algebras $\mathcal{B}(E)$.

Remember. . .

AMENABLE $\approx$ SMALL

Complete positivity

If $\mathfrak{A}$ and $\mathfrak{B}$ are C^*-algebras, then so are $M_n(\mathfrak{A})$ and $M_n(\mathfrak{B})$ for all $n \in \mathbb{N}$. Whenever $T : \mathfrak{A} \to \mathfrak{B}$ is linear, we write $T^{(n)} : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ for its amplification, i.e.,

$$T^{(n)}[a_{j,k}] := [Ta_{j,k}] \qquad ([a_{j,k}] \in M_n(\mathfrak{A})).$$

Definition

$T : \mathfrak{A} \to \mathfrak{B}$ is called completely positive if $T^{(n)} : M_n(\mathfrak{A}) \to M_n(\mathfrak{B})$ is positive for each $n \in \mathbb{N}$.

Example

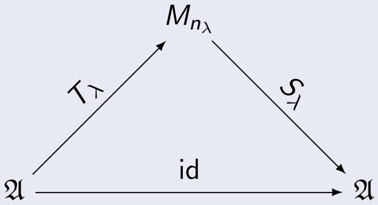
$$M_2 \to M_2, \quad a \mapsto a^t$$

is positive, but not completely positive.

Nuclear C^*-algebras

Definition

A C^*-algebra $\mathfrak{A}$ is called nuclear if there are nets $(n_\lambda)_\lambda$ of positive integers and of completely positive contractions



such that

$$(S_\lambda \circ T_\lambda)a \to a \qquad (a \in \mathfrak{A}).$$

Nuclearity and amenability

Theorem (A. Connes, 1978)

A C^*-algebra $\mathfrak{A}$ is nuclear if it is amenable and $\mathfrak{A}^*$ is separable.

Theorem (U. Haagerup, 1983)

All nuclear C^*-algebras are amenable.

Theorem (A. Connes, U. Haagerup, et al.)

The following are equivalent for a C^*-algebra $\mathfrak{A}$:

1. *$\mathfrak{A}$ is nuclear;*
2. *$\mathfrak{A}$ is amenable.*

Nuclear von Neumann algebras

Theorem (S. Wasserman, 1976)

The following are equivalent for a von Neumann algebra $\mathfrak{M}$:

1. *$\mathfrak{M}$ is nuclear;*
2. *$\mathfrak{M}$ is subhomogeneous, i.e.,*

$$\mathfrak{M} \cong M_{n_1}(\mathfrak{M}_1) \oplus \cdots \oplus M_{n_k}(\mathfrak{M}_k)$$

with $n_1, \ldots, n_k \in \mathbb{N}$ and $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ abelian.

Corollary

$\mathcal{B}(\mathfrak{H})$ is amenable if and only if $\dim \mathfrak{H} < \infty$.

Representations of locally compact groups

Definition

A representation of G on a Hilbert space $\mathfrak{H}$ is a group homomomorphism π from G into the invertible elements of $\mathcal{B}(\mathfrak{H})$ which is continuous with respect to the given topology on G and the weakstrong operator topology on $\mathcal{B}(\mathfrak{H})$.

We call π:

1. unitary if $\pi(G)$ consists of unitaries,
2. unitarizable if π is similar to a unitary representation, i.e., there is an invertible $T \in \mathcal{B}(\mathfrak{H})$ such that $T^{-1}\pi(\cdot)T$ is unitary, and
3. uniformly bounded if

$$\sup_{g \in G} \|\pi(g)\| < \infty.$$

Unitarizability and amenability

Obvious...

$$\pi \text{ unitary} \implies \pi \text{ unitarizable} \implies \pi \text{ uniformly bounded.}$$

Theorem (J. Dixmier, 1950)

Suppose that G is amenable. Then every uniformly bounded representation of G is unitarizable.

Big open question

Does the converse hold, i.e., is any G such that each uniformly bounded represenation is unitarizable already amenable?

Fact

It's false for $\mathbb{F}_2$!

From groups to algebras

Integration of representations

If π is a uniformly bounded representation of G, we can integrate π and obtain a representation of $L^1(G)$ on $\mathfrak{H}$:

$$\pi(f) := \int_G f(g)\pi(g)\,dg \qquad (f \in L^1(G)).$$

Easy

If G is amenable, then $\mathfrak{A} := \overline{\pi(L^1(G))}$ is amenable.

Slightly more difficult

If G is amenable, then π is unitarizable, so that there is an invertible $T \in \mathcal{B}(\mathfrak{H})$ such that $T\mathfrak{A}T^{-1}$ is a C^*-subalgebra of $\mathcal{B}(\mathfrak{H})$.

The similarity problem for amenable operator algebras

Definition

A closed, but not necessarily self-adjoint subalgebra of $\mathcal{B}(\mathfrak{H})$ is called similar to a C^*-algebra if there is an invertible $T \in \mathcal{B}(\mathfrak{H})$ such that $T\mathfrak{A}T^{-1}$ is a C^*-subalgebra of $\mathcal{B}(\mathfrak{H})$.

Big open question

Is every closed, amenable subalgebra of $\mathcal{B}(\mathfrak{H})$ similar to a C^*-subalgebra of $\mathcal{B}(\mathfrak{H})$ (which is necessarily nuclear)?

Some partial results

Theorem (J. A. Gifford, <2006)

Suppose that $\mathfrak{A}$ is a closed, amenable subalgebra of $\mathcal{K}(\mathfrak{H})$. Then $\mathfrak{A}$ is similar to a C^*-subalgebra of $\mathcal{K}(\mathfrak{H})$.

Definition

Let $C \geq 1$. We call $\mathfrak{A}$ C-amenable if $\mathfrak{A}$ has an approximate diagonal bounded by C.

Theorem (D. Blecher & C. LeMerdy, 2004)

Let $\mathfrak{A}$ be a closed, 1-amenable subalgebra of $\mathcal{B}(\mathfrak{H})$. Then $\mathfrak{A}$ is a nuclear C^*-algebra.

Open question

What if $\mathfrak{A}$ is commutative, even generated by one operator?