Amenability of operator algebras on Banach spaces, I

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University of Alberta

NBFAS, Leeds, June 1, 2010

Philosophical musings

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A quote

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$$\text{AMENABLE} \approx \text{SMALL}$$

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AMENABLE ≈ SMALL

Mean things

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Let G be a locally compact group.

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Let G be a locally compact group. A mean on $L^\infty(G)$

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where

$$(L_x \phi)(y) := \phi(xy) \qquad (y \in G).$$

Some amenable groups. . .

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- $\mu(xE) = \mu(E) \qquad (x \in \mathbb{F}_2,\ E \subset \mathbb{F}_2)$.

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Example (continued. . .)

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Example (continued. . .)

For $x \in \{a, b, a^{-1}, b^{-1}\}$ set

$$W(x) := \{w \in \mathbb{F}_2 : w \text{ starts with } x\}.$$

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Similarly,

$$\mathbb{F}_2 = W(b) \cup bW(b^{-1})$$

holds.

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Consequences

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- $G = \mathrm{GL}(N, \mathbb{R})$ with $N \geq 2$, or
- $G = \mathrm{SO}(N, \mathbb{R})$ with $N \geq 3$ equipped with the discrete topology.

Amenable Banach algebras via approximate diagonals

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with $\Delta : \mathfrak{A}\hat{\otimes}\mathfrak{A} \to \mathfrak{A}$ denoting multiplication.

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Let $\mathcal{C}$ be a class of Banach algebras. Characterize the amenable members of $\mathcal{C}$!

More from abstract harmonic analysis

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Theorem (H. G. Dales, F. Ghahramani, & A. Ya. Helemskiĭ, 2002)

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AMENABLE $\approx$ SMALL

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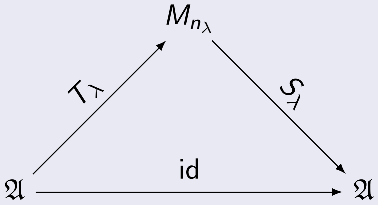

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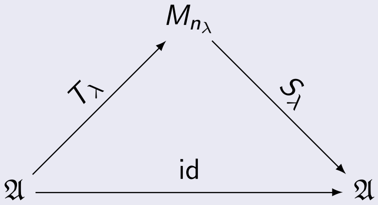

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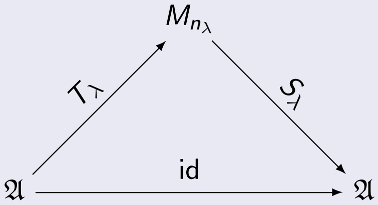

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Nuclear von Neumann algebras

Theorem (S. Wasserman, 1976)

The following are equivalent for a von Neumann algebra $\mathfrak{M}$:

1. $\mathfrak{M}$ *is nuclear;*
2. $\mathfrak{M}$ *is subhomogeneous, i.e.,*

$$\mathfrak{M} \cong M_{n_1}(\mathfrak{M}_1) \oplus \cdots \oplus M_{n_k}(\mathfrak{M}_k)$$

with $n_1, \ldots, n_k \in \mathbb{N}$ *and* $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ *abelian.*

Corollary

$\mathcal{B}(\mathfrak{H})$ *is amenable if and only if* $\dim \mathfrak{H} < \infty$.

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A **representation** of G on a Hilbert space $\mathfrak{H}$ is a group homomomorphism π from G into the invertible elements of $\mathcal{B}(\mathfrak{H})$ which is continuous with respect to the given topology on G and the weak operator topology on $\mathcal{B}(\mathfrak{H})$.

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Fact

It's false for $\mathbb{F}_2$!

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If G is amenable, then π is unitarizable, so that there is an invertible $T \in \mathcal{B}(\mathfrak{H})$ such that $T\mathfrak{A}T^{-1}$ is a C^*-subalgebra of $\mathcal{B}(\mathfrak{H})$.

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What if $\mathfrak{A}$ is commutative, even generated by one operator?